COLLOQUIUM MATHEMATICUM

VOL. 76 1998 NO. 2

THREE-DIMENSIONAL COOPERATIVE IRREDUCIBLE SYSTEMS WITH A FIRST INTEGRAL

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1. Introduction. This paper deals with a class of three-dimensional cooperative irreducible systems of ordinary differential equations (ODEs)

$$\dot{x}^i = F^i(x), \qquad x \in X, \tag{1}$$

where $F = (F^1, F^2, F^3)$ is a vector field of class C^1, defined on an open subset X of the three-dimensional real affine space $A = \{x = (x^1, x^2, x^3)\}$. A system (1) is *cooperative* if $\partial F^i/\partial x^j \geq 0$ for $i \neq j$. A cooperative system is *irreducible* if the matrix $DF = [\partial F^i/\partial x^j]_{i,j=1}^3$ is irreducible.

The symbol $\phi = \{\phi_t\}$ stands for the local flow generated on X by (1): $\phi_t x = y$ if $\varphi(x,t) = y$, where $\varphi(x,\cdot)$ is the unique noncontinuable solution of (1) with initial condition $\varphi(x,0) = x$. We shall usually write $x \cdot t$ instead of $\phi_t x$. For $x \in X$ the domain of $t \mapsto x \cdot t$ is an open interval $(\sigma(x), \tau(x)) \ni 0$, where $\sigma(x)$ [$\tau(x)$] is called the *backward* [*forward*] *escape time* for x. The *backward* [*forward*] *semiorbit* of $x \in X$ is defined as $O_{\mathrm{b}}(x) := \{x \cdot t : t \in (\sigma(x), 0]\}$ [$O_{\mathrm{f}}(x) := \{x \cdot t : t \in [0, \tau(x)\}$]. The *orbit* $O(x)$ of x is the union of its backward and forward semiorbits. A set $Y \subset X$ is called *backward* [*forward*] *invariant* if for each $x \in Y$, $O_{\mathrm{b}}(x) \subset Y$ [$O_{\mathrm{f}}(x) \subset Y$]. Y is *invariant* if it is both backward and forward invariant.

A point $y \in X$ is called an ω*-limit point* of $x \in X$ if there is a sequence $t_n \to \infty$ such that $x \cdot t_n \to y$ as $n \to \infty$. The set of all ω-limit points of x is called the ω*-limit set* of x and denoted by $\omega(x)$. An ω-limit set is invariant. An *equilibrium* is a point $x \in X$ such that $F(x) = 0$, or equivalently, $x \cdot t = x$ for all t. We denote the set of all equilibria by E. By a *cycle* we mean the orbit of a point $x \in X$ such that $x \cdot t \neq x$ for $t \in (0, T)$ and $x \cdot T = x$. (Notice that for an equilibrium y its orbit $\{y\}$ is not a cycle.)

By a *first integral* for (1) we shall understand a C^1 function $H : \operatorname{dom} H \to \mathbb{R}$ such that $\langle dH(x), F(x) \rangle = 0$ for all $x \in \operatorname{dom} H$, where the domain $\operatorname{dom} H$

1991 *Mathematics Subject Classification*: Primary 34C05.

Research supported by KBN grant 2 P03A 076 08.

of H is an invariant open subset of X, $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $dH(x)$ denotes the derivative of H at x. Observe that H need not be defined on the whole of X. If H is constant on some $Y \subset \operatorname{dom} H$, the symbol $\mathcal{H}(Y)$ stands for the level set of H passing through Y.

In the existing literature on cooperative systems, the first integral has almost always had some monotonicity properties. That has far-reaching consequences, for example, if the gradient of the C^1 first integral is everywhere positive then each ω-limit set either is empty or consists of a single equilibrium (see the author's note [6]). The only exception is Theorem 4.7 in M. W. Hirsch's paper [3]: If there are countably many equilibria and X is a connected open subset of a real affine space such that almost every point in X has compact forward semiorbit closure then every continuous first integral is constant on X. The proof relies heavily on results about generic convergence to equilibria.

The motivation for the present research came from the paper [1] by Grammaticos *et al.*, where a wide class of first integrals was found for three-dimensional Lotka–Volterra systems. Those first integrals are compositions of rational functions and exponentials, and there is no hope for them to be defined everywhere, not to speak of having positive gradient. I am grateful to Professor J.-M. Strelcyn for calling my attention to [1].

We now state our main theorem.

THEOREM 1.1. *Let* (1) *be a* C^1 *three-dimensional cooperative irreducible system of ODEs defined on the positive orthant* $A_{++} := \{x \in A : x^i > 0$ *for* $i = 1, 2, 3\}$. *Assume that* H *is a first integral for* (1). *If* $x \in A_{++} \setminus E$ *does not belong to a cycle and* $y \in \omega(x) \cap \operatorname{dom} H$ *is such that* $dH(y)$ *is nonzero*, *then* $\omega(x) = \{y\}$.

See Theorems 2.2 and 2.3 in Section 2.

We will distinguish between the affine space A (consisting of points) and the three-dimensional real Euclidean space V, that is, the set of all (free) vectors $v = (v^1, v^2, v^3)$. The dual space to V will be denoted by V^*. As we use only the standard bases in V and V^*, we will freely identify vectors [functionals] with their representations in the respective bases.

The set $C := \{v \in V : v^i \geq 0$ for $i = 1, 2, 3\}$ is called the (*nonnegative*) *cone*. Evidently the interior C° of C equals $\{v \in C : v^i > 0$ for $i = 1, 2, 3\}$. D stands for the nonnegative cone in the dual space V^*, and D° stands for the interior of D.

For any two points $x, y \in A$, we write $x \leq y$ if $y - x \in C$, and $x \ll y$ if $y - x \in C^\circ$. Also, $x < y$ means that $x \leq y$ and $x \neq y$. The reversed inequalities are obvious. We say that $Y \subset A$ is *p-convex* if for any two points $x, y \in Y$ with $x < y$ the line segment joining them is contained in Y. A set $Y \subset A$ is *strongly balanced* if there are no $x, y \in Y$ with $x < y$. The symbol $B(x;r)$ will stand for the closed ball in A with center x and radius $r > 0$.

Denote by $\Phi = \{\Phi_t\}$ the *derivative* (*local*) *flow* of (1),

$$\Phi_t(x,v) = (\phi_t x, \psi_t(x)v) \quad \text{for } x \in X, v \in V \text{ and } t \in (\sigma(x), \tau(x)).$$

Here $\psi_t(x)v$ is the unique noncontinuable solution of the variational system of ODEs $dv/dt = DF(\phi_t x)v$ with initial condition $v(0) = v$.

The derivative flow is a local linear skew-product flow on the tangent bundle $X \times V$ of X: for each $x \in X$ and $t \in (\sigma(x), \tau(x))$ the linear operator $\psi_t(x)$ acts from the tangent space at x onto the tangent space at $\phi_t x$.

We can associate with Φ the *dual* local linear skew-product flow Φ^* on the cotangent bundle $X \times V^*$ of X, defined abstractly as

$$\Phi_t^*(x,v^*) = (\phi_{-t}x, (\psi_t(\phi_{-t}x))^* v^*) \qquad \text{for } x \in X,\ v^* \in V^* \text{ and } t \in (-\tau(x), -\sigma(x)),$$

where $(\psi_t(\phi_{-t}x))^*$ is the linear operator from the cotangent space at x onto the cotangent space at $\phi_{-t}x$ dual to the linear operator $\psi_t(\phi_{-t}x)$ from the tangent space at $\phi_{-t}x$ onto the tangent space at x.

For $v^* \in V^*$ the linear functional $(\psi_t(\phi_{-t}x))^* \in V^*$ equals the value at time $s = t$ of the solution of the three-dimensional system of linear ODEs

$$\frac{dv^*}{ds} = (DF(\phi_{-s}(x)))^{\mathrm{T}} v^*$$

with initial condition $v^*(0) = v^*$, where $^{\mathrm{T}}$ stands for the matrix transpose.

Notice that if $x \in \operatorname{dom} H$, where H is a C^1 first integral, then $dH(x) = (\psi_t(x))^* dH(\phi_t x)$ for all $t \in (\sigma(x), \tau(x))$. As $(\psi_t(x))^*$ are linear automorphisms, the open set of those $x \in \operatorname{dom} H$ at which $dH(x)$ is nonzero is invariant.

An important property of cooperative irreducible systems of ODEs is contained in the following result, due essentially to M. Müller and E. Kamke.

Proposition 1.2. *Assume that* (1) *is a* C^1 *cooperative irreducible system of ODEs defined on an open set* $X \subset A$. *Then*

(a) *If* $x \in X$ *then* $\psi_t(x)(C \setminus \{0\}) \subset C^\circ$ *for* $0 < t < \tau(x)$.
(b) *If* $x \in X$ *then* $(\psi_t(\phi_{-t}x))^*(D \setminus \{0\}) \subset D^\circ$ *for* $0 < t < -\sigma(x)$.

The next result is a consequence of the above proposition.

Proposition 1.3. *Assume that* (1) *is a* C^1 *cooperative irreducible system of ODEs defined on an open p-convex set* $X \subset A$. *If* $x < y$ *then* $\phi_t x \ll \phi_t y$ *for* $0 < t < \min\{\tau(x), \tau(y)\}$.

For proofs see e.g. Hirsch's paper [2].

Henceforward our standard assumption will be:

(1) *is a* C^1 *cooperative irreducible system of ODEs defined on an open* p*-convex subset* X *of* A *admitting a* C^1 *first integral* $H : \operatorname{dom} H \to \mathbb{R}$.

Finally, we state a slightly more general version of Theorem 1.1.

THEOREM 1.4. *For* $x \in X \setminus E$ *not belonging to a cycle*, *if* $y \in \omega(x) \cap \operatorname{dom} H$ *is such that* $dH(y)$ *is nonzero*, *then* $\omega(x) = \{y\}$.

2. Proof of Theorem 1.4. We begin by formulating an easy consequence of the implicit function theorem.

LEMMA 2.1. *Let* $x \in \operatorname{dom} H$ *be such that* $dH(x)$ *is nonzero. Then there is an open neighborhood* U *of* x *in* X *with the following properties*:

(i) $\mathcal{H}(x) \cap U$ *is a* C^1 *embedded two-dimensional submanifold.*

(ii) *At each* $z \in \mathcal{H}(x) \cap U$ *the tangent space of* $\mathcal{H}(x)$ *at* z *equals the nullspace of the linear functional* $dH(z)$.

The proof of Theorem 1.4 will be carried out in two steps. First, we exclude the case of $\omega(x)$ containing points not being equilibria (Theorem 2.2). Next, in Theorem 2.3 we prove that equilibria must be isolated in $\omega(x) \cap E$.

THEOREM 2.2. *Assume that* $x \in X \setminus E$ *does not belong to a cycle. Let* $y \in \omega(x) \cap \operatorname{dom} H$ *be such that* $dH(y)$ *is nonzero. Then* $y \in E$.

Proof. Consider first the case $dH(y) \in D$. We can assume $dH(y) \in D^\circ$, since otherwise in view of Proposition 1.2(b) we can replace y by $y \cdot s$ for some $\sigma(y) < s < 0$. Again from Proposition 1.2(b) we deduce that $dH(x \cdot t) \in D^\circ$ for all $t \in (\sigma(x), \infty)$. An application of the Main Theorem in Mierczyński [6] gives that $y \in E$.

Assume now that $dH(y) \notin D$, that is, $dH(y) = (w^1, w^2, w^3)$ with some nonzero w^i and w^j of different signs. It is easy to see that the intersection of the nullspace of $dH(y)$ with the interior C° of the cone C is nonempty. Suppose by way of contradiction that $y \notin E$. Then there is a vector $v \in C^\circ$ not collinear with $F(y)$ and tangent at y to $\mathcal{H}(y)$. Take a C^1 arc $M \subset \mathcal{H}(y)$ tangent at y to v and so small as to be linearly ordered by $\ll$. The arc M is a transversal for the vector field $F|\mathcal{H}(y)$. As $y \in \omega(x)$, there is a sequence $t_n \to \infty$ such that $x \cdot t_n \in M$ and $x \cdot t_n \to y$ as $n \to \infty$. Assume for definiteness $x \cdot t_1 \ll x \cdot t_2$. By choosing a subsequence, if necessary, we can assume $x \ll x \cdot t_1 \ll \ldots \ll x \cdot t_n \ll x \cdot t_{n+1} \ll \ldots \ll y$, and $\lim_{n\to\infty} x \cdot t_n = y$. In order not to have too complicated indices, we do now some relabeling. Put $x_1 := x \cdot t_1$, $x_2 := x \cdot t_2$. Also, we write t_{n-1} instead of $t_n - t_1$. Therefore, we now have $x_2 = x_1 \cdot t_1$. In the new notation

$$x_1 \ll x_1 \cdot t_1 \ll \ldots \ll x_1 \cdot t_n \ll x_1 \cdot t_{n+1} \ll \ldots \ll y.$$

As a consequence of strong monotonicity,

$$x_2 \ll x_2 \cdot t_1 \ll \ldots \ll x_2 \cdot t_n \ll x_2 \cdot t_{n+1} \ll \ldots$$

Put

$$\eta := \inf\{\|x_1 \cdot t_n - x_2 \cdot t_n\|\}.$$

We claim $\eta > 0$. Indeed, if not then (after possibly taking a subsequence) $\|x_1 \cdot t_n - x_2 \cdot t_n\| \to 0$ as $n \to \infty$. From this it follows that $\lim_{n\to\infty} x_2 \cdot t_n = y$, which by the Colimiting Principle (see e.g. Smith [9]) gives $y \in E$.

Suppose that $x_2 \cdot t_n$ converges to some $y' \in X$. From the above paragraph we get $y < y'$. We can assume $y \ll y'$, since otherwise we replace x_i, $i = 1, 2$, by $x_i \cdot s$ for some $0 < s < \tau(y)$. We have thus obtained two points, $y, y' \in \omega(x)$ being in the $\ll$ relation, which contradicts Theorem 3.3.2 of [9]. (That theorem is formulated for $\omega(x)$ compact, but it holds without assuming compactness.)

Finally, suppose that the sequence $x_2 \cdot t_n$ does not have a limit in X. Since the function $\sigma(\cdot)$ is upper semicontinuous, we can find a closed neighborhood $U \subset X$ of M and a number $\theta > 0$ such that $\sigma(z) < -\theta$ for each $z \in U$.

Consider the set

$$\mathcal{S} := \left\{ \frac{\psi_\theta(z)v}{\|\psi_\theta(z)v\|} : z \in \phi_{-\theta}U,\ v \in C,\ \|v\| = 1 \right\}.$$

From Proposition 1.2(a) we derive that $\mathcal{S}$ is compact and contained in $\{v \in C^\circ : \|v\| = 1\}$. Consequently, there exists a closed convex cone $\mathcal{C}$ in V such that $\mathcal{C} \setminus \{0\} \subset C^\circ$ and $\mathcal{S} \subset \mathcal{C}$. The cone $\mathcal{C}$ is defined in the tangent space V, but the natural exponential mapping allows us to define a new partial order relation on A:

$$z_1 \preceq z_2 \qquad \text{if } z_2 - z_1 \in \mathcal{C},$$

and $z_1 \prec z_2$ if $z_1 \preceq z_2$ and $z_1 \neq z_2$. Obviously $z_1 \preceq z_2$ implies $z_1 \leq z_2$, and $z_1 \prec z_2$ implies $z_1 \ll z_2$. Because $x_1 \cdot (t_n - \theta) \to y \cdot (-\theta)$ as $n \to \infty$, there are n_0 and $\delta > 0$ such that $B(x_1 \cdot (t_n - \theta); \delta) \subset \phi_{-\theta}U$ for all $n \geq n_0$. Define, for each n,

$$\mathcal{K}_n := (x_1 \cdot t_n + \mathcal{C}) \cap (A \setminus (y + C^\circ)).$$

The closed sets $\mathcal{K}_n$ are easily seen to be bounded and starshaped. As the sequence $x_1 \cdot t_n$ converges in a strongly monotone way to y, the intersection $\bigcap_{n=1}^\infty \mathcal{K}_n$ equals $\{y\}$. As a consequence, $\lim_{n\to\infty} \operatorname{diam} \mathcal{K}_n = 0$.

The mapping $\phi_{-\theta}$ is locally Lipschitz, so there is $\lambda > 0$ such that

$$\|z_1 \cdot \theta - z_2 \cdot \theta\| \geq \lambda \|z_1 - z_2\| \qquad \text{for any } z_1, z_2 \in \phi_{-\theta}U. \tag{2.1}$$

Fix $l \geq n_0$ such that $t_l > \theta$ and $\operatorname{diam} \mathcal{K}_l < \min\{\lambda\delta, \eta\}$. Denote by I the line segment with endpoints $x_1 \cdot (t_l - \theta)$ and $x_2 \cdot (t_l - \theta)$. The segment I is linearly ordered by $\ll$.

Assume first that we have $\eta/\lambda \leq \|x_1 \cdot (t_l - \theta) - x_2 \cdot (t_l - \theta)\| \leq \delta$. The convexity of closed balls in A implies $I \subset B(x_1 \cdot (t_l - \theta); \delta) \subset \phi_{-\theta}U$. By integration we obtain $x_1 \cdot t_l \prec x_2 \cdot t_l$, that is, $x_2 \cdot t_l \in x_1 \cdot t_l + (\mathcal{C} \setminus \{0\})$.

An application of 2.1 yields $\|x_1 \cdot t_l - x_2 \cdot t_l\| \ge \eta > \operatorname{diam} \mathcal{K}_l$. Consequently, $x_2 \cdot t_l \notin \mathcal{K}_l$, hence $x_2 \cdot t_l \gg y$.

Now, assume $\|x_1 \cdot (t_l - \theta) - x_2 \cdot (t_l - \theta)\| > \delta$. Denote by ζ the unique point in I at distance δ from $x_1 \cdot (t_l - \theta)$, and $\zeta' := \zeta \cdot \theta$. Proceeding as in the above paragraph we obtain $\|x_1 \cdot t_l - \zeta'\| \ge \lambda\delta > \operatorname{diam} \mathcal{K}_l$, hence $\zeta' \gg y$. As $\zeta \ll x_2 \cdot (t_l - \theta)$, strong monotonicity yields $x_2 \cdot t_l \gg \zeta' \gg y$.

Pick a natural m such that $t_m > t_l + t_1$. We have $x_1 \cdot 0 \ll x_1 \cdot t_m \ll y \ll x_2 \cdot t_l = x_1 \cdot (t_l + t_1)$ with $0 < t_l + t_1 < t_m$, which contradicts Lemma 3.3.1 of [9]. ∎

THEOREM 2.3. *Let $x \notin E$ and $y \in \omega(x) \cap \operatorname{dom} H \cap E$ be such that $dH(y)$ is nonzero. Then $\omega(x) = \{y\}$.*

Before proving the above theorem we note a couple of results.

For an equilibrium y we denote by $\varrho(y)$ the *stability modulus* of the matrix $DF(y)$, that is, the maximum of the real parts of the eigenvalues of $DF(y)$.

The following result is a corollary of the well-known Perron–Frobenius theorem.

LEMMA 2.4. *Let y be an equilibrium for a cooperative irreducible system of ODEs. Then the stability modulus $\varrho(y)$ is a simple eigenvalue exceeding in modulus the remaining two eigenvalues (counted with multiplicity). Moreover, any eigenvector corresponding to $\varrho(y)$ belongs to $C^\circ \cup (-C^\circ)$, and the $DF(y)$-invariant two-dimensional subspace corresponding to the remaining eigenvalues meets C only at 0.*

For $y \in E$, denote by $V^{\mathrm{s}}(y)$, $V^{\mathrm{u}}(y)$ and $V^{\mathrm{c}}(y)$ the sums of the generalized eigenspaces corresponding to the eigenvalues of $DF(y)$ with negative, positive and zero real parts, respectively. It is well known (see e.g. Hirsch, Pugh and Shub [5]) that there exist locally invariant C^1 embedded submanifolds $M^{\mathrm{s}}(y)$, $M^{\mathrm{u}}(y)$ and $M^{\mathrm{c}}(y)$, tangent at y to $V^{\mathrm{s}}(y)$, $V^{\mathrm{u}}(y)$ and $V^{\mathrm{c}}(y)$ (the local stable, unstable and center manifolds at y). The stable and unstable manifolds are locally unique, whereas the center manifold need not be unique in general. Also, $M^{\mathrm{s}}(y)$ [$M^{\mathrm{u}}(y)$] can be made forward [backward] invariant.

Recall that an equilibrium y is *hyperbolic* if no eigenvalue of $DF(y)$ has zero real part.

LEMMA 2.5. *No equilibrium $y \in \operatorname{dom} H$ such that $dH(y) \neq 0$ is hyperbolic.*

Proof. Suppose that y is a hyperbolic equilibrium with $dH(y) \neq 0$. Since $\omega(z) = \{y\}$ for each $z \in M^{\mathrm{s}}(y)$ and $\alpha(z) = \{y\}$ for each $z \in M^{\mathrm{u}}(y)$, we have $M^{\mathrm{s}}(y) \cup M^{\mathrm{u}}(y) \subset \mathcal{H}(y)$. But the latter set is a two-dimensional manifold (Lemma 2.1), whereas the former is either a three-dimensional manifold or the union of a two-dimensional manifold and a one-dimensional manifold transverse to each other, a contradiction. ■

An easy implication of Lemmas 2.4 and 2.5 is that for each $y \in E$ such that $dH(y) \neq 0$ the dimension of the center manifold $M^{\mathrm{c}}(y)$ is either one or two.

We now state the following easy lemma, which will be instrumental in proving Theorem 2.3.

LEMMA 2.6. *Assume that x is not an equilibrium and $y \in \omega(x) \cap E$. Then either $\omega(x) = \{y\}$ or there is $\delta > 0$ such that for each $0 < \varepsilon \le \delta$ there is $z \in \omega(x)$ with $\|y - z\| = \varepsilon$.*

Proof of Theorem 2.3. Take $U \subset X$ to be an open neighborhood of y as in Lemma 2.1. Also, we can assume U to be such that $M^{\mathrm{s}}(y) \cap U$ $[M^{\mathrm{u}}(y) \cap U]$ is forward [backward] invariant.

Assume first $\dim M^{\mathrm{c}}(y) = 1$. As for each $z \in M^{\mathrm{s}}(y)$ $[z \in M^{\mathrm{u}}(y)]$ we have $x \cdot t \to y$ as $t \to \infty$ $[x \cdot t \to y$ as $t \to -\infty]$, it follows that $M^{\mathrm{s}}(y) \cup M^{\mathrm{u}}(y) \subset \mathcal{H}(y)$. In the local C^1 dynamical system $\{\phi_t | \mathcal{H}(y) \cap U\}$ the point y is a hyperbolic equilibrium, hence it is isolated in the set $E \cap \mathcal{H}(y) \cap U$. In view of Theorem 2.2, $\omega(x) \cap U \subset E \cap \mathcal{H}(y) \cap U$. Lemma 2.6 shows that $\omega(x) = \{y\}$.

Assume now $\dim M^{\mathrm{c}}(y) = 2$. By Lemma 2.4, $\dim M^{\mathrm{u}}(y) = 1$. As $\dim V^{\mathrm{c}}(y) + \dim V^{\mathrm{u}}(y) = 3$ and $V^{\mathrm{u}}(y)$ is contained in the nullspace of $dH(y)$, we see that the two-dimensional locally invariant C^1 manifolds $\mathcal{H}(y) \cap U$ and $M^{\mathrm{c}}(y)$ are transverse. Consequently, their intersection I is a locally invariant C^1 submanifold of dimension one, passing through y. By Palis and Takens [7], the local dynamical system $\{\phi_t | \mathcal{H}(y) \cap U\}$ has local product structure: After possibly taking a smaller U, there is an orbit-preserving homeomorphism P of $\mathcal{H}(y) \cap U$ onto the neighborhood $N := \{(\xi^1, \xi^2) : -1 \le \xi^i \le 1\}$ of 0 in the two-dimensional real affine space, where $P(I) = N \cap \{(\xi^1, 0)\}$ and the orbits of $\{\phi_t | \mathcal{H}(y) \cap U\}$ are taken into orbits of an ODE system $\dot{\xi}^1 = g(\xi^1)$, $\dot{\xi}^2 = \varrho(y)\xi^2$, with g a C^1 function satisfying $g(0) = g'(0) = 0$.

Suppose to the contrary that $\omega(x) \neq \{y\}$. Lemma 2.6 gives the existence of $\delta > 0$ such that for each $0 < \varepsilon \le \delta$ there is $z \in \omega(x)$ with $\|z - y\| = \varepsilon$. From the local product structure it follows that $\omega(x) \cap U \subset E \cap \mathcal{H}(y) \cap U$ is contained in the one-dimensional submanifold I. Therefore, there is a set $J \subset I$ homeomorphic to the real interval $[0, 1]$, consisting entirely of equilibria from $\omega(x)$, and such that y is a boundary point (in the sense of manifolds-with-boundary) of J. For definiteness, assume that $P(I') = \{(\xi^1, 0) : \xi^1 \in [0, 1]\}$. Consider the set $S := P^{-1}([0, 1] \times [-1, 1])$. S is easily seen to be backward invariant. Also, if for some $z \in S$ its forward semiorbit $O_{\mathrm{f}}(z)$ is contained in S then $z \in J$ (hence z is an equilibrium). Now, take $y' := P^{-1}(1/2, 0)$. Since $y' \in \omega(x)$, there is a sequence $t_n \to \infty$ such that $x \cdot t_n \to y'$. As y' belongs to the interior of S relative to $\mathcal{H}(y) \cap U$, we can assume that all $x \cdot t_n \to y'$ are in S. But S is backward invariant, so $O_{\mathrm{f}}(x \cdot t_1) \subset S$. Therefore $x \cdot t_1 \in E$, a contradiction. ∎

3. Cycles. The existence of a first integral (in our sense) does not preclude occurrence of cycles, as the following example shows.

EXAMPLE. Consider the linear three-dimensional cooperative irreducible system of ODEs:

$$\begin{aligned} \dot{x}^1 &= x^1 + 2x^2, \\ \dot{x}^2 &= x^2 + 2x^3, \\ \dot{x}^3 &= 2x^1 + x^3. \end{aligned} \tag{3.1}$$

The matrix of system (3.1) has eigenvalue 3 corresponding to the eigenvector $(1,1,1)$ and a pair of purely imaginary eigenvalues corresponding to the invariant subspace $\Sigma := \{x \in A : x^1 + x^2 + x^3 = 0\}$. Consequently, $\Sigma \setminus \{0\}$ is filled with cycles. A straightforward computation shows that $H(x) := \|x\|^2 - (x^1x^2 + x^2x^3 + x^3x^1)$ is an analytic first integral with $\operatorname{dom} H = A$. Also, $dH(x) = 0$ if and only if $x \in \operatorname{span}(1,1,1)$.

By a *closed order-interval* we mean the set $[a,b] := \{x \in A : a \le x \le b\}$, where $a \le b$. By an *open order-interval* we mean the set $[[a,b]] := \{x \in A : a \ll x \ll b\}$, where $a \ll b$. We say $Y \subset A$ is *order-convex* if $[x,y] \subset Y$ for any two $x, y \in Y$, $x \le y$.

The next result should be well known to everybody working in cooperative systems; I was unable, however, to locate it anywhere in the literature.

PROPOSITION 3.1. *Assume that $X = [[a,b]]$ and $\gamma \subset X$ is a cycle. Then there exists a strongly balanced invariant C^1 embedded two-dimensional disk $G \supset \gamma$ having γ as its* (*manifold*) *boundary.*

Proof. Following Smith [9], define K to be the set of those $x \in A$ which are not $<$-related to any point $z \in \gamma$. Proposition 4.3 of [9] states that K has two components, one bounded and one unbounded, and the bounded one, $K(\gamma)$, is homeomorphic to an open ball in A, contained in X and backward invariant. It is straightforward that $K(\gamma)$ is order-convex. Now we apply results in Hirsch [4] to conclude that there is an invariant (with respect to the local flow $\{\phi_t | K(\gamma)\}$) strongly balanced surface $M \subset K(\gamma)$ containing γ, such that the orthogonal projection Π along $(1,1,1)$ is a Lipeomorphism onto its image. Also, this image $\Pi(M)$ is homeomorphic to the open two-dimensional unit ball. By the Jordan curve theorem, there are two closed sets, $\widetilde{G}$ and $\widetilde{G}'$, such that $\Pi(M) = \widetilde{G} \cup \widetilde{G}'$, $\widetilde{G} \cap \widetilde{G}' = \Pi(\gamma)$, $\widetilde{G}$ is compact and $\widetilde{G}'$ is not compact. Put $G := \Pi^{-1}(\widetilde{G})$. Then G is a Lipschitz manifold-with-boundary. A result of I. Tereščák [10] shows that G is in fact C^1. ∎

THEOREM 3.2. *Assume that $dH(x)$ is nonzero at each x in some $[[a,b]] \subset X$. Then there is no cycle contained in $[[a,b]]$.*

Proof. Assume that there is a cycle $\gamma \subset [[a,b]]$. As we will consider the local flow restricted to $[[a,b]]$, we put $X = [[a,b]]$. Let G be a C^1 disk as in Proposition 3.1. Denote by $\widehat{H}$ the restriction of H to G. Evidently $\widehat{H}$ is a C^1 function on the C^1 manifold-with-boundary G. We claim that if $x \in G$ belongs to some cycle γ_1 then the derivative $d\widehat{H}(x)$ is nonzero. Indeed, by Theorem 1.3 of Smith [8] there exists a backward invariant two-dimensional strong unstable manifold L of γ_1 such that $z \cdot t$ approaches γ_1 as $t \to -\infty$, for any $z \in L$. Consequently, H is constant on L. The two-dimensional submanifolds L and G intersect transversely at γ_1, hence the derivative $d\widehat{H}(x)$ is nonzero at each $x \in \gamma_1$.

Consider the set S of those $x \in G$ for which $d\widehat{H}(x)$ is zero. Evidently, S is nonempty, compact and invariant. By the Poincaré–Bendixson theorem, S contains equilibria or cycles. But from the above paragraph it follows that S cannot contain any cycle. So, let $y \in S \cap E$. Then the stability modulus $\varrho(y)$ is zero. Indeed, by Lemmas 2.5 and 2.4, $\varrho(y)$ cannot be negative. On the other hand, if $\varrho(y)$ were positive then there would exist a locally invariant one-dimensional strongly unstable C^1 manifold $M^{\mathrm{uu}}(y)$, tangent at y to some vector $w \in C^\circ$ (see e.g. [5]). For each $z \in M^{\mathrm{uu}}(y)$ we have $z \cdot t \to y$ as $t \to -\infty$, hence H is constant on $M^{\mathrm{uu}}(y)$ and $\langle dH(y), w\rangle = 0$. As $M^{\mathrm{uu}}(y)$ is transverse to G, we deduce that $dH(y)$ is zero, a contradiction.

We have thus proved that $\varrho(y) = 0$. But from Lemma 2.4 it follows that the eigenvalues of the restriction of $DF(y)$ to the tangent space of G at y are negative. Therefore, there is a relative neighborhood U of y in G such that $\omega(z) = \{y\}$ for each $z \in U$. This yields that H is constant on U, hence $d\widehat{H}$ is zero on U. Now, for any $x \in S$ we have $\omega(x) \subset E$, and therefore S is relatively open in G. As the latter is connected, this gives a contradiction. ■

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Received 1 July 1997