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ON THE EQUIVALENCE OF RICCI-SEMISYMMETRY AND SEMISYMMETRY

BY

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Dedicated to the memory of our friend Dr. Şahnur Yaprak

Introduction. A semi-Riemannian manifold (M,g), $n=\dim M\geq 3$, is said to be *semisymmetric* [28] if

$$R\cdot R=0 \tag{1}$$

holds on M. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R=0$) as a proper subset. Recently the theory of Riemannian semisymmetric manifolds has been presented in the monograph [1]. It is clear that every semisymmetric manifold satisfies

$$R\cdot S=0. \tag{2}$$

The semi-Riemannian manifold (M,g), $n\geq 3$, satisfying (2) is called *Ricci-semisymmetric*. There exist non-semisymmetric Ricci-semisymmetric manifolds. However, under some additional assumptions, (1) and (2) are equivalent for certain manifolds. For instance, we have the following statement.

REMARK 1.1. (1) and (2) are equivalent on every 3-dimensional semi-Riemannian manifold as well as at all points of any semi-Riemannian manifold (M,g), of dimension ≥ 4, at which the Weyl tensor C of (M,g) vanishes (see e.g. [15, Lemma 2]). In particular, (1) and (2) are equivalent for every conformally flat manifold.

It is a long standing question whether (1) and (2) are equivalent for hypersurfaces of Euclidean spaces; cf. Problem P 808 of [27] by P. J. Ryan, and references therein. More generally, one can ask the same question for hypersurfaces of semi-Riemannian space forms. It was proved in [29] that

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(1) and (2) are equivalent for hypersurfaces which have positive scalar curvature in Euclidean space $\mathbb{E}^{n+1}$, $n \geq 3$. In [26] this result was generalized to hypersurfaces of $\mathbb{E}^{n+1}$, $n \geq 3$, which have non-negative scalar curvature and also to hypersurfaces of constant scalar curvature. [26] also proves that (1) and (2) coincide for hypersurfaces of Riemannian space forms with non-zero constant sectional curvature.

Further, in [24] it was proved that (1) and (2) are equivalent for hypersurfaces of $\mathbb{E}^{n+1}$, $n \geq 3$, under the additional global condition of completeness. In [4] it was shown that (1) and (2) are equivalent for Lorentzian hypersurfaces of a Minkowski space $\mathbb{E}_1^{n+1}$, $n \geq 4$. [4] also proves that (1) and (2) are equivalent for para-Kähler hypersurfaces of a semi-Euclidean space $\mathbb{E}_s^{2m+1}$, $m \geq 2$. The problem of equivalence of (1) and (2) was solved in the 4-dimensional case. More precisely, we have the following statement.

THEOREM 1.1 ([3, Theorem 4.1]). (1) *and* (2) *are equivalent for hypersurfaces of semi-Riemannian spaces of constant curvature* $N^5(c)$.

The problem of equivalence of (1) and (2) for hypersurfaces with pseudosymmetric Weyl tensor of semi-Euclidean spaces was considered in [5].

THEOREM 1.2 ([5, Theorem 4.1]). *Let M be a Ricci-semisymmetric hypersurface of a semi-Euclidean space $\mathbb{E}_s^{n+1}$ of index s, $n \geq 4$. If M has pseudosymmetric Weyl tensor then* (1) *holds on the set U_S consisting of all points of M at which the Ricci tensor S of M is not proportional to the metric tensor of M.*

Hypersurfaces with pseudosymmetric Weyl tensor were studied in [7], [20] and [21]. In particular, the following curvature property of semisymmetric hypersurfaces was found.

THEOREM 1.3 ([20, Theorem 7.3(ii)]; [21, Theorem 4.1]). *Every semisymmetric hypersurface M isometrically immersed in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, is a hypersurface with pseudosymmetric Weyl tensor.*

Our main result (see Theorem 5.2) is related Theorem 1.2. Namely, we prove that if (M, g), $\dim M \geq 4$, is a Riemannian Ricci-semisymmetric manifold with pseudosymmetric Weyl tensor, satisfying

$$R \cdot R = Q(S, R), \tag{3}$$

then (1) holds on U_S. Theorem 1.2, in the case when the ambient space is a Euclidean space, is an immediate consequence of Theorem 5.2. We recall that every hypersurface M isometrically immersed in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 3$, satisfies (3) ([18, Corollary 3.1]).

The paper is organized as follows. In Section 2 we fix the notations and give precise definitions of the symbols used. Moreover, we give a short presentation of classes of semi-Riemannian manifolds satisfying curvature conditions of pseudosymmetry type. In Section 3 we give preliminary results. Finally, in Sections 4 and 5 we present our main results.

2. Certain curvature conditions. Let (M, g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞. We define on M the endomorphisms $\widetilde{\mathcal{R}}(X, Y)$ and $X \wedge Y$ by

$$\widetilde{\mathcal{R}}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y,$$

where ∇ is the Levi-Civita connection of (M, g) and $X, Y, Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. Furthermore, we define the Riemann–Christoffel curvature tensor R and the $(0, 4)$-tensor G of (M, g) by

$$R(X_1, X_2, X_3, X_4) = g(\widetilde{\mathcal{R}}(X_1, X_2)X_3, X_4),$$
$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4).$$

We denote by S and κ the Ricci tensor and the scalar curvature of (M, g), respectively. For a $(0, k)$-tensor field T on M, $k \geq 1$, we define the $(0, k+2)$-tensors $R \cdot T$ and $Q(g, T)$ by

$$\begin{aligned}(R \cdot T)(X_1, \ldots, X_k; X, Y) &= (\widetilde{\mathcal{R}}(X, Y) \cdot T)(X_1, \ldots, X_k)\\ &= -T(\widetilde{\mathcal{R}}(X, Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, \widetilde{\mathcal{R}}(X, Y)X_k),\\ Q(g, T)(X_1, \ldots, X_k; X, Y) &= ((X \wedge Y) \cdot T)(X_1, \ldots, X_k)\\ &= -T((X \wedge Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge Y)X_k).\end{aligned}$$

A semi-Riemannian manifold (M, g) is said to be *pseudosymmetric* ([11], [30]) if

$(*)_1$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent

at every point of M. This is equivalent to the equality

$$R \cdot R = L_R Q(g, R) \tag{4}$$

holding on

$$U_R = \left\{x \in M \;\middle|\; R - \frac{\kappa}{n(n-1)}G \neq 0 \text{ at } x\right\},$$

for some function L_R on U_R. It is clear that every semisymmetric manifold is pseudosymmetric. The condition $(*)_1$ arose in the study of totally umbilical submanifolds of semisymmetric manifolds as well as when considering geodesic mappings of semisymmetric manifolds ([11], [30]). There exist pseudosymmetric manifolds which are non-semisymmetric. For instance, in [12, Example 3.1 and Theorem 4.1] it was shown that the warped product $S^p \times_F S^{n-p}$, $p \geq 2$, $n - p \geq 1$, of the standard spheres S^p and S^{n-p}, with a certain function F, is such a manifold.

A semi-Riemannian manifold (M, g) is said to be *Ricci-pseudosymmetric* ([8], [16]) if

$(*)_2$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent

at every point of M. Thus (M, g) is Ricci-pseudosymmetric if and only if

$$R \cdot S = L_S Q(g, S) \tag{5}$$

on

$$U_S = \left\{ x \in M \,\middle|\, S - \frac{\kappa}{n} g \neq 0 \text{ at } x \right\},$$

for some function L_S on U_S. Note that $U_S \subset U_R$. It is clear that if $(*)_1$ holds at x then so does $(*)_2$. The converse is not true. E.g. every warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 = n-1 \geq 3$, of a manifold $(M_1, \overline{g})$ and a non-pseudosymmetric Einstein manifold $(M_2, \widetilde{g})$ is a non-pseudosymmetric, Ricci-pseudosymmetric manifold (cf. [16, Remark 3.4] and [13, Theorem 4.1]).

Remark 2.1. In [10, Theorem 4] it was shown that $(*)_1$ and $(*)_2$ are equivalent on the subset U_S of a 4-dimensional warped product $M_1 \times_F M_2$. In particular, (1) and (2) are equivalent on the subset U_S of a 4-dimensional warped product $M_1 \times_F M_2$. We also note that there exist non-semisymmetric Einsteinian 4-dimensional warped products $M_1 \times_F M_2$, e.g. the Schwarzschild spacetimes as well as the Kerr spacetimes. Moreover, the Schwarzschild spacetimes are pseudosymmetric manifolds.

For any $X, Y \in \Xi(M)$ we define the endomorphism $\widetilde{\mathcal{C}}(X, Y)$ by

$$\widetilde{\mathcal{C}}(X, Y) = \widetilde{\mathcal{R}}(X, Y) - \frac{1}{n-2}\left(X \wedge \widetilde{\mathcal{S}} Y + \widetilde{\mathcal{S}} X \wedge Y - \frac{\kappa}{n-1} X \wedge Y \right).$$

The *Ricci operator* $\widetilde{\mathcal{S}}$ and the *Weyl conformal curvature tensor* C of (M, g) are defined by

$$g(\widetilde{\mathcal{S}} X, Y) = S(X, Y), \qquad C(X_1, X_2, X_3, X_4) = g(\widetilde{\mathcal{C}}(X_1, X_2) X_3, X_4).$$

Now we define the $(0, 6)$-tensor $C \cdot C$ by

$$\begin{aligned}(C \cdot C)(X_1, X_2, X_3, X_4; X, Y) &= (\widetilde{\mathcal{C}}(X, Y) \cdot C)(X_1, X_2, X_3, X_4) \\ &= -C(\widetilde{\mathcal{C}}(X, Y) X_1, X_2, X_3, X_4) - \ldots - C(X_1, X_2, X_3, \widetilde{\mathcal{C}}(X, Y) X_4).\end{aligned}$$

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be a *manifold with pseudosymmetric Weyl tensor* ([11], [23], [30]) if

$(*)_3$ the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent

at every point of M. The manifold (M, g) has pseudosymmetric Weyl tensor if and only if

$$C \cdot C = L_C Q(g, C) \tag{6}$$

on

$$U_C = \{x \in M \mid C \neq 0 \text{ at } x\},$$

for some function L_C on U_C. It is known that every warped product $M_1 \times_F M_2$, $\dim M_1 = \dim M_2 = 2$, satisfies $(*)_3$ ([10, Theorem 2]). An example of a 4-dimensional Riemannian manifold satisfying $(*)_3$, which is not a warped product, was found in [23]. Manifolds satisfying $(*)_1$ and $(*)_3$ were investigated in [23].

For a symmetric $(0,2)$-tensor A we define the endomorphism $X \wedge_A Y$ of $\Xi(M)$ by $(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y$. Furthermore, for a $(0,k)$-tensor field T, $k \geq 1$, and the tensor field A we define the tensor $Q(A,T)$ by

$$\begin{aligned} Q(A,T)(X_1, \ldots, X_k; X, Y) &= ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k) \\ &= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k). \end{aligned}$$

In particular, in this way we obtain the $(0,6)$-tensor field $Q(S,R)$.

Semi-Riemannian manifolds satisfying $(*)$, $(*)_1$, $(*)_2$ or $(*)_3$ are called *manifolds of pseudosymmetry type* ([11], [30]). We finish this section with some examples of Ricci-pseudosymmetric manifolds.

EXAMPLE 2.1. It is known that the Cartan hypersurfaces in the sphere $S^{n+1}(c)$, $n = 3, 6, 12$ or 24, are compact, minimal hypersurfaces with constant principal curvatures $-(3c)^{1/2}$, 0, $(3c)^{1/2}$ having the same multiplicity. More precisely, the Cartan hypersurfaces are the tubes of constant radius over the standard Veronese embeddings $i : \mathbb{F}P^2 \to S^{3d+1}(c) \to \mathbb{E}^{3d+2}$, $d = 1, 2, 4, 8$, of the projective plane $\mathbb{F}P^2$ in the sphere $S^{3d+1}(c)$ in $\mathbb{E}^{3d+2}$, where $\mathbb{F} = \mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{Q}$ (quaternions) or $\mathbb{O}$ (Cayley numbers), respectively. The Cartan hypersurfaces satisfy certain curvature condition of pseudosymmetry type. In [22, Theorem 1] it was shown that every Cartan hypersurface in $S^{n+1}(c)$, $n = 6, 12, 24$, is a non-pseudosymmetric, Ricci-pseudosymmetric manifold with non-pseudosymmetric Weyl tensor satisfying the relations

$$\begin{aligned} R \cdot S &= \frac{\widetilde{\kappa}}{n(n+1)} Q(g, S), \\ R \cdot R - Q(S, R) &= -\frac{(n-2)\widetilde{\kappa}}{n(n+1)} Q(g, C) \end{aligned}$$

on M, where $\widetilde{\kappa}$ is the scalar curvature of $S^{n+1}(c)$. The Cartan hypersurface in $S^4(c)$ is a pseudosymmetric manifold satisfying

$$R \cdot R = \frac{\widetilde{\kappa}}{12} Q(g, R).$$

3. Preliminary results. Let (M,g), $n \geq 3$, be a semi-Riemannian manifold covered by a system of coordinate neighbourhoods $\{\mathcal{U}; x^h\}$. We denote by g_{ij}, R_{hijk}, S_{ij} and C_{hijk} the local components of the tensors g, R, S and C, respectively. Further, we denote by $S^2_{ij} = S_{ip}S_j{}^p$ and $S_j{}^k = g^{ks}S_{js}$ the local components of the tensor S^2 defined by $S^2(X,Y) = S(\widetilde{\mathcal{S}}X, Y)$, $X, Y \in \Xi(M)$, and of the Ricci operator $\widetilde{\mathcal{S}}$, respectively.

Let U and $\overline{S}$ be the $(0,4)$-tensor fields on (M,g) defined by

$$\begin{aligned} U(X_1,X_2,X_3,X_4) &= g(X_1,X_4)S(X_2,X_3) - g(X_1,X_3)S(X_2,X_4) \\ &\quad + g(X_2,X_3)S(X_1,X_4) - g(X_2,X_4)S(X_1,X_3), \end{aligned} \tag{7}$$

$$\overline{S}(X_1,X_2,X_3,X_4) = S(X_1,X_4)S(X_2,X_3) - S(X_1,X_3)S(X_2,X_4). \tag{8}$$

Lemma 3.1. *The following identities hold on any semi-Riemannian manifold* (M,g), $n \geq 3$:

$$Q(g,U) = -Q(S,G), \tag{9}$$

$$Q(S,U) = -Q(g,\overline{S}), \tag{10}$$

$$Q(g,C) = Q(g,R) + \frac{1}{n-2}Q(S,G), \tag{11}$$

$$Q(S,C) = Q(S,R) - \frac{1}{n-2}Q(S,U) + \frac{\kappa}{(n-1)(n-2)}Q(S,G), \tag{12}$$

$$Q(S,C) = Q(S,R) + \frac{1}{n-2}Q(g,\overline{S}) + \frac{\kappa}{(n-1)(n-2)}Q(S,G). \tag{13}$$

Proof. The identities (9) and (10) are immediate consequences of the definitions of the tensors G, U and $\overline{S}$. (11) was shown in [2, Remark 2.1]. Using the definition of the Weyl tensor C we easily get (12). Finally, putting (10) in (12) we obtain (13).

Lemma 3.2. *Let* (M,g), $n \geq 4$, *be a semi-Riemannian manifold satisfying*

$$R(\widetilde{S}(X),Y,Z,W) = \tau R(X,Y,W,Z), \tag{14}$$

where τ *is a function on* M *and* $X,Y,Z,W \in \Xi(M)$. *Then*

$$C \cdot C = R \cdot R - \frac{1}{n-2}Q(S,R) + \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \tau\right)Q(g,C). \tag{15}$$

Proof. First of all, it is easy to verify that (14) implies

$$R \cdot S = 0, \tag{16}$$

$$S^2 = \tau S. \tag{17}$$

Now, using (16), we get

$$R \cdot C = R \cdot R. \tag{18}$$

Further, we can check that the following identities hold on any semi-Riemannian manifold:

$$(C\cdot C)_{hijklm} = (R\cdot C)_{hijklm} + \frac{1}{n-2}Q\left(\frac{\kappa}{n-1}g - S, C\right)_{hijklm} - \frac{1}{n-2}\,(g_{hl}S_m^{\ p}C_{pijk} - g_{hm}S_l^{\ p}C_{pijk} - g_{il}S_m^{\ p}C_{phjk} + g_{im}S_l^{\ p}C_{phjk} + g_{jl}S_m^{\ p}C_{pkhi} - g_{jm}S_l^{\ p}C_{pkhi} - g_{kl}S_m^{\ p}C_{pjhi} + g_{km}S_l^{\ p}C_{pjhi}), \tag{19}$$

$$S_m^{\ p}C_{pijk} = S_m^{\ p}R_{pijk} - \frac{1}{n-2}\,(S_{mk}S_{ij} - S_{mj}S_{ik}) - \frac{1}{n-2}(g_{ij}S^2_{mk} - g_{ik}S^2_{mj}) + \frac{\kappa}{(n-1)(n-2)}\,(g_{ij}S_{mk} - g_{ik}S_{mj}). \tag{20}$$

The relation (20), by (14) and (17), turns into

$$S_m^{\ p}C_{pijk} = \tau R_{mijk} - \frac{1}{n-2}\,(S_{mk}S_{ij} - S_{mj}S_{ik}) - \frac{\tau}{n-2}\,(g_{ij}S_{mk} - g_{ik}S_{mj}) + \frac{\kappa}{(n-1)(n-2)}\,(g_{ij}S_{mk} - g_{ik}S_{mj}). \tag{21}$$

Applying (18) and (21) in (19) we find

$$C\cdot C = R\cdot R + \frac{\kappa}{(n-1)(n-2)}\,Q(g,C) - \frac{1}{n-2}\,Q(S,C) - \frac{\tau}{n-2}\,Q(g,R) + \frac{1}{(n-2)^2}\,Q(g,\overline{S}) + \frac{1}{(n-2)^2}\left(\frac{\kappa}{n-1} - \tau\right)Q(S,G). \tag{22}$$

This, by making use of (13), gives

$$C\cdot C = R\cdot R - \frac{1}{n-2}\,Q(S,R) + \frac{\kappa}{(n-1)(n-2)}\,Q(g,C) - \frac{\tau}{n-2}\,Q(g,R) - \frac{\tau}{(n-2)^2}\,Q(S,G), \tag{23}$$

which, by (11), yields (15). Our lemma is thus proved.

Lemma 3.3. *If (M,g), $n \geq 4$, is a semi-Riemannian Ricci-pseudosymmetric manifold then at any point $x \in U_S \subset M$,*

$$(R\cdot S)_{hijk} = L_S(g_{hj}S_{ik} + g_{ij}S_{hk} - g_{hk}S_{ij} - g_{ik}S_{hj}), \tag{24}$$

(25) $$S_h{}^p R_{pijk} + S_j{}^p R_{pikh} + S_k{}^p R_{pihj} = 0,$$

(26) $$(R \cdot S^2)_{hijk} = L_S(g_{hj} S^2_{ik} + g_{ij} S^2_{hk} - g_{hk} S^2_{ij} - g_{ik} S^2_{hj}),$$

(27) $$S^2_{ph} R^p{}_{ijk} + S^2_{pj} R^p{}_{ikh} + S^2_{pk} R^p{}_{ihj} = 0,$$

(28) $$A_{ij} = S^{pq} R_{pijq} = S^2_{ij} - n L_S S_{ij} + \kappa L_S g_{ij}.$$

Moreover, if (3) *is satisfied at* x *then*

(29) $$S^2_{ij} = \lambda S_{ij} + ((n-1)L_S - \kappa) L_S g_{ij}, \qquad \lambda \in \mathbb{R},$$

holds at x.

Proof. (24) is an immediate consequence of (5). Summing cyclically (24) in h, j, k and using the identity

(30) $$(R \cdot S)_{hijk} = S_h{}^p R_{pijk} + S_i{}^p R_{phjk},$$

we obtain (25). We note that

$$(R \cdot S^2)_{hijk} = S_h{}^p (S_p{}^a R_{aijk} + S_i{}^a R_{apjk}) + S_i{}^p (S_p{}^a R_{ahjk} + S_h{}^a R_{apjk}),$$

whence

$$(R \cdot S^2)_{hijk} = S_h{}^p (R \cdot S)_{pijk} + S_i{}^p (R \cdot S)_{phjk}.$$

Substituting here (24) we easily get (26). (27) follows from (26). Contracting (24) with g^{hk} and using (30) we obtain (28). From (28) we get

$$(R \cdot S)_{pqhk} R^p{}_{ij}{}^q + S^{pq} (R \cdot R)_{pijqhk} = (R \cdot S^2)_{ijhk} - n L_S (R \cdot S)_{ijhk}.$$

Substituting here (3), (5), (24), (26) and

(31) $$\begin{aligned} Q(S,R)_{pijqhk} = {} & S_{ph} R_{kijq} + S_{ih} R_{pkjq} + S_{jh} R_{pikq} + S_{qh} R_{pijk} \\ & - S_{pk} R_{hijq} - S_{ik} R_{phjq} - S_{jk} R_{pihq} - S_{qk} R_{pijh}, \end{aligned}$$

we obtain

$$\begin{aligned} L_S (S_{kp} R^p{}_{jih} + S_{kp} R^p{}_{ijh} - S_{hp} R^p{}_{jik} - S_{hp} R^p{}_{ijk}) & \\ - S^2_{kp} R^p{}_{jih} - S^2_{kp} R^p{}_{ijh} + S^2_{hp} R^p{}_{jik} + S^2_{hp} R^p{}_{ijk} & \\ + S_{ih} A_{jk} + S_{jh} A_{ik} - S_{ik} A_{jh} - S_{jk} A_{ih} & \\ = L_S (g_{ih} S^2_{jk} + g_{jh} S^2_{ik} - g_{ik} S^2_{jh} - g_{jk} S^2_{ih}) & \\ - n L_S^2 (g_{ih} S_{jk} + g_{jh} S_{ik} - g_{ik} S_{jh} - g_{jk} S_{ih}). & \end{aligned}$$

Applying (25) and (27) we find

$$\begin{aligned} -L_S (R \cdot S)_{ijhk} + (R \cdot S^2)_{ijhk} + S_{ih} A_{jk} + S_{jh} A_{ik} - S_{ik} A_{jh} - S_{jk} A_{ih} & \\ = L_S (g_{ih} S^2_{jk} + g_{jh} S^2_{ik} - g_{ik} S^2_{jh} - g_{jk} S^2_{ih}) & \\ - n L_S^2 (g_{ih} S_{jk} + g_{jh} S_{ik} - g_{ik} S_{jh} - g_{jk} S_{ih}), & \end{aligned}$$

which, by making use of (24), (26) and (28), turns into

$$L_S((n-1)L_S-\kappa)(g_{ih}S_{jk}+g_{jh}S_{ik}-g_{ik}S_{jh}-g_{jk}S_{ih}) \\ = S_{ik}S^2_{jh}+S_{jk}S^2_{ih}-S_{ih}S^2_{jk}-S_{jh}S^2_{ik}.$$

The last equality can be rewritten as

$$S_{jh}(S^2_{ik}-((n-1)L_S-\kappa)L_Sg_{ik})+S_{ih}(S^2_{jk}-((n-1)L_S-\kappa)L_Sg_{jk}) \\ -S_{jk}(S^2_{ih}-((n-1)L_S-\kappa)L_Sg_{ih}) \\ -S_{ik}(S^2_{jh}-((n-1)L_S-\kappa)L_Sg_{jh})=0,$$

or briefly

$$Q(S,S^2-((n-1)L_S-\kappa)L_Sg)=0,$$

from which, in view of Lemma 2.4(i) of [18], we get (29), completing the proof.

4. Ricci-semisymmetric manifolds with $\kappa=0$

PROPOSITION 4.1. *If* (M,g), $n\geq 4$, *is a semi-Riemannian Ricci-pseudo-symmetric manifold satisfying* (3) *then*, *at any point* $x\in U_S\subset M$,

$$(nL_S-\kappa)S_m{}^pR_{pijk}=\kappa L_S(g_{mk}S_{ij}-g_{jm}S_{ik})+nL^2_S(g_{ij}S_{km}-g_{ki}S_{jm}) \\ +(\kappa L_S-\mathrm{tr}(S^2))R_{mijk}-\kappa L^2_SG_{mijk} \\ +nL_S(S_{km}S_{ij}-S_{jm}S_{ik}). \tag{32}$$

P r o o f. From (3) we have $(R\cdot R)_{hijklq}=Q(S,R)_{hijklq}$, i.e.

$$R_{pijk}R^p{}_{hlq}-R_{phjk}R^p{}_{ilq}+R_{pkhi}R^p{}_{jlq}-R_{pjhi}R^p{}_{klq} \\ =S_{hl}R_{qijk}+S_{il}R_{hqjk}+S_{jl}R_{hiqk}+S_{kl}R_{hijq} \\ -S_{hq}R_{lijk}-S_{iq}R_{hljk}-S_{jq}R_{hilk}-S_{kq}R_{hijl}.$$

Transvecting this with $S_m{}^q$ we obtain

$$R_{pijk}R^p{}_{hlq}S_m{}^q-R_{phjk}R^p{}_{ilq}S_m{}^q+R_{pkhi}R^p{}_{jlq}S_m{}^q-R_{pjhi}R^p{}_{klq}S_m{}^q \\ =S_{hl}S_m{}^qR_{qijk}+S_{il}S_m{}^qR_{hqjk}+S_{jl}S_m{}^qR_{hiqk}+S_{kl}S_m{}^qR_{hijq} \\ -S^2_{hm}R_{lijk}-S^2_{im}R_{hljk}-S^2_{jm}R_{hilk}-S^2_{km}R_{hijl}.$$

Symmetrizing in l,m and using (24) we get

$$L_S(g_{hl}S_m{}^pR_{pijk}+g_{hm}S_l{}^pR_{pijk}-g_{il}S_m{}^pR_{phjk}-g_{im}S_l{}^pR_{phjk} \\ +g_{jl}S_m{}^pR_{pkhi}+g_{jm}S_l{}^pR_{pkhi}-g_{kl}S_m{}^pR_{pjhi}-g_{km}S_l{}^pR_{pjhi} \\ -S_{hm}R_{lijk}-S_{hl}R_{mijk}+S_{im}R_{lhjk}+S_{il}R_{mhjk} \\ -S_{jm}R_{lkhi}-S_{jl}R_{mkhi}+S_{km}R_{ljhi}+S_{kl}R_{mjhi}) \\ =S_{jl}S_m{}^pR_{pkhi}+S_{jm}S_l{}^pR_{pkhi}-S_{kl}S_m{}^pR_{pjhi}-S_{km}S_l{}^pR_{pjhi} \\ -S^2_{hm}R_{lijk}-S^2_{hl}R_{mijk}-S^2_{im}R_{hljk}-S^2_{il}R_{hmjk} \\ -S^2_{jm}R_{hilk}-S^2_{jl}R_{himk}-S^2_{km}R_{hijl}-S^2_{kl}R_{hijm}.$$

Further, contracting with g^{lh} and applying (25), (24) and (27), we find

$$\begin{aligned}(33)\quad & L_S(nS_m{}^p R_{pijk} - g_{mj}A_{ik} + g_{mk}A_{ij} - \kappa R_{mijk} \\ &\quad + S_{jm}S_{ik} - S_{km}S_{ij} - g_{jm}S_{ik} - g_{ij}S_{mk} + g_{mk}S_{ij} + g_{ik}S_{mj}) \\ &= \kappa S_m{}^p R_{pijk} + L_S(S_{jm}S_{ik} + g_{ij}S^2_{mk} - S_{mk}S_{ij} - g_{ik}S^2_{mj}) \\ &\quad - S_{jm}A_{ik} + S_{km}A_{ij} + S_{ik}S^2_{jm} - S_{ij}S^2_{km} \\ &\quad - \mathrm{tr}(S^2)R_{mijk} - S^2_{ip}R^p{}_{mjk} - S^2_{mp}R^p{}_{ijk}.\end{aligned}$$

We note that (28), by (29), turns into

$$A_{ij} = (\lambda - nL_S)S_{ij} + (n-1)L_S^2 g_{ij}. \tag{34}$$

Substituting (29) and (34) into (33) we get (30), completing the proof.

PROPOSITION 4.2. *Let* (M,g), $n \geq 4$, *be a semi-Riemannian Ricci-pseudosymmetric manifold. If*

$$\beta R = nL_S^2 U - \kappa L_S^2 G + nL_S \overline{S}, \qquad \beta \in \mathbb{R}, \tag{35}$$

at a point $x \in U_S \subset M$, *then, at* x,

$$\beta(R\cdot R - L_S Q(g,R)) = 0. \tag{36}$$

Proof. First of all we note that (35) implies

$$\beta R\cdot R = nL_S^2 R\cdot U + nL_S R\cdot \overline{S}. \tag{37}$$

We now prove that, at x,

$$R\cdot U = L_S Q(g,U), \tag{38}$$

$$R\cdot \overline{S} = L_S Q(g,\overline{S}). \tag{39}$$

We have

$$\begin{aligned}(R\cdot U)_{hijklm} &= g_{ij}(R\cdot S)_{hklm} - g_{hj}(R\cdot S)_{iklm} \\ &\quad + g_{hk}(R\cdot S)_{ijlm} - g_{ik}(R\cdot S)_{jhlm} \\ &= L_S(g_{ij}(g_{hl}S_{km} + g_{kl}S_{hm} - g_{hm}S_{kl} - g_{km}S_{hl}) \\ &\quad - g_{hj}(g_{il}S_{km} + g_{kl}S_{im} - g_{im}S_{kl} - g_{km}S_{il}) \\ &\quad + g_{hk}(g_{il}S_{jm} + g_{jl}S_{im} - g_{im}S_{jl} - g_{jm}S_{il}) \\ &\quad - g_{ki}(g_{hl}S_{jm} + g_{jl}S_{hm} - g_{hm}S_{jl} - g_{jm}S_{hl})) \\ &= -L_S Q(S,G)_{hijklm},\end{aligned}$$

or, briefly,

$$R\cdot U = -L_S Q(S,G). \tag{40}$$

Applying (9) in (40) we get (38). Furthermore, we have

$$\begin{aligned}(R\cdot \overline{S})_{hijklm} &= S_{ij}(R\cdot S)_{hklm} - S_{hj}(R\cdot S)_{iklm}\\ &\quad + S_{hk}(R\cdot S)_{ijlm} - S_{ik}(R\cdot S)_{jhlm}\\ &= L_S(S_{ij}(g_{hl}S_{km} + g_{kl}S_{hm} - g_{hm}S_{kl} - g_{km}S_{hl})\\ &\quad - S_{hj}(g_{il}S_{km} + g_{kl}S_{im} - g_{im}S_{kl} - g_{km}S_{il})\\ &\quad + S_{hk}(g_{il}S_{jm} + g_{jl}S_{im} - g_{im}S_{jl} - g_{jm}S_{il})\\ &\quad - S_{ki}(g_{hl}S_{jm} + g_{jl}S_{hm} - g_{hm}S_{jl} - g_{jm}S_{hl}))\\ &= L_S Q(g,\overline{S})_{hijklm},\end{aligned}$$

i.e. (39) holds at x. Applying now (38) and (39) in (37) we get

$$\beta R\cdot R = nL_S^2(L_S Q(g,U) + Q(g,\overline{S})).$$

But on the other hand, from (35) we also obtain

$$\beta Q(g,R) = nL_S^2 Q(g,U) + nL_S Q(g,\overline{S}).$$

The last two relations complete the proof of our proposition.

THEOREM 4.1. *Let (M,g), $n\geq 4$, be a semi-Riemannian manifold and let $x\in U_S\subset M$.*

(i) *If (M,g) is a Ricci-pseudosymmetric manifold satisfying* (3) *then* (32) *is satisfied on U_S.*

(ii) *If the conditions*: (32) *and $nL_S-\kappa\neq 0$ are satisfied at x then $R\cdot S = L_S Q(g,S)$ at x.*

(iii) *If the conditions*: (32) *and $nL_S-\kappa = 0$ are satisfied at x then*, *at x*,

$$\left(\operatorname{tr}(S^2) - \frac{\kappa^2}{n}\right)(R\cdot R - L_S Q(g,R)) = 0.$$

P r o o f. The proof of (i) is presented in Lemma 3.2. Next, (32), by symmetrization in m,i, implies (5). Finally, (iii) is a consequence of Proposition 4.2.

As an immediate consequence of Theorem 4.1(iii) we have the following.

THEOREM 4.2. *Let (M,g), $n\geq 4$, be a Riemannian Ricci-semisymmetric manifold satisfying* (3). *If the scalar curvature κ of the manifold (M,g) vanishes on $U_S\subset M$ then $R\cdot R = 0$ on U_S.*

5. Ricci-semisymmetric manifolds with $\kappa\neq 0$. Let (M,g), $\dim M\geq 3$, be a semi-Riemannian manifold. We denote by U_κ the set of all points of M at which the scalar curvature κ of (M,g) is non-zero. We note that if (M,g), $\dim M\geq 4$, is a Ricci-semisymmetric manifold satisfying (3) then, in view of Lemma 3.3 and Proposition 4.1, (29) and (32) take on

$U_S \cap U_\kappa$ the forms

$$S^2 = \tau S, \tag{41}$$

$$R(\widetilde{S}(X), Y, Z, W) = \tau R(X, Y, Z, W), \tag{42}$$

respectively, where $X, Y, Z, W \in \Xi(M)$ and

$$\tau = \frac{\operatorname{tr}(S^2)}{\kappa}. \tag{43}$$

Now Lemma 3.2 yields immediately the following.

Proposition 5.1. *If* (M, g), $n \geq 4$, *is a semi-Riemannian Ricci-semi-symmetric manifold satisfying* (3) *then*

$$C \cdot C = \frac{n-3}{n-2} R \cdot R + \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \tau\right) Q(g, C) \tag{44}$$

on $U_S \cap U_\kappa$.

Theorem 5.1. *Let* (M, g), $n \geq 4$, *be a semi-Riemannian Ricci-semi-symmetric manifold satisfying* (3). *If* (M, g) *has pseudosymmetric Weyl tensor then* $R \cdot R = 0$ *on* $U_S \cap U_\kappa$.

Proof. Let $x \in M$. If $x \in M - U_C$ then our assertion follows from Remark 1.1. Let $x \in U_S \cap U_\kappa \cap U_C$. Thus (44), by (6), turns into

$$R \cdot R = \mu Q(g, C), \tag{45}$$

which, by (2), gives

$$R \cdot C = \mu Q(g, C), \tag{46}$$

where

$$\mu = L_C - \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \tau\right).$$

We consider two cases.

I. First we assume that dim $M \geq 5$. Now (46), in view of [14, Theorem 1], implies $R \cdot R = \mu Q(g, R)$, whence

$$R \cdot S = \mu Q(g, S). \tag{47}$$

But this, by (2), yields $\mu Q(g, S) = 0$. Since $x \in U_S$, the last relation gives $\mu = 0$, which reduces (47) to (1).

II. We now assume that dim $M = 4$. It is well known that the following identity is satisfied for every 4-dimensional semi-Riemannian manifold (M, g) ([25]):

$$\begin{aligned} 0 = {} & g_{hm} C_{lijk} + g_{lm} C_{ihjk} + g_{im} C_{hljk} + g_{hj} C_{likm} + g_{lj} C_{ihkm} \\ & + g_{ij} C_{hlkm} + g_{hk} C_{limj} + g_{lk} C_{ihmj} + g_{ik} C_{hlmj}. \end{aligned} \tag{48}$$

From this we get immediately

$$
\begin{aligned}
(49)\qquad 0 = {} & g_{hm}(R\cdot C)_{lijkab} + g_{lm}(R\cdot C)_{ihjkab} + g_{im}(R\cdot C)_{hljkab} \\
& + g_{hj}(R\cdot C)_{likmab} + g_{lj}(R\cdot C)_{ihkmab} + g_{ij}(R\cdot C)_{hlkmab} \\
& + g_{hk}(R\cdot C)_{limjab} + g_{lk}(R\cdot C)_{ihmjab} + g_{ik}(R\cdot C)_{hlmjab},
\end{aligned}
$$

which, in virtue of (46), turns into

$$
\begin{aligned}
(50)\qquad 0 = \mu(& g_{hm}Q(g,C)_{lijkab} + g_{lm}Q(g,C)_{ihjkab} + g_{im}Q(g,C)_{hljkab} \\
& + g_{hj}Q(g,C)_{likmab} + g_{lj}Q(g,C)_{ihkmab} + g_{ij}Q(g,C)_{hlkmab} \\
& + g_{hk}Q(g,C)_{limjab} + g_{lk}Q(g,C)_{ihmjab} + g_{ik}Q(g,C)_{hlmjab}).
\end{aligned}
$$

But on the other hand, (45), by (3), gives

$$
(51)\qquad Q(S,R) = \mu Q(g,C).
$$

Applying (51) in (50) we obtain

$$
\begin{aligned}
(52)\qquad 0 = {} & g_{hm}Q(S,R)_{lijkab} + g_{lm}Q(S,R)_{ihjkab} + g_{im}Q(S,R)_{hljkab} \\
& + g_{hj}Q(S,R)_{likmab} + g_{lj}Q(S,R)_{ihkmab} + g_{ij}Q(S,R)_{hlkmab} \\
& + g_{hk}Q(S,R)_{limjab} + g_{lk}Q(S,R)_{ihmjab} + g_{ik}Q(S,R)_{hlmjab}.
\end{aligned}
$$

Further, using the definition of the tensor $Q(S,R)$ and (41) and (42), we can easily check the following identities on $U_S \cap U_\kappa$:

$$
(53)\qquad S_c{}^p Q(S,R)_{pijkab} = \tau Q(S,R)_{cijkab},
$$

$$
(54)\qquad S_c{}^p Q(S,R)_{hijkap} = \tau Q(S,R)_{hijkac}.
$$

Transvecting now (52) with $S_p{}^h$ and using (53) and (54) we get

$$
\begin{aligned}
(55)\qquad 0 = {} & S_{pm}Q(S,R)_{lijkab} + S_{pj}Q(S,R)_{likmab} + S_{pk}Q(S,R)_{limjab} \\
& + \tau(g_{lm}Q(S,R)_{ihjkab} + g_{im}Q(S,R)_{hljkab} + g_{lj}Q(S,R)_{ihkmab} \\
& + g_{ij}Q(S,R)_{hlkmab} + g_{lk}Q(S,R)_{ihmjab} + g_{ik}Q(S,R)_{hlmjab}).
\end{aligned}
$$

Subtracting (52) we find

$$
\begin{aligned}
(S_{pm} - \tau g_{pm})Q(S,R)_{lijkab} + (S_{pj} - \tau g_{pj})Q(S,R)_{likmab} & \\
+ (S_{pk} - \tau g_{pk})Q(S,R)_{limjab} & = 0.
\end{aligned}
$$

From this, by transvection with $S_c{}^m$ and making use of (41) and (53), we obtain

$$
\tau((S_{pj} - \tau g_{pj})Q(S,R)_{likcab} + (S_{pk} - \tau g_{pk})Q(S,R)_{licjab}) = 0.
$$

Further, transvecting this with $S_d{}^j$ and using again (41) and (53), we find

$$
\tau^2(S_{pk} - \tau g_{pk})Q(S,R)_{licdab} = 0,
$$

or, briefly,

$$
\tau^2(S - \tau g)Q(S,R) = 0.
$$

Since $x \in U_S$, this reduces to

$$\tau Q(S,R) = 0. \tag{56}$$

If τ is non-zero at a point $x \in U_S \cap U_\kappa$ then (56) implies $Q(S,R) = 0$ and by (3) we get (1). If $\tau = 0$ at x then (41), (42), (53) and (54) reduce to

$$S^2_{ij} = 0, \tag{57}$$

$$S_m^{\ p} R_{pijk} = 0, \tag{58}$$

$$S_a^{\ p} Q(S,R)_{pijklm} = 0, \tag{59}$$

$$S_a^{\ p} Q(S,R)_{hijkpm} = 0, \tag{60}$$

respectively. Further, from (51) we have

$$Q(S,R)_{hijklm} = \mu Q(g,C)_{hijklm}. \tag{61}$$

Transvecting this with $S_c^{\ l}$ and using (60) we get

$$\mu S_a^{\ p} Q(g,C)_{hijkpm} = 0. \tag{62}$$

If μ vanishes at x then (61) reduces to $Q(S,R) = 0$. Thus (3) implies again (1). If μ is non-zero at x then (62) yields

$$S_a^{\ p} Q(g,C)_{hijkpm} = 0,$$

whence we obtain

$$\begin{aligned} 0 = {} & S_{ah}C_{mijk} - S_{ai}C_{mhjk} + S_{aj}C_{mkhi} - S_{ak}C_{mjhi} \\ & - g_{mh}S_a^{\ l}C_{lijk} + g_{mi}S_a^{\ l}C_{lhjk} - g_{mj}S_a^{\ l}C_{lkhi} + g_{mk}S_a^{\ l}C_{ljhi}. \end{aligned}$$

Contracting this with g^{ah} we find

$$\begin{aligned} 0 = {} & \kappa C_{mijk} + S_i^{\ p}C_{pmjk} + S_j^{\ p}C_{pimk} - S_k^{\ p}C_{pimj} \\ & - S_m^{\ l}C_{lijk} + g_{mj}S^{pq}C_{pkiq} - g_{mk}S^{pq}C_{pjiq}. \end{aligned} \tag{63}$$

Since $\tau = 0$, (21) reduces to

$$\begin{aligned} T_{mijk} = S_m^{\ p}C_{pijk} = {} & -\frac{1}{n-2}(S_{mk}S_{ij} - S_{mj}S_{ik}) \\ & + \frac{\kappa}{(n-1)(n-2)}(g_{ij}S_{mk} - g_{ik}S_{mj}). \end{aligned} \tag{64}$$

Contracting this with g^{mk} and using (57) we obtain

$$P_{ij} = S^{pq}C_{pijq} = -\frac{n\kappa}{(n-1)(n-2)}S_{ij} + \frac{\kappa^2}{(n-1)(n-2)}g_{ij}. \tag{65}$$

Let T and P be the tensors with local components T_{mijk} and P_{ij} defined by (64) and (65), respectively. From (64) and (65), in virtue of (2), we obtain $R \cdot T = 0$ and $R \cdot P = 0$, respectively. Applying the last two relations in (63) we find $\kappa R \cdot C = 0$, and by the assumption that $\kappa \neq 0$, we have $R \cdot C = 0$, which reduces (46) to $\mu Q(g,C) = 0$. Evidently, if $\mu \neq 0$ then $Q(g,C) = 0$, which, in view of Lemma 1.1 of [6], implies $C = 0$, a contradiction. Thus we have $\mu = 0$. Now (45) completes the proof.

Finally, using Theorems 4.2 and 5.1 we get our main result.

THEOREM 5.2. *Let* (M, g), $n \geq 4$, *be a Riemannian Ricci-semisymmetric manifold satisfying* (3). *If* (M, g) *is a manifold with pseudosymmetric Weyl tensor then* $R \cdot R = 0$ *on* $U_S \subset M$.

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