COLLOQUIUM MATHEMATICUM

VOL. 76 1998 NO. 2

MINIMAL BIPARTITE ALGEBRAS OF INFINITE PRINJECTIVE TYPE WITH PRIN-PREPROJECTIVE COMPONENT

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1. Introduction. Let k be an algebraically closed field and let R be the path k-algebra of a finite quiver Q modulo an admissible ideal. We assume that R is *triangular*, that is, the quiver Q does not have oriented cycles. By a *bipartite algebra* we mean an algebra R together with a *bipartition*, that is, a presentation in an upper triangular matrix form

$$R = \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix} \tag{1.1}$$

where A and B are k-algebras, and ${}_AM_B$ is an A-B-bimodule.

All R-modules considered are right finitely generated; the category of finitely generated right R-modules is denoted by $\operatorname{mod}(R)$.

We shall use the terminology and notation on prinjective modules over bipartite algebras introduced in [13].

Following [13], [24] an R-module X, viewed as a triple $(X'_A, X''_B, \phi : X'_A \otimes_A M_B \to X''_B)$, is called ${}_AM_B$-*prinjective* provided X'_A is a projective A-module and X''_B is an injective B-module. By $\operatorname{prin}(R)^A_B$ we denote the full subcategory of $\operatorname{mod}(R)$ formed by ${}_AM_B$-prinjective modules. If the bipartition (1.1) of the algebra R is fixed we shall often write $\operatorname{prin}(R)$ instead of $\operatorname{prin}(R)^A_B$ and ${}_AM_B$-prinjective modules will be called prinjective.

We say that a bipartite algebra R of the form (1.1) is of *infinite prinjective type* if the category $\operatorname{prin}(R)$ is of infinite representation type, that is, there exists an infinite family of pairwise non-isomorphic indecomposable prinjective R-modules.

We recall from [13, Section 2], [17, Section 5], [24] that prinjective modules over bipartite algebras enable us to give a useful module-theoretical interpretation of bipartite bimodule matrix problems in the sense of Drozd [4]. They also play an important role in the study of representation types

1991 *Mathematics Subject Classification*: 16G20, 16G60.

Supported by Polish KBN Grant 2 P03A 007 12.

of categories latt(Λ) of lattices over classical orders Λ (see [19], [22]) and in constructing suitable functorial embeddings of module categories [20].

In a number of papers various criteria for finite representation type for certain classes of matrix problems are given. For instance a criterion for finite prinjective type of posets is obtained in [19]. Analogous criteria for bipartite posets and for a class of right peak algebras are given in [7] and [25]. Each criterion includes a list of "critical configurations", that is, minimal problems of infinite representation type in a given class. One can observe that the critical configurations are related to tame concealed algebras (this was remarked by Weichert in [25]). One of our aims is to understand this phenomenon for bipartite algebras. It seems that Theorem 3.10 below gives a satisfactory explanation. We follow ideas of description of minimal algebras of infinite representation type with a preprojective component and we obtain results analogous to the well-known classifications of minimal algebras of infinite representation type (see [15, 2.3]).

In Section 2 we collect basic facts about the category of prinjective modules over bipartite algebras which will be used later. Next in Section 3 we investigate prin-critical bipartite algebras in the sense of Definition 3.1 below. The prin-critical algebras are minimal of infinite prinjective type and such that the Auslander–Reiten quiver of the category of prinjective modules has a preprojective component. In other words, they are minimal of infinite prinjective type and have a "prin-preprojective" component. We relate them to critical algebras described by Bongartz [3] and Happel and Vossieck [5]. The main results of the paper are Theorems 3.10 and 3.12, which assert in particular that a bipartite prin-critical algebra (up to simple exceptions) is tame concealed and the Auslander–Reiten quivers of prin(R) and of mod(R) coincide up to a finite number of vertices. In Corollary 3.13 we give a description of the Auslander–Reiten quiver of the category of prinjective modules over a prin-critical algebra.

The author acknowledges Professor Daniel Simson's many helpful remarks and suggestions concerning the paper.

2. Preliminaries. Throughout, R is a bipartite algebra with a fixed bipartition (1.1).

2.1. Lemma. (a) *The subcategory* prin(R) *of* mod(R) *is closed under taking direct summands and extensions*, *and it has the unique decomposition property.*

(b) $\mathrm{Ext}^i_R(X,Y) = 0$ *for any pair of prinjective modules* X, Y *and all* $i \geq 2$.

(c) prin(R) *has enough relative projective objects and enough relative injective objects.*

Proof. See [13, Prop. 2.4], [17, Sec. 5]. ∎

It follows from the results of [13] that the category $\mathrm{prin}(R)$ has relative Auslander–Reiten sequences. By Δ_R and $\Gamma(\mathrm{prin}(R))$ we shall denote the Auslander–Reiten translate and the Auslander–Reiten quiver of $\mathrm{prin}(R)$, respectively. As usual, τ_R and Γ_R denote the Auslander–Reiten translate and the Auslander–Reiten quiver of $\mathrm{mod}(R)$. (See [1], [18].)

Given a finite-dimensional k-algebra Λ and a Λ-module X let

$$p_X : P_\Lambda(X) \to X \quad \text{and} \quad u_X : X \to E_\Lambda(X)$$

be the Λ-projective cover and the Λ-injective envelope of X respectively.

Let $e_1, \ldots, e_n$ (resp. $e_{n+1}, \ldots, e_{n+m}$) be a complete set of primitive orthogonal idempotents of the algebra A (resp. B). Let $S_j = \operatorname{top} e_j R$ be the simple R-module corresponding to e_j and let $P_i = e_i A \cong P_A(S_i)$ for $i \le n$ and $E_j = E_B(S_j)$ for $n < j \le n+m$. An R-module X is called *sincere* provided $Xe_i \ne 0$ for $i = 1, \ldots, n+m$.

For a prinjective module $X = (X'_A, X''_B, \phi)$, its *coordinate vector* $\mathbf{cdn}(X) \in \mathbb{Z}^{n+m}$ is defined as follows. We fix unique decompositions

$$X'_A = \bigoplus_{i=1}^{n} P_i^{t_i}, \qquad X''_B = \bigoplus_{i=n+1}^{n+m} E_i^{t_i}$$

and we set $\mathbf{cdn}(X) = (t_1, \ldots, t_{n+m})$ (see [13]).

2.2. Lemma [19, Lemma 2.2]. *The homomorphism $X \mapsto \mathbf{cdn}(X)$ induces an isomorphism of the Grothendieck group $\mathbf{K}_0(\mathrm{prin}(R))$ of $\mathrm{prin}(R)$ and the free abelian group $\mathbb{Z}^{n+m}$.* ∎

Fix the following notation:

$$\begin{aligned} a_{ij} &= \dim_k(e_j A e_i) && \text{for } i,j = 1, \ldots, n, \\ c_{ij} &= \dim_k(e_i M e_j) && \text{for } i = 1, \ldots, n;\ j = n+1, \ldots, n+m, \\ b_{ij} &= \dim_k(e_j B e_i) && \text{for } i,j = n+1, \ldots, n+m. \end{aligned} \tag{2.3}$$

Following [13] we associate with the algebra R and the fixed set of idempotents $e_1, \ldots, e_{n+m}$ the bilinear form $\langle -, - \rangle_R : \mathbb{Z}^{n+m} \times \mathbb{Z}^{n+m} \to \mathbb{Z}$ defined by

$$\langle x, y \rangle_R = \sum_{i,j=1}^{n} a_{ij} x_i y_j + \sum_{i,j=n+1}^{n+m} b_{ij} x_i y_j - \sum_{i=1}^{n} \sum_{j=n+1}^{n+m} c_{ij} x_i y_j. \tag{2.4}$$

We also set $(x,y)_R = \frac{1}{2}(\langle x,y \rangle_R + \langle y,x \rangle_R)$ and $\mathbf{q}_R^{\mathrm{prin}}(x) = (x,x)_R$.

The quadratic form $\mathbf{q}_R^{\mathrm{prin}} : \mathbb{Z}^{n+m} \to \mathbb{Z}$ is called the *Tits prinjective quadratic form* of the bipartite algebra R. Note that since R is a triangular algebra, we have $a_{ii} = b_{ss} = 1$ for $i = 1, \ldots, n$, $s = n+1, \ldots, n+m$. Thus $\mathbf{q}_R^{\mathrm{prin}}$ is a *unit form* in the sense of [15, 1.0].

The Cartan matrices of the algebras A and B are the following:

$$C_A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

$$C_B = \begin{pmatrix} b_{n+1,n+1} & b_{n+1,n+2} & \dots & b_{n+1,n+m} \\ b_{n+2,n+1} & b_{n+2,n+2} & \dots & b_{n+2,n+m} \\ \vdots & & & \vdots \\ b_{m+n,n+1} & b_{m+n,n+2} & \dots & b_{m+n,n+m} \end{pmatrix},$$

where a_{ij}, b_{st} are defined by formula (2.3). We set

$$C_R = \begin{pmatrix} C_A & 0 \\ C_M & C_B \end{pmatrix}, \quad C_B^A = \begin{pmatrix} C_A & 0 \\ 0 & C_B^{\mathrm{tr}} \end{pmatrix}$$

where

$$C_M = \begin{pmatrix} c_{1,n+1} & c_{2,n+1} & \dots & c_{n,n+1} \\ c_{1,n+2} & c_{2,n+2} & \dots & c_{n,n+2} \\ \vdots & & & \vdots \\ c_{1,n+m} & c_{2,n+m} & \dots & c_{n,n+m} \end{pmatrix}.$$

We denote by $\mathbf{q}_R : \mathbb{Z}^{n+m} \to \mathbb{Z}$ the usual Tits quadratic form of the algebra R (see [2]) defined by $\mathbf{q}_R(x) = xC_R^{-\mathrm{tr}}x^{\mathrm{tr}}$.

For any vector $v \in \mathbb{N}^{n+m}$ the vector $d_R^v \in \mathbb{N}^{n+m}$ is defined by

$$(d_R^v)^{\mathrm{tr}} = C_B^A v^{\mathrm{tr}}. \tag{2.5}$$

2.6. Lemma. *For any prinjective R-module X,*

$$\mathbf{dim}(X) = d_R^{\mathbf{cdn}(X)},$$

where $\mathbf{dim}(X)$ is the dimension vector of X.

Proof. See [8], [13], [21, Section 3]. ∎

Recall that the *dimension vector* $\mathbf{dim}(X) \in \mathbb{Z}^{n+m}$ of an R-module X is defined by $\mathbf{dim}(X)(i) = \dim_k Xe_i$ for $i = 1, \dots, n+m$.

2.7. Lemma [12, Prop. 4.4]. *For any prinjective R-modules X, Y,*

$$\langle \mathbf{cdn}(X), \mathbf{cdn}(Y) \rangle_R = \dim_k \mathrm{Hom}_R(X, Y) - \dim_k \mathrm{Ext}_R^1(X, Y). \ ∎$$

2.8. Lemma. *Assume that R is a bipartite triangular algebra of the form (1.1) and let $\mathbf{q}_R^{\mathrm{prin}}$, $\mathbf{q}_R$, $d_R^{(-)}$ be as above. Then:*

(a) *The homomorphism $v \mapsto d_R^v$ is an automorphism of the group $\mathbb{Z}^{n+m}$.*

(b) *For any $v \in \mathbb{Z}^{n+m}$ the equality $\mathbf{q}_R^{\mathrm{prin}}(v) = \mathbf{q}_R(d_R^v)$ holds.*

Proof. To prove (a) note that our assumptions imply that the determinant of the matrix C_B^A equals 1 (compare with [21, Lemma 3.2]). In order to show (b) observe that it is enough to prove the required equality for $v \in \mathbb{N}^{n+m}$. But this follows from the fact that if $v \in \mathbb{N}^{n+m}$ then $\mathbf{cdn}(X) = v$ for some X in $\mathrm{prin}(R)$ and

$$\mathbf{q}_R^{\mathrm{prin}}(v) = \dim \mathrm{End}_R(X) - \dim \mathrm{Ext}_R^1(X, X) = \mathbf{q}_R(\mathbf{dim}(X)) = \mathbf{q}_R(d_R^v).$$

The first equality follows from Lemma 2.7, the second from [2] and the fact that $\mathrm{Ext}_R^2(X, X) = 0$. The third is a consequence of Lemma 2.6. ∎

2.9. Definition [13]. A prinjective module X is called *prin-projective* (resp. *prin-injective*) provided $\mathrm{Ext}_R^1(X, Y) = 0$ (resp. $\mathrm{Ext}_R^1(Y, X) = 0$) for any prinjective module Y. ∎

Recall from [15, 1.0] that an integral quadratic form $q : \mathbb{Z}^l \to \mathbb{Z}$ is called *weakly positive* if $q(v) > 0$ for any non-zero vector v with all coordinates non-negative. In the following theorem we collect some facts concerning the quadratic form $\mathbf{q}_R^{\mathrm{prin}}$.

2.10. Theorem. *Let R be a bipartite algebra of the form* (1.1) *and let $\mathbf{q}_R^{\mathrm{prin}}$ be the Tits prinjective quadratic form of R.*

(1) *If for any vector v there exist only finitely many isomorphism classes of indecomposable prinjective R-modules X with $\mathbf{cdn}(X) = v$ then the form $\mathbf{q}_R^{\mathrm{prin}}$ is weakly positive. In particular, $\mathbf{q}_R^{\mathrm{prin}}$ is weakly positive provided R is of finite prinjective type.*

(2) *Assume that $\mathcal{P}$ is a preprojective component in $\Gamma(\mathrm{prin}(R))$* (see [1], [15], [18]). *Then $\mathbf{q}_R^{\mathrm{prin}}(\mathbf{cdn}(X)) = 1$ for any X in $\mathcal{P}$.*

(3) *If there exists a preprojective component in $\Gamma(\mathrm{prin}(R))$ and the form $\mathbf{q}_R^{\mathrm{prin}}$ is weakly positive then the algebra R is of finite prinjective type.*

Proof. The statement (1) follows by algebraic geometry arguments. This is proved essentially in [18, Theorem 10.1], although the theorem there is formulated only for a special class of algebras (see also [8]).

For the proof of (2) repeat the well-known arguments (see e.g. [18, Corollary 11.96]), whereas (3) follows from [13, Proposition 4.13]. ∎

Following [13] we describe the prin-projective and prin-injective indecomposable modules. In order to do it given an R-module $X = (X'_A, X''_B, \phi)$ let us define two modules $\widehat{X}$ and $\widetilde{X}$ by the formulae

$$\widehat{X} = (X'_A, E_B(X''_B), \widehat{\phi}), \qquad \widetilde{X} = (P_A(X'_A), X''_B, \widetilde{\phi}), \tag{2.11}$$

where the homomorphism $\widetilde{\phi}$ is the composition

$$P_A(X') \otimes_A M \xrightarrow{p_{X'} \otimes \mathrm{id}_M} X' \otimes_A M \xrightarrow{\phi} X''$$

and the homomorphism $\widehat{\phi}$ is the composition

$$X' \otimes_A M \xrightarrow{\phi} X'' \xrightarrow{u_{X''}} E_B(X'')$$

(compare [13, 2.1]).

There exist canonical R-homomorphisms

$$\varepsilon_X : \widetilde{X} \to X, \quad v_X : X \to \widehat{X}, \tag{2.11'}$$

and ε_X is an epimorphism and v_X is a monomorphism.

We use the following notation:

$$P_i^{\diamondsuit} = \widehat{e_i R}, \quad Q_i^{\diamondsuit} = \widehat{S}_i = (e_i A, 0, 0) \quad \text{for } i = 1, \ldots, n,$$

and

$$P_j^{\diamondsuit} = \widetilde{S_j} = (0, E_B(S_j), 0), \quad Q_j^{\diamondsuit} = \widetilde{E_R(S_j)} \quad \text{for } j = n+1, \ldots, n+m.$$

2.12. LEMMA [13, Proposition 2.4]. *The modules* $P_1^{\diamondsuit}, \ldots, P_{n+m}^{\diamondsuit}$ (*resp.* $Q_1^{\diamondsuit}, \ldots, Q_{n+m}^{\diamondsuit}$) *form a complete set of indecomposable prin-projective* (*resp. prin-injective*) *modules up to isomorphism.* ■

2.13. LEMMA. *Let* $X = (X'_A, X''_B, \phi)$ *be an* R*-module. The following conditions are equivalent*:

(a) *The homomorphism* ϕ *is an epimorphism.*
(b) $\mathrm{Hom}_R(X, P_i^{\diamondsuit}) = 0$ *for any* $i = n+1, \ldots, n+m$.

If this is the case then the module $\widehat{X}$ *is indecomposable provided* X *is indecomposable. Moreover*, *if* R*-modules* X, Y *satisfy* (a) *and* (b) *then* $\widehat{X} \cong \widehat{Y}$ *implies* $X \cong Y$.

Proof. The equivalence of (a) and (b) is easy, we leave it to the reader. To prove the remaining statements assume that $\widetilde{X} = Y \oplus Z$ and $Y = (Y'_A, Y''_B, \psi)$, $Z = (Z'_A, Z''_B, \eta)$. Since ϕ is an epimorphism, we have $X''_B = \mathrm{Im}\,\psi \oplus \mathrm{Im}\,\eta$ and it follows by indecomposability of X that one of Y'_A, Z'_A, say Y'_A, is the zero module. But then also Y''_B is zero, because $\mathrm{Im}\,u_{X''_B}\phi \cap Y''_B = \{0\}$ and $\mathrm{Im}\,u_{X''_B}\phi = \mathrm{Im}\,u_{X''_B}$ is an essential submodule of $E_B(X''_B)$.

Now assume that $X = (X'_A, X''_B, \phi)$, $Y = (Y'_A, Y''_B, \psi)$ and there is an isomorphism $f : \widehat{X} \to \widehat{Y}$. Let $f = (f', f'')$, where $f' : X'_A \to Y'_A$ and $f'' : E_B(X''_B) \to E_B(Y''_B)$. Since the diagram

$$\begin{array}{ccc} X' \otimes_A M & \xrightarrow{f' \otimes \mathrm{id}_M} & Y' \otimes_A M \\ \widehat{\phi}\downarrow & & \downarrow\widehat{\psi} \\ E_B(X''_B) & \xrightarrow{f''} & E_B(Y''_B) \end{array}$$

commutes we see that f'' induces an isomorphism $f''_| : \mathrm{Im}\,\widehat{\phi} \to \mathrm{Im}\,\widehat{\psi}$. But $\mathrm{Im}\,\widehat{\phi} \cong X''_B$, $\mathrm{Im}\,\widehat{\psi} \cong Y''_B$ and we get an isomorphism $X \cong Y$. ■

Dually we obtain the following lemma.

2.14. Lemma. *Let* $X = (X'_A, X''_B, \phi)$ *be an* R*-module. The following conditions are equivalent*:

(a) *The homomorphism* $\overline{\phi}$ *adjoint to* ϕ *is a monomorphism.*
(b) $\mathrm{Hom}_R(Q_i^{\diamondsuit}, X) = 0$ *for any* $i = 1, \ldots, n$.

If this is the case then the module $\widetilde{X}$ *is indecomposable provided* X *is indecomposable. Moreover, if* R*-modules* X, Y *satisfy* (a) *and* (b) *then* $\widetilde{X} \cong \widetilde{Y}$ *implies* $X \cong Y$. ■

2.15. Lemma. *Let* X *be an arbitrary* R*-module. Given any prinjective* R*-modules* Y, Z *and* R*-module homomorphisms* $f : Y \to X$, $g : X \to Z$ *there exist* R*-module homomorphisms* $\widetilde{f}$, $\widehat{f}$, $\widetilde{g}$, $\widehat{g}$ *making the following diagram commutative*:

$$\begin{array}{ccccc} Y & \xrightarrow{\widetilde{f}} & \widetilde{X} & \xrightarrow{\widetilde{g}} & Z \\ {\scriptstyle \mathrm{id}_Y}\downarrow & & \downarrow{\scriptstyle \varepsilon_X} & & \downarrow{\scriptstyle \mathrm{id}_Z} \\ Y & \xrightarrow{f} & X & \xrightarrow{g} & Z \\ {\scriptstyle \mathrm{id}_Y}\downarrow & & \downarrow{\scriptstyle v_X} & & \downarrow{\scriptstyle \mathrm{id}_Z} \\ Y & \xrightarrow{\widehat{f}} & \widehat{X} & \xrightarrow{\widehat{g}} & Z \end{array}$$

Proof. We put $\widetilde{g} = g\varepsilon_X$ and $\widehat{f} = v_X f$. To construct the $\widetilde{f}$, let $Y = (Y'_A, Y''_B, \psi)$ and $f = (f', f'')$, where $f' : Y'_A \to X'_A$ and $f'' : Y''_A \to X''_A$. Since Y'_A is A-projective we can lift f' to a homomorphism $\widetilde{f}' : Y'_A \to P_A(X'_A)$ such that $p_X \widetilde{f}' = f'$, and we put $\widetilde{f} = (\widetilde{f}', f'')$. The homomorphism $\widehat{g}$ is constructed dually. ■

In Lemma 2.16 below we shall use the following notation. For $i = 1, \ldots, n$ we set $\overline{p}_i = \mathbf{dim}(C_i)$, where

$$C_i = \mathrm{Coker}(v_{e_i R} : e_i R \to P_i^{\diamondsuit})$$

and for $i = n+1, \ldots, n+m$ we set $\overline{q}_i = \mathbf{dim}(K_i)$, where

$$K_i = \mathrm{Ker}(\varepsilon_{E_R(S_i)} : Q_i^{\diamondsuit} \to E_R(S_i));$$

see (2.11′).

2.16. Lemma. (a) *Let* X *be a prinjective* R*-module. Then*

$$\dim_k \mathrm{Hom}_R(P_i^{\diamondsuit}, X) = \begin{cases} \mathbf{dim}(X)(i) + \sum\limits_{j=n+1}^{n+m} \overline{p}_i(j)\mathbf{cdn}(X)(j) & \text{if } i \le n, \\ \sum\limits_{j=n+1}^{n+m} b_{ij}\mathbf{cdn}(X)(j) & \text{if } i > n, \end{cases}$$

and

$$\dim_k \operatorname{Hom}_R(X, Q_i^{\diamondsuit}) = \begin{cases} \sum_{j=1}^{n} a_{ji}\mathbf{cdn}(X)(j) & \textit{if } i \le n, \\ \mathbf{dim}(X)(i) + \sum_{j=1}^{n} \overline{q}_i(j)\mathbf{cdn}(X)(j) & \textit{if } i > n. \end{cases}$$

(b) *There exist group automorphisms* $g, h : \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}$ *such that*

$$g(\mathbf{cdn}(X)) = (\dim_k \operatorname{Hom}_R(P_1^{\diamondsuit}, X), \ldots, \dim_k \operatorname{Hom}_R(P_{n+m}^{\diamondsuit}, X)),$$
$$h(\mathbf{cdn}(X)) = (\dim_k \operatorname{Hom}_R(X, Q_1^{\diamondsuit}), \ldots, \dim_k \operatorname{Hom}_R(X, Q_{n+m}^{\diamondsuit}))$$

for any prinjective R-module X.

(c) *If X is a prinjective R-module and*

$$\operatorname{Hom}_R(P_i^{\diamondsuit}, X) = 0 \quad \textit{or} \quad \operatorname{Hom}_R(X, Q_i^{\diamondsuit}) = 0$$

then $\mathbf{cdn}(X)(i) = 0$.

Proof. (a) We only prove the first equality, the remaining one is dual. Let $X = (X'_A, X''_B, \phi)$. Assume that $i \le n$ and note that the canonical homomorphism $v_{e_iR} : e_iR \to P_i^{\diamondsuit}$ induces a homomorphism

$$v_{e_iR}^* : \operatorname{Hom}_R(P_i^{\diamondsuit}, X) \to \operatorname{Hom}_R(e_iR, X),$$

which is an epimorphism by Lemma 2.15. Moreover, we have $\operatorname{Ker} v_{e_iR}^* \cong \operatorname{Hom}_R(C_i, X)$, where C_i is the cokernel of v_{e_iR}. It is easy to check that

$$\dim_k \operatorname{Hom}_R(C_i, X) = \sum_{j=n+1}^{n+m} \overline{p}_i(j)\mathbf{cdn}(X)(j).$$

Since obviously $\dim_k \operatorname{Hom}_R(e_iR, X) = \mathbf{dim}(X)(i)$, our formula holds for $i \le n$.

Now assume that $i > n$ and note that

$$\begin{aligned} \operatorname{Hom}_R(P_i^{\diamondsuit}, X) &\cong \operatorname{Hom}_B(E_B(S_i), X''_B) \\ &\cong \bigoplus_{j=n+1}^{n+m} \operatorname{Hom}_B(E_B(S_i), E_B(S_j))^{\mathbf{cdn}(X)(j)} \\ &\cong \bigoplus_{j=n+1}^{n+m} (e_jBe_i)^{\mathbf{cdn}(X)(j)}; \end{aligned}$$

thus our formula follows by the definition (2.3) of the numbers b_{ij}.

The assertions (b) and (c) are direct consequences of (a). ■

2.17. Lemma. *Assume that*

$$e : 0 \to X \xrightarrow{u} Y \xrightarrow{w} Z \to 0$$

is an Auslander–Reiten sequence in $\mathrm{prin}(R)$ *and*

(a) $\mathrm{Hom}_R(Z, P_i^{\diamondsuit}) = 0$ *for any* $i = n+1, \ldots, n+m$,

(b) $\mathrm{Hom}_R(Q_i^{\diamondsuit}, Y) = 0$ *for any* $i = 1, \ldots, n$.

Then e *is an Auslander–Reiten sequence in* $\mathrm{mod}(R)$.

Proof. Assume that a homomorphism $f : U \to Z$ in $\mathrm{mod}(R)$ is not a splitting epimorphism. We shall prove that f factorizes through w.

Let $U = (U', U'', \phi_U)$ and $Z = (Z', Z'', \phi_Z)$. Consider the module $\widehat{U} = (U', E_B(U''), \widehat{\phi}_U)$ and let $v_U : U \to \widehat{U}$ be the natural embedding (2.11′). By Lemma 2.15 there exists a homomorphism $\widehat{f} : \widehat{U} \to Z$ such that $\widehat{f} v_U = f$.

Suppose that $\widehat{f}$ is a splitting epimorphism and let $r : Z \to \widehat{U}$ be a homomorphism such that $\widehat{f} r = \mathrm{id}_Z$. If $\mathrm{Im}\, r \subseteq v_U(U)$ then f is a splitting epimorphism, a contradiction. Hence r induces a non-zero homomorphism $\overline{r} : Z \to \widehat{U}/U = (0, E_B(U'')/U'', 0)$ and there is a non-zero homomorphism from Z to the module $(0, Q, 0)$, where $Q = E_B(E_B(U'')/U'')$ is an injective B-module, a contradiction with (a).

Consider the homomorphisms

$$\widetilde{\widetilde{\widehat{U}}} \xrightarrow{\varepsilon_{\widehat{U}}} \widehat{U} \xrightarrow{\widehat{f}} Z,$$

where $\widetilde{\widetilde{\widehat{U}}} = (P_A(U'), E_B(U''), \widetilde{\widetilde{\widehat{\phi_U}}})$ and $\varepsilon_{\widehat{U}}$ is the natural projection. The module $\widetilde{\widetilde{\widehat{U}}}$ is prinjective and $\widehat{f}\varepsilon_{\widehat{U}}$ is not a splitting epimorphism because $\widehat{f}$ is not a splitting epimorphism. Since e is an Auslander–Reiten sequence in $\mathrm{prin}(R)$, there is a map $h : \widetilde{\widetilde{\widehat{U}}} \to Y$ such that $wh = \widehat{f}\varepsilon_{\widehat{U}}$. Let $K = \mathrm{Ker}\, \varepsilon_{\widehat{U}} = (K', 0, 0)$. If $h(K) \neq 0$ then there exists a non-zero homomorphism from $(P_A(K'), 0, 0)$ to Y, a contradiction with (b). Hence h induces a homomorphism $\overline{h} : \widehat{U} \to Y$ such that $\overline{h}\varepsilon_{\widehat{U}} = h$. Note that $w\overline{h}v_U = f$. Indeed: $w\overline{h}\varepsilon_{\widehat{U}} = wh = \widehat{f}\varepsilon_{\widehat{U}}$, but $\varepsilon_{\widehat{U}}$ is an epimorphism, thus $w\overline{h} = \widehat{f}$ and $w\overline{h}v_U = \widehat{f}v_U = f$. Hence $\overline{h}v_U$ is the required homomorphism from U to Y and the lemma follows. ∎

Consider a subset $I \subseteq \{1, \ldots, n+m\}$ and an idempotent $e_I = \sum_{i \in I} e_i$. Let $\xi_I = \sum_{i \in I,\, i \le n} e_i$ and $\eta_I = e_I - \xi_I$. Let

$$R_I = e_I R e_I = \begin{pmatrix} A_I & M_I \\ 0 & B_I \end{pmatrix}$$

where $A_I = \xi_I A \xi_I$, $M_I = \xi_I M \eta_I$ and $B_I = \eta_I B \eta_I$. We define the *induction functor*

$$T_{R_I}^R : \mathrm{mod}(R_I) \to \mathrm{mod}(R) \tag{2.18}$$

by the formula (compare [18, 11.85], [7, 2.2])

$$T^R_{R_I}(X'_{A_I}, X''_{B_I}, \phi) = (X' \otimes_{A_I} \xi_I A, \mathrm{Hom}_{B_I}(B\eta_I, X''), \widetilde{\phi}),$$

where

$$\widetilde{\phi} : X' \otimes_{A_I} \xi_I A \ \otimes_A M \to \mathrm{Hom}_{B_I}(B\eta_I, X'')$$

is the homomorphism adjoint to the composition of the natural isomorphism

$$X' \otimes_{A_I} \xi_I A \otimes_A M \otimes_B B\eta_I \cong X' \otimes_{A_I} \xi_I M \eta_I$$

with the homomorphism ϕ. The functor $T^R_{R_I}$ is defined on homomorphisms in a natural way. The following lemma is an analogue of [18, Proposition 11.84].

2.19. Lemma. (a) *The functor* $T^R_{R_I}$ *is full and faithful.*

(b) *The functor* (2.18) *induces a functor*

$$T^R_{R_I} : \mathrm{prin}(R_I) \to \mathrm{prin}(R),$$

and $\mathbf{cdn}(T^R_{R_I}(X)) = t_I(\mathbf{cdn}(X))$ *for any prinjective* R_I*-module* X, *where* $t_I : \mathbb{Z}^I \to \mathbb{Z}^{n+m}$ *is the natural embedding. Moreover*, *a prinjective* R*-module* X *belongs to the image of* $T^R_{R_I}$ *if and only if* $\mathbf{cdn}(X) \in t_I(\mathbb{Z}^I)$.

(c) *If the category* prin(R_I) *is of infinite representation type then so is the category* prin(R).

The proof is routine.

3. Prin-critical algebras. From now on we assume that R is a bipartite prin-critical algebra in the sense of the following definition.

3.1. Definition. A bipartite algebra R of the form (1.1) is called *prin-critical* provided:

(a) the category prin(R) is of infinite representation type, but for any proper subset $I \subsetneq \{1, \ldots, n+m\}$ the category prin(R_I) is of finite representation type, where R_I is the bipartite algebra $e_I R e_I$ with $e_I = \sum_{i\in I} e_i$,

(b) the Auslander–Reiten quiver $\Gamma(\mathrm{prin}(R))$ of prin(R) contains a preprojective component (see [1], [18] for definition). ■

Examples of prin-critical algebras are incidence algebras of critical posets (see [19]) and critical bipartite posets (see [7]).

The name "prin-critical" is justified by the following result (compare [15, 4.3(6)]).

3.2. Lemma. *Assume that* R *is a bipartite algebra of the form* (1.1) *with a complete set* $e_1, \ldots, e_{n+m}$ *of primitive orthogonal idempotents. If* R *is of infinite prinjective type and the quiver* $\Gamma(\mathrm{prin}(R))$ *has a preprojective component then there exists a set* $I \subseteq \{1, \ldots, n+m\}$ *such that the algebra* $R_I = e_I R e_I$ *is prin-critical.*

Proof. Let J be the set of elements i such that the prin-projective module $P_i^{\diamondsuit}$ lies in a preprojective component. It follows from Lemma 2.16 that for each preprojective module X in $\operatorname{prin}(R)$ the equality $\mathbf{cdn}(X)(i) = 0$ holds for $i \notin J$. All components of $\Gamma(\operatorname{prin}(R))$ are infinite (see [1], [18, Corollary 11.54]), hence the algebra R_J is of infinite prinjective type by Lemma 2.19(c).

Let I be a minimal subset of J such that the bipartite algebra R_I is of infinite prinjective type. We claim that R_I is prin-critical. To prove this it is enough to show that the quiver $\Gamma(\operatorname{prin}(R_I))$ has a preprojective component.

We follow an idea of [15, 4.3(6)]. Recall that given a Krull–Schmidt category $\mathcal{K}$ the sequence $\mathcal{K}_{-1}, \mathcal{K}_0, \mathcal{K}_1, \ldots$ is defined inductively as follows: $\mathcal{K}_{-1} = \{0\}$ and for $d \geq 0$ an object X belongs to $\mathcal{K}_d$ if and only if any object Y of $\mathcal{K}$ such that $\operatorname{rad}(Y, X) \neq 0$ belongs to $\mathcal{K}_{d-1}$. By rad we denote the Jacobson radical of the category $\mathcal{K}$ (see [1], [18]). We define $\mathcal{K}_\infty$ to be the union of all $\mathcal{K}_d$, $d \in \mathbb{N}$.

We shall prove that each prin-projective R_I-module is in $\operatorname{prin}(R_I)_\infty$. It will follow that $\Gamma(\operatorname{prin}(R_I))$ has a preprojective component.

First consider prin-projective modules of the form $Y = (0, E_{B_I}(S_i), 0)$. We keep the notation from Lemma 2.19, that is, we set $R_I = e_I R e_I$ and

$$R_I = \begin{pmatrix} A_I & M_I \\ 0 & B_I \end{pmatrix}$$

where $A_I = \xi_I A \xi_I$, $B_I = \eta_I B \eta_I$, $M_I = \xi_I M \eta_I$ and $e_I = \xi_I + \eta_I$. Note that $T^R_{R_I}(Y) \cong (0, E_B(S_i), 0) = P_i^{\diamondsuit}$ is preprojective in $\Gamma(\operatorname{prin}(R))$ because $i \in J$, and hence belongs to $\operatorname{prin}(R)_\infty$. One can prove by induction on d that if $T^R_{R_I}(Y)$ belongs to $\operatorname{prin}(R)_d$ then Y belongs to $\operatorname{prin}(R_I)_d$. It follows that Y belongs to $\operatorname{prin}(R_I)_\infty$. Let d_0 be a number such that any prin-projective R_I-module of the form $(0, E_{B_I}(S_i), 0)$ belongs to $\operatorname{prin}(R_I)_{d_0}$.

Now we prove by induction on d that given an R_I-module $Y = (Y', Y'', \phi)$ if the module $\widehat{Y} = (Y', E_{B_I}(Y''), \widehat{\phi})$ is an indecomposable prinjective R_I-module then $\widehat{Y}$ belongs to $\operatorname{prin}(R_I)_{d_0+d+1}$ provided the module $(Y \otimes_{R_I} e_I R)^\wedge$ belongs to $\operatorname{prin}(R)_d$. We write $(U)^\wedge$ for $\widehat{U}$ in case U is a long expression.

The statement is clear for $d = -1$. Assume now that $d \geq 0$.

If there is a non-zero homomorphism from $\widehat{Y}$ to a module of the form $(0, E_{B_I}(S_i), 0)$ then $\widehat{Y}$ belongs to $\operatorname{prin}(R_I)_{d_0}$ and the claim follows. Thus we can assume by Lemma 2.13 that the homomorphism $\widehat{\phi} : Y' \otimes M_I \to E_{B_I}(Y'')$ is an epimorphism. It follows that $Y = \widehat{Y}$ and ϕ is an epimorphism. This means that Y is a quotient of the projective R_I-module $P_{R_I}(Y) = (Y', Y' \otimes_{A_I} M_I, \mathrm{id}_{Y' \otimes_{A_I} M_I})$ by a submodule Z of the form $Z = (0, Z'', 0)$. The sequence

$$0 \to Z \to P_{R_I}(Y) \to Y \to 0$$

induces an exact sequence

$$Z \otimes_{R_I} e_I R \to P_{R_I}(Y) \otimes_{R_I} e_I R \to Y \otimes_{R_I} e_I R \to 0$$

and $P_{R_I}(Y) \otimes_{R_I} e_I R$ is a projective R-module and $Z \otimes_{R_I} e_I R = (0, Z'' \otimes_{B_I} \eta_I B, 0)$. It follows that if we write $Y \otimes_{R_I} e_I R$ in the form (U', U'', ψ) then U' is a projective A-module and ψ is an epimorphism. Hence by Lemma 2.13 the prinjective module $(Y \otimes_{R_I} e_I R)^\wedge$ is indecomposable.

Let $(Y \otimes_{R_I} e_I R)^\wedge$ belong to $\mathrm{prin}(R)_d$ and assume that X is an indecomposable prinjective module and $f : X \to Y$ is a non-zero non-isomorphism. If there is a non-zero homomorphism from X to a module of the form $(0, E_B(S_i), 0)$ then X is in $\mathrm{prin}(R)_{d_0}$. Now assume that this is not the case.

The properties of the functor $(-) \otimes_{R_I} e_I R : \mathrm{mod}(R_I) \to \mathrm{mod}(R)$ (see e.g. [18, Theorem 17.46]) imply that $f \otimes \mathrm{id}_{e_I R} : X \otimes_{R_I} e_I R \to Y \otimes_{R_I} e_I R$ is a non-zero non-isomorphism and the modules $X \otimes_{R_I} e_I R$ and $Y \otimes_{R_I} e_I R$ are indecomposable. By applying the above arguments to X we see that also $(X \otimes_{R_I} e_I R)^\wedge$ is indecomposable and there exists a non-zero non-isomorphism $(f \otimes \mathrm{id}_{e_I R})^\wedge : (X \otimes_{R_I} e_I R)^\wedge \to (Y \otimes_{R_I} e_I R)^\wedge$ by Lemmata 2.13 and 2.15. It follows that $(X \otimes_{R_I} e_I R)^\wedge$ belongs to $\mathrm{prin}(R)_{d-1}$ and hence X belongs to $\mathrm{prin}(R_I)_{d_0+d}$ by the induction hypothesis.

We have shown that if $f : X \to Y$ belongs to the radical of $\mathrm{prin}(R_I)$ then X belongs to $\mathrm{prin}(R_I)_{d_0+d}$. Hence Y is in $\mathrm{prin}(R_I)_{d_0+d+1}$.

In order to finish the proof of the lemma observe that if Y is a prin-projective R_I-module of the form $\widehat{e_i R_I}$ then $(e_i R_I \otimes_{R_I} e_I R)^\wedge \cong \widehat{e_i R}$ is a prin-projective R-module because $i \in J$, thus it belongs to $\mathrm{prin}(R)_\infty$. Hence $\widehat{e_i R_I}$ belongs to $\mathrm{prin}(R_I)_\infty$ and the lemma follows. ∎

Recall that a vector $v \in \mathbb{Z}^l$ is *sincere* if it has all the coordinates positive. The quadratic form q is called *critical* if any vector $v \neq 0$ with only non-negative coordinates such that $q(v) = 0$ is sincere [15, 1.0].

3.3. LEMMA. *Assume that R is a bipartite prin-critical algebra* (1.1).

(a) *There exists a unique preprojective component* $\mathcal{P}(\mathrm{prin}(R))$ *of the quiver* $\Gamma(\mathrm{prin}(R))$ *containing all indecomposable prin-projective modules and no prin-injective modules. Moreover, for all but a finite number of modules* X *in* $\mathcal{P}(\mathrm{prin}(R))$ *the vector* $\mathbf{cdn}(X)$ *is sincere.*

(b) *The Tits prinjective form* $\mathbf{q}_R^{\mathrm{prin}}$ *is a critical form.*

Proof. (a) Let $\mathcal{P}$ be a preprojective component in $\Gamma(\mathrm{prin}(R))$ and let I' be the set of all indices $i = 1, \ldots, n+m$ such that the prin-projective module $P_i^\diamondsuit$ does not lie in $\mathcal{P}$ or the corresponding prin-injective module $Q_i^\diamondsuit$ belongs to $\mathcal{P}$. Assume that I' is not empty and put $I = \{1, \ldots, n+m\} \setminus I'$ and $e_I = \sum_{i \in I} e_i$. It follows from Lemma 2.16 that $\mathbf{cdn}(X)(i) = 0$ holds for $i \in I'$ and all but a finite number of modules in $\mathcal{P}$. Since $\mathcal{P}$ is an infinite component the algebra $R_I = e_I R e_I$ is of infinite prinjective type by Lemma 2.19, a contradiction. This shows in particular that $\mathcal{P}$ is the unique preprojective component of $\Gamma(\mathrm{prin}(R))$; we shall denote it by $\mathcal{P}(\mathrm{prin}(R))$. If there exist infinitely many modules X in $\mathcal{P}(\mathrm{prin}(R))$ with $\mathbf{cdn}(X)(i) = 0$ for some i then the algebra $(1-e_i)R(1-e_i)$ is of infinite prinjective type; again a contradiction.

(b) Since $\mathrm{prin}(R)$ is of infinite representation type and $\Gamma(\mathrm{prin}(R))$ has a preprojective component, it follows from Theorem 2.10(3) that $\mathbf{q}_R^{\mathrm{prin}}$ is not weakly positive. Any quadratic form q_i defined by $q_i(x_1,\ldots,x_{n+m-1}) = \mathbf{q}_R^{\mathrm{prin}}(x_1,\ldots,x_{i-1},0,x_i,\ldots,x_{n+m-1})$ is the Tits prinjective form of the bipartite algebra $(1-e_i)R(1-e_i)$, which is of finite prinjective type, and thus by Theorem 2.10(1), q_i is weakly positive and hence $\mathbf{q}_R^{\mathrm{prin}}$ is critical. ∎

Throughout this paper we shall use the generalized Kronecker algebra

$$\Lambda_r = \begin{pmatrix} k & k^r \\ 0 & k \end{pmatrix}, \tag{3.4}$$

$r \ge 2$, where k^r is viewed as a k-k-bimodule in a natural way (see [20]).

3.5. Corollary. *Assume that R is a bipartite prin-critical algebra* (1.1) *and let n and m be the ranks of the Grothendieck groups* $\mathbf{K}_0(A)$ *and* $\mathbf{K}_0(B)$ *respectively. Then one of the following conditions holds*:

(1) $n = m = 1$ *and* $R \cong \Lambda_r$ *for some* $r \ge 2$.

(2) $n + m \ge 3$ *and* $\mathbf{q}_R^{\mathrm{prin}}$ *is non-negative*, *that is*, $\mathbf{q}_R^{\mathrm{prin}}(v) \ge 0$ *for any* $v \in \mathbb{Z}^{n+m}$.

Proof. Clearly, $n, m \ge 1$. If $n = m = 1$ then R is of the form Λ_r and $r \ge 2$, since $\mathrm{prin}(R)$ is of infinite representation type. If $n + m \ge 3$ then by the results of Ovsienko in [10] (see also [15, 1.0]) the criticality of $\mathbf{q}_R^{\mathrm{prin}}$ implies (2). ∎

3.6. Lemma. *Assume that* $R \cong \Lambda_r$ (cf. (3.4)).

(a) $\mathrm{prin}(R) = \mathrm{mod}(R)$ *and the quivers* $\Gamma(\mathrm{prin}(R))$ *and* Γ_R *are isomorphic as translation quivers.*

(b) *R is of tame prinjective type if and only if* $r = 2$, *otherwise it is of fully wild prinjective type* (see [9] for definitions).

Proof. The lemma follows from the well-known representation theory of the hereditary algebra Λ_r (see [1]). ∎

3.7. Lemma. *Assume R is a bipartite prin-critical algebra*, $\mathcal{P}(\mathrm{prin}(R))$ *is the unique preprojective component in* $\Gamma(\mathrm{prin}(R))$ *and X is an indecomposable module in* $\mathcal{P}(\mathrm{prin}(R))$ *such that its translate $\Delta_R X$ is not a predecessor of a prin-projective module in* $\Gamma(\mathrm{prin}(R))$. *Then* $\mathrm{pd}_R X \le 1$ *and* $\mathrm{id}_R X \le 1$, *where* $\mathrm{pd}_R X$ *and* $\mathrm{id}_R X$ *are the projective and the injective dimension of* X *respectively.*

P r o o f. Observe first that any finitely generated injective R-module is an epimorphic image of a prin-injective R-module. Indeed, consider an indecomposable injective R-module $E_R(S_i)$. In case $i \geq n+1$ it is a quotient of $Q_i^{\diamondsuit} = \widetilde{E_R(S_i)}$. If $i \leq n$ it is enough to take the canonical projection of $(P_A(E_A(S_i)), 0, 0)$ onto $(E_A(S_i), 0, 0) \cong E_R(S_i)$. Similarly, any projective R-module is a submodule of a prin-projective one.

Secondly it follows by Lemma 2.17 that $\Delta_R X \cong \tau_R X$ and $\Delta_R^- X \cong \tau_R^- X$. Since for any prin-injective module $Q^{\diamondsuit}$ we have $\mathrm{Hom}_R(Q^{\diamondsuit}, \tau_R X) = 0$ it follows that $\mathrm{Hom}_R(Q, \tau_R X) = 0$ for any injective R-module Q and then $\mathrm{pd}_R X \leq 1$ by [15, 2.4]. Similarly we obtain $\mathrm{id}_R X \leq 1$. ∎

Following the construction in [15, 4.2(3)] we shall construct in $\mathcal{P}(\mathrm{prin}(R))$ a "relative slice", that is, a set $\mathcal{S}$ of pairwise non-isomorphic prinjective indecomposable R-modules in $\mathcal{P}(\mathrm{prin}(R))$ such that:

(a) If $X_0 \to X_1 \to \ldots \to X_l$ is a sequence of non-isomorphisms between indecomposable prinjective R-modules and $X_0, X_l \in \mathcal{S}$ then $X_j \in \mathcal{S}$ for $j = 1, \ldots, l$.

(b) If X is indecomposable and not prin-projective, then at most one of the modules X, $\Delta_R X$ belongs to $\mathcal{S}$.

(c) If X, Y are indecomposable, $f : X \to Y$ is an irreducible homomorphism in the category $\mathrm{prin}(R)$ and $Y \in \mathcal{S}$ then $X \in \mathcal{S}$ or X is not prin-injective and $\Delta_R^- X \in \mathcal{S}$ (see [15, 4.2]).

Without loss of generality we can assume that any $X \in \mathcal{S}$ is not a prin-projective module and $\Delta_R X$ is not a predecessor of a prin-projective module. This can always be achieved by a suitable shift of $\mathcal{S}$. Note that $\mathcal{S}$ intersects each Δ_R^--orbit in $\mathcal{P}(\mathrm{prin}(R))$ in one module.

3.8. PROPOSITION. *Let* $\mathcal{S}$ *be as above and assume* $\mathcal{S} = \{X_1, \ldots, X_{n+m}\}$. *Let* $\mathcal{Q}_{\mathcal{S}}$ *be the full subquiver of* $\mathcal{P}(\mathrm{prin}(R))$ *with the set* $\mathcal{S}$ *of vertices.*

(a) *The module* $X = \bigoplus_{i=1}^{n+m} X_i$ *is a tilting and cotilting R-module* (see [15, 4.1]).

(b) *The algebra* $H = \mathrm{End}_R(X) \cong k(\mathcal{Q}_{\mathcal{S}}^{\mathrm{op}})$ *is hereditary. Consequently*, R *is a tilted algebra and* $\mathbf{K}_0(\mathrm{mod}(H)) \cong \mathbf{K}_0(\mathrm{prin}(R)) \cong \mathbb{Z}^{n+m}$.

(c) *Assume that* R *is a bipartite prin-critical algebra not isomorphic to* Λ_r, $r \geq 3$ (cf. (3.4)). *Then the quiver* $\mathcal{Q}_{\mathcal{S}}$ *is an extended Dynkin diagram, that is*, H *is a tame algebra in the sense of* [18, Section 14.4].

P r o o f. (a) (Compare [15, 4.2(3)].) By Lemma 3.7, $\mathrm{pd}_R(X) \leq 1$ and $\mathrm{id}_R X \leq 1$. By standard arguments we show that X has no selfextensions (one can use the relative Auslander–Reiten formula [13, 3.15(a)]). Moreover, X has $n+m$ indecomposable direct summands and (a) follows.

For the proof of (b) repeat the arguments from [15, 4.2(3)] (note that by Lemma 2.17 the translates Δ_R and τ_R coincide on $\mathcal{S}$).

In the proof of (c) we follow [6, 3.1], [11, 3.2.2]. The statement is obvious if $R \cong \Lambda_2$. From now on we assume that this is not the case. Let X be a successor of $\mathcal{S}$ in $\mathcal{P}(\mathrm{prin}(R))$, that is, a successor of a module in $\mathcal{S}$. We shall approximate the growth of $\dim_k \Delta_R^{-l}X$, where Δ_R is the Auslander–Reiten translation in prin(R). In order to do it for any $i = 1, \ldots, n+m$ consider the difference $|\dim_k \mathrm{Hom}_R(P_i^\diamondsuit, \Delta_R^- X) - \dim_k \mathrm{Hom}_R(P_i^\diamondsuit, X)|$. Non-zero homomorphisms from $P_i^\diamondsuit$ to X do not factorize through prin-injective modules, because X belongs to the preprojective component containing no prin-injective modules. Thus $\dim_k \mathrm{Hom}(P_i^\diamondsuit, X) = \dim_k \mathrm{Ext}_R^1(\Delta_R^- X, P_i^\diamondsuit)$ by [13, Proposition 3.15(a)]. Note that $\mathrm{Hom}_R(X, P_i^\diamondsuit) = 0$ and $\mathrm{Ext}_R^1(P_i^\diamondsuit, X) = 0$. Thus by Lemma 2.7,

$$|\dim_k \mathrm{Hom}_R(P_i^\diamondsuit, \Delta_R^- X) - \dim_k \mathrm{Hom}_R(P_i^\diamondsuit, X)| = 2|(\mathbf{cdn}(P_i), \mathbf{cdn}(\Delta_R^- X))_R|.$$

By Theorem 2.10(2) the vectors $p = \mathbf{cdn}(P_i^\diamondsuit)$ and $x = \mathbf{cdn}(\Delta_R^- X)$ are positive roots of $\mathbf{q}_R^{\mathrm{prin}}$, that is, $\mathbf{q}_R^{\mathrm{prin}}(p) = \mathbf{q}_R^{\mathrm{prin}}(x) = 1$; hence

$$2(p,x)_R = \mathbf{q}_R^{\mathrm{prin}}(p+x) - \mathbf{q}_R^{\mathrm{prin}}(p) - \mathbf{q}_R^{\mathrm{prin}}(x) \geq -2$$

and

$$-2(p,x)_R = \mathbf{q}_R^{\mathrm{prin}}(p-x) - \mathbf{q}_R^{\mathrm{prin}}(p) - \mathbf{q}_R^{\mathrm{prin}}(x) \geq -2$$

by the non-negativity of $\mathbf{q}_R^{\mathrm{prin}}$. Hence $|(p,x)_R| \leq 1$ (compare [13, Lemma 4.14]) and

$$|\dim_k \mathrm{Hom}_R(P_i^\diamondsuit, \Delta_R^- X) - \dim_k \mathrm{Hom}_R(P_i^\diamondsuit, X)| \leq 2$$

for any $i = 1, \ldots, n+m$. Now it follows by Lemma 2.16(b) and Lemma 2.6 that the difference $|\dim_k(\Delta_R^- X) - \dim_k(X)|$ is bounded by a constant independent of X, hence

$$(*) \qquad \lim_{r\to\infty} \frac{\dim_k(\Delta_R^{-r} X)}{\varrho^r} = 0$$

for any $\varrho > 1$.

Let $s \in \mathbf{K}_0(H) \cong \mathbb{Z}^{n+m}$ (see (b) above) be the vector defined by $s(i) = \dim_k(X_i)$. We assume that the ith standard basis vector of the group $\mathbf{K}_0(H)$ corresponds to the vertex X_i of the quiver $\mathcal{Q}_{\mathcal{S}}$. It is easy to see $\dim_k(\Delta_R^{-l} X_i) = (s\Phi_H^l)(i)$ for $l \geq 0$ and $i = 1, \ldots, n+m$ (comp. [6, 3.1]). Here Φ_H denotes the Coxeter transformation of H (see [15, 2.4]). The set $\{s\Phi_H^l\}_{l\geq 0}$ consists of vectors with non-negative coordinates, thus by [6, Lemma 3.2] and its proof the condition $(*)$ implies that the quiver $\mathcal{Q}_{\mathcal{S}}$ is an extended Dynkin diagram. We remark that in the statement of Lemma 3.2 in [6] it is assumed that the quiver $\mathcal{Q}_{\mathcal{S}}$ is a tree. But by [12, Theorem 3.5] this assumption is not necessary. ∎

3.9. Proposition. *Let R be a bipartite prin-critical algebra not isomorphic to Λ_r, $r \geq 3$* (cf. (3.4)). *Then*

(a) *R is a tame concealed algebra* (see [15, 4.3]).

(b) *$\Gamma(\mathrm{prin}(R))$ has a unique preinjective component $\mathcal{Q}(\mathrm{prin}(R))$ containing all prin-injective indecomposable objects. Moreover, the modules from $\mathcal{P}(\mathrm{prin}(R))$ (resp. $\mathcal{Q}(\mathrm{prin}(R))$) are preprojective (resp. preinjective) in Γ_R.*

(c) *There exists a sincere vector $v \in \mathbb{N}^{n+m}$ such that $\mathbf{q}_R(d_R^v) = 0$, where d_R^v is defined in* (2.5) *and $\mathbf{q}_R$ is the Tits quadratic form of the algebra R. Moreover, if the largest common divisor of the coordinates v_i of v equals* 1 *then* $\mathrm{Ker}\, \mathbf{q}_R = \mathbb{Z} d_R^v$, *where* $\mathrm{Ker}\, \mathbf{q}_R = \{u \in \mathbb{Z}^{n+m} : \mathbf{q}_R(u) = 0\}$.

Proof. (a) We know from Proposition 3.8 that R is a tilted algebra of extended Dynkin type. It is enough to show that the direct summands of a tilting module $T = T_H$ such that $R = \mathrm{End}_H(T)$ are all preprojective or all preinjective (comp. [11, 3.2.2]). Since the algebra R is of infinite representation type it follows by [15, 4.2(8)] that T does not have both preprojective and preinjective direct summands. Now it is enough to show that T does not have regular direct summands. Let $T = \bigoplus_{i=1}^{n+m} T_i$, T_i indecomposable, and let e_i be the idempotent of R corresponding to the summand T_i. Assume that T_1 is a regular H-module.

Given a number $d \in \mathbb{N}$ for all but a finite number of indecomposable H-modules M of dimension d we have $\mathrm{Hom}_H(T_1, M) = \mathrm{Ext}_H^1(T_1, M) = 0$. It follows that for $d \in \mathbb{N}$ all but a finite number of indecomposable R-modules of dimension d are annihilated by e_1 (see [15, 4.2(8)]).

Since the form $\mathbf{q}_R^{\mathrm{prin}}$ is not weakly positive it follows by Theorem 2.10(1) that there exists a vector $v \in \mathbb{N}^{n+m}$ and an infinite family $\{X_\lambda\}_\lambda$ of pairwise non-isomorphic indecomposable prinjective R-modules such that $\mathbf{cdn}(X_\lambda) = v$ for any λ. The algebra R is prin-critical so v is sincere. Hence the R-modules X_λ are not annihilated by e_1, a contradiction.

(b) For all but a finite number of modules X in $\mathcal{P}(\mathrm{prin}(R))$ the translates $\Delta_R^- X$ and $\tau_R^- X$ coincide by Lemma 2.17. It follows that for those modules X the module $\tau_R^{-m} X$ is defined for all $m \geq 0$ and X is not τ_R-periodic. Thus all modules in $\mathcal{P}(\mathrm{prin}(R))$ lie in the preprojective component $\mathcal{P}$ of the Auslander–Reiten quiver Γ_R of $\mathrm{mod}(R)$. The modules X_λ constructed in the proof of (a) above are regular. Take an arbitrary indecomposable prin-injective R-module $Q^\diamondsuit$. Since $\mathbf{cdn}(X_\lambda)$ is a sincere vector for any index λ we get $\mathrm{Hom}_R(X_\lambda, Q^\diamondsuit) \neq 0$ (see Lemma 2.16). Thus $Q^\diamondsuit$ lies in the preinjective component $\mathcal{Q}$ of the quiver Γ_R.

Let $Q_i^\diamondsuit$ be the prin-injective indecomposable module having no prin-injective predecessors in Γ_R. It follows from Lemma 2.17 that $\Delta_R Q_i^\diamondsuit \cong \tau_R Q_i^\diamondsuit$; the same can be said about all the predecessors of $Q_i^\diamondsuit$ in $\mathcal{Q}$. It follows that all but a finite number of modules in $\mathcal{Q}$ are prinjective. It is easy to check that those modules form a unique preinjective component $\mathcal{Q}(\mathrm{prin}(R))$ of $\Gamma(\mathrm{prin}(R))$.

(c) Put $v = \mathbf{cdn}(X_\lambda)$, where the modules X_λ form the infinite family constructed in the proof of (a). Clearly, the modules X_λ are regular and $\mathbf{q}_R(\mathbf{dim}(X_\lambda)) = 0$ by [15, 4.3(8)]. But $\mathbf{dim}(X_\lambda) = d_R^v$ and $\mathbf{q}_R^{\mathrm{prin}}(v) = \mathbf{q}_R(d_R^v)$ by Lemmata 2.6 and 2.8. Since the form $\mathbf{q}_R^{\mathrm{prin}}$ is critical the vector v is sincere and (c) follows. The remaining statement is a consequence of the results of [10]. ■

Note that it follows from the above proposition that if R is a bipartite prin-critical algebra then a prinjective R-module X is preprojective (resp. preinjective) in $\Gamma(\mathrm{prin}(R))$ if and only if X is preprojective (resp. preinjective) in Γ_R.

3.10. THEOREM. *Let R be a bipartite algebra of the form* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ (see (1.1)) *and let n, m be the numbers of the isomorphism classes of simple modules in* $\mathrm{mod}(A)$ *and* $\mathrm{mod}(B)$ *respectively. The algebra R is prin-critical if and only if one of the following conditions is satisfied*:

(1) $R = \Lambda_r$ (see (3.4)) *for some* $r \geq 2$.

(2) $n + m \geq 3$ *and R is tame concealed and there exists a sincere vector* $v \in \mathbb{N}^{n+m}$ *such that the largest common divisor of the coordinates v_i of v equals* 1 *and* $\mathbf{q}_R(d_R^v) = 0$.

If this is the case then $\mathrm{Ker}\,\mathbf{q}_R = \mathbb{Z}d_R^v$, *where* $\mathrm{Ker}\,\mathbf{q}_R = \{u \in \mathbb{Z}^{n+m} : \mathbf{q}_R(u) = 0\}$.

Proof. When $n + m = 2$ the statement follows by Corollary 3.5. If $n + m \geq 3$ then if R is prin-critical the condition (2) follows from Proposition 3.9. To prove the converse implication we show first that the algebra satisfying (2) is of infinite prinjective type. By Lemma 2.8(b) and our assumption $\mathbf{q}_R^{\mathrm{prin}}(v) = \mathbf{q}_R(d_R^v) = 0$. Thus the form $\mathbf{q}_R^{\mathrm{prin}}$ is not weakly positive and therefore by Theorem 2.10(1), $\mathrm{prin}(R)$ is of infinite representation type.

Now we prove that the quiver $\Gamma(\mathrm{prin}(R))$ has a preprojective component. Since $\mathbf{q}_R^{\mathrm{prin}}$ is not weakly positive we conclude by Theorem 2.10 that there is an infinite family of pairwise non-isomorphic indecomposable prinjective R-modules $\{X_\lambda\}_\lambda$ having the same coordinate vector v'. It follows that all modules X_λ are regular R-modules and then $\mathbf{q}_R(\mathbf{dim}(X_\lambda)) = \mathbf{q}_R(d_R^{v'}) = 0$. The form $\mathbf{q}_R$ is critical, hence, by Ovsienko's Theorem [10], the vectors d_R^v and $d_R^{v'}$ are linearly dependent. Since the homomorphism $v \mapsto d_R^v$ is invertible by Lemma 2.8(a) the vector v' is a multiple of v and hence v' is a sincere vector in $\mathbb{Z}^{n+m}$. Using Lemma 2.16 one can prove that all prinprojective indecomposable modules lie in the preprojective component $\mathcal{P}$ of Γ_R. It follows that $\Gamma(\mathrm{prin}(R))$ has a preprojective component.

By Lemma 3.2 there exists a subset $I \subseteq \{1, \ldots, n+m\}$ such that the bipartite algebra R_I is prin-critical. It follows that there exists a vector $v' \in \mathbb{N}^I \subseteq \mathbb{N}^{n+m}$ such that $\mathbf{q}_R^{\mathrm{prin}}(v') = 0$. Thus $\mathbf{q}_R(d_R^{v'}) = 0$ and as above we conclude that v' is sincere and $I = \{1, \ldots, n+m\}$. Hence the algebra $R = R_I$ is prin-critical ∎

Note that condition (2) of Theorem 3.10 together with the list of all the tame concealed algebras provides a description of all prin-critical algebras. In particular, we prove the following lemma.

3.11. LEMMA. *If $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is a bipartite prin-critical algebra which is tame concealed of type $\widetilde{\mathbb{A}}_n$ then R is isomorphic to the path algebra $k\widetilde{\mathbb{A}}_n^*$, where*

$$\widetilde{\mathbb{A}}_n^*:\quad \begin{array}{ccccccccc} a_1 & & a_2 & & & & a_{t-1} & & a_t \\ \downarrow & \searrow & \downarrow & \searrow & & \searrow & \downarrow & \searrow & \downarrow \\ b_1 & & b_2 & & \ldots & & b_{t-1} & & b_t \\ \uparrow & & & & & & & & \end{array}$$

and $A = ke_{a_1} \times ke_{a_2} \times \ldots \times ke_{a_t}$, $B = ke_{b_1} \times ke_{b_2} \times \ldots \times ke_{b_t}$. If this is the case then $\mathrm{mod}(R) = \mathrm{prin}(R)$ *and the Auslander–Reiten quivers* $\Gamma(\mathrm{prin}(R))$ *and* Γ_R *coincide.*

Proof. It follows from the classification of tame concealed algebras ([5], [15]) that R is the path algebra of the quiver Q of type $\widetilde{\mathbb{A}}_n$. Let $A = kQ_A$ and $B = kQ_B$, where Q_A and Q_B are subquivers of Q. There is no oriented path from Q_B to Q_A. By Theorem 3.10 there exists a sincere vector $v \in \mathbb{N}^{n+m}$ such that $\mathbf{q}_R(d_R^v) = 0$. Under our assumptions on R it follows that $d_R^v(i) = c$ for a constant c and all $i \in Q_0$. Then $v(i) = c$ if and only if i is a source in Q_A or a sink in Q_B, and $v(i) = 0$ otherwise. Since v is sincere the first part of the lemma follows. In order to finish the proof it is enough to note that each $k\widetilde{\mathbb{A}}_n^*$-module is prinjective if the bipartition of $k\widetilde{\mathbb{A}}_n^*$ is as above. ∎

3.12. THEOREM. *If R is a bipartite prin-critical algebra then all but a finite number of indecomposable R-modules are prinjective and the Auslander–Reiten quiver* $\Gamma(\mathrm{prin}(R))$ *is obtained from* Γ_R *by deleting a finite number of preprojective and preinjective vertices.*

Proof. It follows easily by Lemma 2.17 and Proposition 3.9(b) that all but finitely many of preprojective and preinjective indecomposable R-modules are prinjective. We shall prove that all regular R-modules are prinjective. Let $X = (X'_A, X''_B, \phi)$ be an indecomposable regular R-module and

$\overline{\phi}$ the homomorphism adjoint to ϕ. There exist infinitely many indecomposable preprojective R-modules Y and infinitely many indecomposable preinjective R-modules Z such that $\mathrm{Hom}_R(Y,X) \neq 0 \neq \mathrm{Hom}_R(X,Z)$. We can assume that all Y's and Z's are prinjective. Since all prin-projective (resp. prin-injective) modules lie in the preprojective (resp. preinjective) component it follows by Lemma 2.13 that the module $\widetilde{X}$ is indecomposable and by Lemma 2.15, $\mathrm{Hom}_R(Y,\widetilde{X}) \neq 0 \neq \mathrm{Hom}_R(\widetilde{X},Z)$ for infinitely many preprojective modules X and infinitely many preinjective modules Z. Hence $\widetilde{X}$ is regular. Note that the natural projection $\varepsilon_X : \widetilde{X} \to X$ is a monomorphism, for otherwise there is a non-zero map $(K,0,0) = \mathrm{Ker}\,\varepsilon_X \to \widetilde{X}$ and consequently a non-zero homomorphism from a prin-injective module to $\widetilde{X}$, which is impossible. Hence $X \cong \widetilde{X}$. Analogously we prove that $X \cong \widehat{X}$ and X is prinjective.

The rest of the statement follows from Lemma 2.17. ∎

3.13. Corollary. *Assume that R is a bipartite prin-critical algebra not isomorphic to Λ_r, $r \geq 3$.*

(a) *The Auslander–Reiten quiver* $\Gamma(\mathrm{prin}(R))$ *of* $\mathrm{prin}(R)$ *consists of the preprojective component* $\mathcal{P}(\mathrm{prin}(R))$, *the preinjective component* $\mathcal{Q}(\mathrm{prin})$ *and a* 1*-parametric standard stable tubular family* $\mathcal{T}$ *separating* $\mathcal{P}(\mathrm{prin}(R))$ *from* $\mathcal{Q}(\mathrm{prin})$ (see [15]).

(b) *The category* $\mathrm{prin}(R)$ *is of tame representation type and domestic.* ∎

3.14. Remark. It is easy to observe that under the assumptions of Corollary 3.13 all components of the quiver $\Gamma(\mathrm{prin}(R))$ are generalized standard in the sense of [23], that is, given two indecomposable modules X, Y in the same component we have $\mathrm{rad}^\infty(X,Y) = 0$, where rad^∞ is the infinite radical of the category $\mathrm{mod}(R)$ (see [1], [23]). Moreover, if we denote by $\mathrm{rad}^\infty_{\mathrm{prin}}$ the infinite radical of the category $\mathrm{prin}(R)$ then $\mathrm{rad}^\infty(X,Y) = \mathrm{rad}^\infty_{\mathrm{prin}}(X,Y)$ for arbitrary prinjective modules X, Y. It would be interesting to know the relation between $\mathrm{rad}^\infty_{\mathrm{prin}}$ and the restriction of rad^∞ to the category $\mathrm{prin}(R)$ in the case of an arbitrary bipartite algebra R. ∎

The next corollary follows by the arguments used in the proof of Theorem 3.12 and Lemmata 2.13, 2.14.

3.15. Corollary. *Assume that R is a bipartite prin-critical algebra. All but a finite number of preprojective and preinjective indecomposable R-modules belong to* $\mathrm{prin}(R) \cap \mathrm{mod}_{ic}(R)^A_B \cap \mathrm{mod}^{pg}(R)^A_B \cap \mathrm{adj}(R)^A_B$. (*For the definitions of the above categories we refer to* [13]). ∎

We finish the paper with the following simple observation.

3.16. Lemma. *Let R be a bipartite prin-critical algebra not isomorphic to Λ_r* (see (3.4)) *for $r \geq 3$. Let X be a preprojective* (*resp. preinjective*)

R-module. Then

$$\lim_{s\to\infty}\frac{\mathbf{cdn}(\varDelta_R^{-s}X)}{|\mathbf{cdn}(\varDelta_R^{-s}X)|}=\frac{\mu_R}{|\mu_R|}\qquad\Big(\textit{resp.}\ \lim_{s\to\infty}\frac{\mathbf{cdn}(\varDelta_R^{s}X)}{|\mathbf{cdn}(\varDelta_R^{s}X)|}=\frac{\mu_R}{|\mu_R|}\Big)$$

where $\mu_R\in\mathbb{N}^{n+m}$ is a non-zero vector such that $\mathbf{q}_R^{\mathrm{prin}}(\mu_R)=0$, and for a vector v we denote by $|v|$ the sum of its coordinates.

Proof. Let X be a module in the preprojective component $\mathcal{P}(\mathrm{prin}(R))$. Then it is clear that

$$\lim_{s\to\infty}|\mathbf{cdn}(\varDelta_R^{-s}X)|=\infty.$$

Moreover, $\mathbf{q}_R^{\mathrm{prin}}(\mathbf{cdn}(\varDelta_R^{-s}X))=1$ for any $s\geq 0$ by Theorem 2.10(2). We shall prove that any subsequence of the sequence $\mathbf{cdn}(\varDelta_R^{-s}X)/|\mathbf{cdn}(\varDelta_R^{-s}X)|$ has a subsequence convergent to $\mu_R/|\mu_R|$ and hence

$$\lim_{n\to\infty}\frac{\mathbf{cdn}(\varDelta_R^{-s}X)}{|\mathbf{cdn}(\varDelta_R^{-s}X)|}=\frac{\mu_R}{|\mu_R|}.$$

The vectors $v_s=\mathbf{cdn}(\varDelta_R^{-s}X)/|\mathbf{cdn}(\varDelta_R^{-s}X)|$ belong to the compact set $\{v\in\mathbb{R}^{n+m}:|v|=1,\ v(1),\dots,v(n+m)\geq 0\}$. Let a subsequence $(v_{s_t})_t$ of the sequence $(v_s)_s$ converge to v_0. Then

$$\mathbf{q}_R^{\mathrm{prin}}(v_0)=\lim_{t\to\infty}\mathbf{q}_R^{\mathrm{prin}}(v_{s_t})=\lim_{t\to\infty}\frac{\mathbf{q}_R^{\mathrm{prin}}(\mathbf{cdn}(\varDelta_R^{-s_t}X))}{|\mathbf{cdn}(\varDelta_R^{-s_t}X)|^2}=0,$$

thus since the quadratic form $\mathbf{q}_R^{\mathrm{prin}}$ is critical and by the results of [10] the vector v_0 is a multiple of μ_R, but $|v_0|=1$, hence $v_0=\mu_R/|\mu_R|$.

In the case when X is a preinjective module the proof is analogous. ∎

3.17. Corollary. *Let R be a bipartite prin-critical algebra of tame prinjective type. Let $l:\mathbf{K}_0(\mathrm{prin}(R))\cong\mathbb{Z}^{n+m}\to\mathbb{Z}$ be a $\mathbb{Z}$-linear function such that $l(\mu_R)>0$. Then for any number M there exists an indecomposable preprojective* (*resp. preinjective*) *prinjective R-module Y such that $l(\mathbf{cdn}(Y))>M$.*

Proof. We prove the existence of a prepojective module satisfying the conditions of the corollary; the existence of a preinjective one follows analogously. Let X be an arbitrary indecomposable module in the preprojective component of $\Gamma(\mathrm{prin}(R))$. Then it follows from Lemma 3.16 that $\lim_{s\to\infty}l(\mathbf{cdn}(\varDelta_R^{s}X))=\infty$. We put $Y=\varDelta_R^{s}X$ for s large enough. ∎

3.18. Remark. The above corollary gives a simplification of the proof of one of the main results in [9], namely that hypercritical posets are of fully wild prinjective type. Indeed, it is enough to put $l=\widehat{l_a}$ defined in (3.9) in the proof of Lemma 3.8 in [9] and $M=3$.

For example, let R be the incidence algebra of the poset

$$\begin{array}{ccccccc}
2 & & & & & & \\
\downarrow & & & & & & \\
3 & & & & 5 & & \\
\downarrow & & & \swarrow & \downarrow & & \\
4 & & 6 & & 7 & & 8 \\
\downarrow & \swarrow & & \searrow & \downarrow & \swarrow & \\
9 & & & & 10 & &
\end{array}$$

that is, R is the path algebra of the above quiver divided by the commutativity relation. We consider R with a bipartition (1.1) such that $B = (e_9 + e_{10})R(e_9 + e_{10})$, where e_i denotes the standard idempotent corresponding to the vertex i. It follows from [19] that R is a prin-critical algebra and it is easy to check that this is a concealed algebra of type $\widetilde{\mathbb{E}}_8$.

Let $\mu_R = (1, 1, 1, 2, 3, 1, 2, 4, 4) \in \mathbb{Z}^{\{2,\ldots,10\}}$. Then μ_R generates the kernel of the Tits prinjective quadratic form of R. Consider the linear function $l : \mathbb{Z}^{\{2,\ldots,8\}} \to \mathbb{Z}$ given by $l(v) = v(9) - v(2) - v(3) - v(4)$. Observe that $l(\mu_R) > 0$. By Corollary 3.17 there exists an indecomposable module X in the preprojective component of $\Gamma(\mathrm{prin}(R))$ such that $l(\mathbf{cdn}(X)) \geq 3$.

Now consider the one-point extension $\widetilde{R}$ of R by a prin-projective R-module $P_2^{\diamondsuit}$ associated with the vertex 2; that is, $\widetilde{R}$ is the path algebra of the quiver

$$\begin{array}{ccccccc}
1 & & & & & & \\
\downarrow & & & & & & \\
2 & & & & & & \\
\downarrow & & & & & & \\
3 & & & & 5 & & \\
\downarrow & & & \swarrow & \downarrow & & \\
4 & & 6 & & 7 & & 8 \\
\downarrow & \swarrow & & \searrow & \downarrow & \swarrow & \\
9 & & & & 10 & &
\end{array}$$

modulo the commutativity relation. We consider $\widetilde{R}$ together with a bipartion such that $\widetilde{R}_I \cong R$ if $I = \{2, \ldots, 10\}$. It follows by results of [9] that if we put $U = Q_1^{\diamondsuit}$ and $V = T_R^{\widetilde{R}}(X)$ then the prinjective $\widetilde{R}$-modules U and V satisfy the following conditions:

(i) $\mathrm{End}_{\widetilde{R}}(U) \cong \mathrm{End}_{\widetilde{R}}(V) \cong K$,
(ii) $\mathrm{Hom}_{\widetilde{R}}(U, V) = \mathrm{Hom}_{\widetilde{R}}(V, U) = 0$,
(iii) $\dim_K(\mathrm{Ext}^1_{\widetilde{R}}(U, V)) \geq 3$.

It follows from Lemmata 1.5 and 8.6 in [14] that this implies the existence of a full faithful exact functor $T_{U,V} : \mathrm{mod}(\Lambda_3) \to \mathrm{mod}(\widetilde{R})$, where Λ_3 is defined in (3.4), such that $\operatorname{Im} T_{U,V} \subseteq \operatorname{prin}(\widetilde{R})$. Thus $\operatorname{prin}(\widetilde{R})$ is of fully wild representation type in the sense of [9].

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Received 2 June 1997;
revised 29 October 1997