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It is proved that a real-valued function $f(x) = \exp(\pi i \chi_I(x))$, where I is an interval contained in $[0,1)$, is not of the form $f(x) = \overline{q(2x)}q(x)$ with $|q(x)| = 1$ a.e. if I has dyadic endpoints. A relation of this result to the uniform distribution mod 2 is also shown.

1. Introduction Let (X,μ) be a probability measure space. A measurable transformation $T : X \to X$ is said to be *measure preserving* if $\mu(T^{-1}E) = \mu(E)$ for every measurable subset E. A measure preserving transformation T on X is called *ergodic* if $f(Tx) = f(x)$ holds only for constant functions f on X. Throughout the paper all set equalities, set inclusions and function equalities are understood modulo measure zero sets, and all subsets are measurable unless otherwise stated. For example, we say that I is an *interval* if the Lebesgue measure of $I \triangle [a,b]$ equals zero for some a,b, where $\triangle$ denotes symmetric difference.

Let χ_E be the characteristic function of a set E and consider the behavior of the sequence $\sum_{k=0}^{n-1} \chi_E(T^k x)$ which counts the number of times the points $T^k x$ visit E. The Birkhoff Ergodic Theorem applied to the ergodic transformation $T : x \mapsto \{2x\}$ on $[0,1)$, where $\{t\}$ is the fractional part of t, gives the classical Borel Theorem on normal numbers:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{[1/2,1)}(T^k x) = \frac{1}{2}.$$

This implies that a.e. x is normal, i.e., the relative frequency of the digit 1 in the binary expansion of x is $1/2$ (see [7]).

In this paper we are interested in the uniform distribution of the sequence $y_n \in \{0,1\}$ defined by

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$$y_n(x) \equiv \sum_{k=0}^{n-1} \chi_E(T^k x) \pmod 2,$$

where $T : x \mapsto \{2x\}$. When $E = [1/2, 1)$ it is shown that $\{y_n(x)\}$ is evenly distributed in L^2-sense [1]. If $\{y_n(x)\}$ is evenly distributed for a fixed set E, that is, the limit of $N^{-1}\sum_{n=1}^N y_n$ exists and equals 1/2, then we call x a *normal number mod* 2 with respect to E. Contrary to our intuition, the limit might not exist and even when it exists it may not be equal to 1/2. This type of problem was first studied by Veech [6]. He considered the case when the transformations are given by irrational rotations on the unit circle, and obtained results which showed that the length of the interval E and the rotation angle θ are closely related. For example, he proved that when the irrational number θ has bounded partial quotients in its continued fraction expansion, then the sequence $\{y_n\}$ is evenly distributed if the length of the interval is not an integral multiple of θ modulo 1. For a related result, see [2].

We investigate the problem from the viewpoint of spectral theory. Let (X, μ) be a probability space and T an ergodic transformation on X which is not necessarily invertible. Consider the behavior of the sequence $2y_n(x)-1 = \exp(\pi i y_n)$, and check whether the limit is zero in a suitable sense. Define an isometry U on $L^2(X)$ by

$$(Uf)(x) = \exp(\pi i \chi_E(x)) f(Tx).$$

Then for $n \geq 1$ and the constant function 1,

$$(U^n 1)(x) = \exp\left(\pi i \sum_{k=0}^{n-1} \chi_E(T^k x)\right) = \exp(\pi i y_n(x)),$$

and the problem is to study the existence of

$$(*) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (U^n 1)(x).$$

Thus we ask whether the limit of $(*)$ equals 0. By the von Neumann Mean Ergodic Theorem, the L^2-limit of $N^{-1}\sum_{n=1}^N U^n f$ exists and equals Pf, where P is the orthogonal projection onto the U-invariant subspace.

We briefly summarize the related results of [1]. Recall that a function $f(x)$ is called a *coboundary* if $f(x) = \overline{q(x)} q(Tx)$ with $|q(x)| = 1$ a.e. on X. Let $\mathcal{M} = \{h \in L^2(X) : Uh = h\}$. Then the dimension of $\mathcal{M}$ is 0 or 1. If $\dim \mathcal{M} = 0$, then $N^{-1}\sum_{n=1}^N U^n 1 \to 0$ in L^2. If $\dim \mathcal{M} = 1$, then (i) $\exp(\pi i \chi_E)$ is a coboundary, (ii) there exists q such that $q(x) = \exp(\pi i \chi_F(x))$ for some F, $\exp(\pi i \chi_E(x)) = q(x) q(Tx)$, $E = F \vartriangle T^{-1}F = F^{\mathrm{c}} \vartriangle T^{-1}F^{\mathrm{c}}$, and (iii) $N^{-1}\sum_{n=1}^N U^n 1 \to Cq$ in L^2, where $C = \int_X q(x)\, d\mu$. In fact, the convergence is better than L^2 since the Birkhoff Ergodic Theorem implies that at a.e. $x \in X$,

$$\frac{1}{N}\sum_{n=1}^{N} U^n 1 = \frac{1}{N}\sum_{n=1}^{N} q(x)q(T^n x) = q(x)\frac{1}{N}\sum_{n=1}^{N} q(T^n x) \to q(x)\int_X q(y)\, d\mu(y).$$

Hence the convergence is pointwise, which was not indicated in [1].

Suppose $\lambda\overline{q(2x)}q(x) = \pm 1$ for some $|q| = 1$. Then $1 = \lambda^2\overline{q^2(2x)}q^2(x)$ and $\lambda^2 q^2(x) = q^2(2x)$. Since 1 is the only eigenvalue of $x \mapsto \{2x\}$, we see that $\lambda^2 = 1$ and q^2 is constant. Thus $\lambda = \pm 1$.

Let F be a Lebesgue measurable subset of $\mathbb{R}$ and m be the Lebesgue measure on $\mathbb{R}$. For a point $x \in \mathbb{R}$ the *metric density* of F at x is defined to be

$$d_F(x) \equiv \lim_{r\to 0+} \frac{m(F\cap(x-r, x+r))}{2r}$$

provided that this limit exists. The metric density of F equals 1 and 0 at a.e. point of F and F^{c}, respectively. If $(x-r, x+r)$ and $2r$ are replaced by $[x, x+r)$ and r respectively in the above limit, then we call the corresponding limit $d_F^+(x)$ the *right metric density* of F at x. Recall that for $f \in L^1(\mathbb{R})$, a point $x \in \mathbb{R}$ is called a *Lebesgue point* of f if

$$\lim_{r\to 0+} \frac{1}{2r} \int_{(x-r,x+r)} |f(y) - f(x)|\, dm(y) = 0.$$

We know that for $f \in L^1(\mathbb{R})$ almost every $x \in \mathbb{R}$ is a Lebesgue point of f. If x is a Lebesgue point of χ_F, then $d_F(x) = d_F^+(x)$. Similarly the left metric density $d_F^-(x)$ is defined.

The metric density of F at a specific point may not be well defined. Then the point is not a Lebesgue point of χ_F [3]: Given κ and η, $0 \le \kappa \le \eta \le 1$, there exists $F \subset \mathbb{R}$ so that the upper and lower limits of $m(F\cap(-\delta,\delta))/(2\delta)$ are η and κ, respectively, as $\delta \to 0$. Recall that for a point x a sequence $A_1, A_2, \ldots$ of measurable sets is said to *shrink to x nicely* if there is a constant $c > 0$ for which there is a sequence of positive numbers $r_1, r_2, \ldots$ with $\lim r_n = 0$ such that $A_n \subset (x - r_n, x + r_n)$ and $m(A_n) \ge cr_n$. If a sequence $\{A_n\}_n$ shrinks to x nicely and x is a Lebesgue point of χ_F, then

$$d_F(x) = \lim_{n\to\infty} \frac{m(F\cap A_n)}{m(A_n)}$$

(see p. 140 of [5]).

Throughout the paper a rational number of the form $\sum_{i=1}^{k} a_i\, 2^{-i}$, $a_i \in \{0, 1\}$ for $1 \le i \le k$ with $a_k = 1$, is called a *dyadic number* and denoted by $[a_1, \ldots, a_k]$. By convention, 0 and 1 are also regarded as dyadic numbers.

Note that for the set $E = [1/6, 5/6]$, $\exp(\pi i\chi_E)$ is a coboundary since $E = F \vartriangle T^{-1}F$ for $F = [1/3, 2/3]$. The numbers 1/6, 5/6 are not dyadic and the sequence $N^{-1}\sum_{n=1}^{N} y_n(x)$ converges to $f(x)$, where $f(x) = 1/3$ if $x \in F$ and $f(x) = 2/3$ if $x \notin F$ almost everywhere. In this paper, we will show that $\exp(\pi i\chi_{[a,b]})$ with a, b dyadic is a coboundary if and only if $a = 1/4$ and $b = 3/4$. The interval $E = [1/4, 3/4]$ satisfies the condition since $E = F \vartriangle T^{-1}F$ for $F = [0, 1/2]$. But $\int \exp(\pi i\chi_F)\, d\mu = 0$, so the sequence converges to 0, hence we see that Borel's theorem mod 2 holds for every interval with dyadic endpoints.

2. Lemmas on metric density. Note that $T^{-1}F \cap [0, r] = \frac{1}{2}F \cap [0, r]$ for $0 < r \le 1/2$, and $T^{-1}F \cap [r, 1] = \left(\frac{1}{2}F + \frac{1}{2}\right) \cap [r, 1)$ for $1/2 \le r < 1$.

For a fixed set F and real $0 \le t < 1$ define a continuous function $h_{F,t}(r)$ on $(0, 1-t)$ by

$$h_{F,t}(r) \equiv h_t(r) = \frac{m(F \cap [t, t+r])}{r}.$$

Similarly for real $0 < t \le 1$ define a function $g_{F,t}(r)$ on $(0, t)$ by

$$g_{F,t}(r) \equiv g_t(r) = \frac{m(F \cap [t-r, t])}{r}.$$

Note that $d_F^+(t) = \lim_{r\to 0+} h_{F,t}(r)$ and $d_F^-(t) = \lim_{r\to 0+} g_{F,t}(r)$.

LEMMA 1. *If two dyadic numbers* $0 < a < b < 1$ *satisfy* $[a, b] = F \vartriangle T^{-1}F$ *for some set* F, *then*:

(i) $h_0(r/2^n) = h_0(r)$ *for all* $n \in \mathbb{N}$ *and all* $0 < r \le \min\{2a, 1\}$.

(ii) *If* $d_F^+(0)$ *exists*, *then* $d_F^+(0) = h_0(r) = 0$ *or* 1.

(iii) *If* $d_F^+(0) = 1$, *then* F *contains an interval of the form* $[0, r]$, $r > 0$, *and if* $d_F^+(0) = 0$, *then* F^c *contains such an interval.*

Proof. (i) Take r with $0 < r \le \min\{2a, 1\}$. Since $(F \vartriangle T^{-1}F) \cap [0, r/2] = \emptyset$, we have $F \cap [0, r/2] = T^{-1}F \cap [0, r/2]$. Thus $m(F \cap [0, r/2]) = m(T^{-1}F \cap [0, r/2]) = m\left(\frac{1}{2}F \cap [0, r/2]\right) = \frac{1}{2}m(F \cap [0, r])$ and $h_0(r/2) = h_0(r)$. Hence $h_0(r/2^n) = h_0(r/2^{n-1}) = \ldots = h_0(r)$.

(ii) Put $c = \min\{2a, 1\}$. Since $h_0(r/2^n) = h_0(r)$ for all $n \in \mathbb{N}$ and $0 \le r < c$ by (i), we have

$$d_F^+(0) = \lim_{s\to 0+} \frac{m(F \cap [0, s])}{s} = \lim_{n\to\infty} h_0\left(\frac{r}{2^n}\right) = h_0(r).$$

Assume that $d_F^+(0) = \alpha$, $0 < \alpha < 1$. Since for every $0 \le r < c$, there exists a sufficiently small $\delta(r) > 0$ such that $0 \le r + \varepsilon < c$ for all $0 < \varepsilon < \delta(r)$, i.e.,

$$\frac{m(F \cap [0, r+\varepsilon])}{r+\varepsilon} = \alpha,$$

we have $m(F\cap[r,r+\varepsilon]) = m(F\cap[0,r+\varepsilon])-m(F\cap[0,r]) = \alpha(r+\varepsilon)-\alpha r = \alpha\varepsilon$. Hence $m(F \cap [r, r + \varepsilon])/\varepsilon = \alpha$ so F has right metric density α at r, for all $0 \le r < c$. Since $0 < \alpha < 1$, this contradicts the fact that almost everywhere the metric density is 0 or 1.

(iii) Assume that $d_F^+(0) = 1$ and F does not contain any interval. Then for every $0 < r \le \min\{2a, 1\}$,

$$h_0(r) = \frac{m(F \cap [0,r])}{r} < 1.$$

But $h_0(r) = d_F^+(0) = 1$. This is a contradiction. Thus F contains an interval of the form $[0,r]$, $r > 0$. The other case is similarly proved. ■

Remark. If a, b and F satisfy the conditions of Lemma 1, then similar results also hold for $d_F^-(1)$ and $g_1(r)$:

(i) $g_1(r/2^n) = g_1(r)$ for all $n \in \mathbb{N}$ and all $0 < r \le 1 - b/2$.

(ii) If $d_F^-(1)$ exists, then $d_F^-(1) = g_1(r) = 0$ or 1.

(iii) If $d_F^-(1) = 1$, then F contains an interval of the form $[s, 1]$, $s < 1$, and if $d_F^-(1) = 1$, then F^{c} contains such an interval.

Hence we investigate the existence of $d_F^+(0)$ in Lemmas 2 and 3. The existence of $d_F^-(1)$ is similarly proved.

Lemma 2. *Let* $a = [a_1, \ldots, a_p]$, $b = [b_1, \ldots, b_q]$ *and* F *satisfy the conditions of Lemma* 1. *Put* $r_0 = 1/2^k$, *where* $k = \max\{p, q\}$. *Then for* $t = [c_1, \ldots, c_l]$, $h_t(r/2^n) = h_t(r)$ *and either* $h_t = h_0$ *or* $h_t = 1 - h_0$ *for* $n \in \mathbb{N}$ *and* $0 < r \le r_0/2^l$. *Hence the right metric density of* F *exists at* 0 *if and only if it exists at every dyadic point* t; *in that case either* $d_F^+(t) = d_F^+(0)$ *or* $d_F^+(t) = 1 - d_F^+(0)$.

Proof. *Step 1.* We consider the case of $l = 1$. Put $E = [a, b]$. Then $h_0(r) = h_0(r/2^n)$ for $n \in \mathbb{N}$ and $0 < r \le r_0$ by Lemma 1, and either $E \cap [1/2, 1/2 + r_0/2] = \emptyset$ or $E \cap [1/2, 1/2 + r_0/2] = [1/2, 1/2 + r_0/2]$.

Case 1. If $E \cap [1/2, 1/2 + r_0/2] = \emptyset$, then $m(E \cap [1/2, 1/2 + r]) = 0$ for $0 < r \le r_0/2$. Since $E = F \vartriangle T^{-1}F$, it follows that $m(F \cap [1/2, 1/2 + r]) = m(T^{-1}F \cap [1/2, 1/2 + r]) = m(T^{-1}F \cap [0, r]) = \frac{1}{2}m(F \cap [0, 2r])$. Thus

$$h_{1/2}(r) = h_0(2r) = h_0(r).$$

Furthermore,

$$h_{1/2}(r/2^n) = h_0(r/2^n) = h_0(r) = h_{1/2}(r)$$

for all n and $0 < r \le r_0/2$.

Case 2. If $E \cap [1/2, 1/2 + r_0/2] = [1/2, 1/2 + r_0/2]$, then $m(E \cap [1/2, 1/2+r]) = r$ for $0 < r \le r_0/2$. So $m(F\cap[1/2, 1/2+r]) = r - m(T^{-1}F\cap[1/2,

$1/2 + r]) = r - m(T^{-1}F \cap [0, r]) = r - \frac{1}{2}m(F \cap [0, 2r])$. Thus

$$h_{1/2}(r) = 1 - h_0(2r) = 1 - h_0(r)$$

and

$$h_{1/2}(r/2^n) = 1 - h_0(r/2^n) = 1 - h_0(r) = h_{1/2}(r)$$

for $n \in \mathbb{N}$ and $0 < r \le r_0/2$.

Hence

$$h_{1/2}(r/2^n) = h_{1/2}(r) = h_0(r) \quad \text{or} \quad 1 - h_0(r)$$

for $n \in \mathbb{N}$ and $0 < r \le r_0/2$.

Step 2. By induction assume that if $s = [s_1, \ldots, s_{l-1}]$ then $h_s(r/2^n) = h_s(r)$ and $h_s = h_0$ or $1 - h_0$ for all $0 < r \le r_0/2^{l-1}$.

Let $t = [c_1, \ldots, c_l]$ and $s = [c_2, \ldots, c_l]$. Then either $t = [0, c_2, \ldots, c_l]$ or $t = [1, c_2, \ldots, c_l]$. If $t = [0, c_2, \ldots, c_l]$ then $t = \frac{1}{2}s$, and if $t = [1, c_2, \ldots, c_l]$ then $t = \frac{1}{2}s + \frac{1}{2}$. Note that either $E \cap [t, t + r_0/2^l] = \emptyset$ or $E \cap [t, t + r_0/2^l] = [t, t + r_0/2^l]$.

Case 1. If $E \cap [t, t + r_0/2^l] = \emptyset$, then $m(E \cap [t, t + r]) = 0$ for $0 < r \le r_0/2^l$. Since $E = F \bigtriangleup T^{-1}F$, it follows that $m(F \cap [t, t + r]) = m(T^{-1}F \cap [t, t + r]) = \frac{1}{2}m(F \cap [s, s + 2r])$. Thus $h_t(r) = h_s(2r) = h_s(r) = h_0(r)$ or $1 - h_0(r)$ and $h_t(r/2^n) = h_s(r/2^n) = h_s(r) = h_t(r)$ for $n \in \mathbb{N}$ and $0 < r \le r_0/2^l$.

Case 2. If $E \cap [t, t + r_0/2^l] = [t, t + r_0/2^l]$, then $m(E \cap [t, t + r]) = r$ for $0 < r \le r_0/2^l$. Since $m(F \cap [t, t + r]) = r - m(T^{-1}F \cap [t, t + r]) = r - \frac{1}{2}m(F \cap [s, s + 2r])$ we have $h_t(r) = 1 - h_s(2r) = 1 - h_s(r) = h_0(r)$ or $1 - h_0(r)$ and $h_t(r/2^n) = 1 - h_s(r/2^n) = 1 - h_s(r) = h_t(r)$ for $n \in \mathbb{N}$ and $0 < r \le r_0/2^l$.

Hence for $t = [c_1, \ldots, c_l]$ we have

$$h_t(r/2^n) = h_t(r) = h_0(r) \quad \text{or} \quad 1 - h_0(r)$$

for $n \in \mathbb{N}$ and $0 < r \le r_0/2^l$. From this the second assertion follows. ■

Lemma 3. *If a, b and F satisfy the conditions of Lemma* 1, *then the right metric density of F exists at every dyadic point.*

Proof. By Lemma 2 it is sufficient to show that the right metric density of F exists at 0. Assume that $\lim_{r \to 0+} h_0(r)$ does not exist. Let $E = [a, b]$ with $a = [a_1, \ldots, a_p]$, $b = [b_1, \ldots, b_q]$ and r_0 be as in Lemma 2. From Lemma 2 we see that for $t = [c_1, \ldots, c_l]$, $h_t(r/2^n) = h_t(r) = h_0(r)$ or $1 - h_0(r)$ for $n \in \mathbb{N}$ and $0 < r \le r_0/2^l$.

Take a Lebesgue point ξ of χ_F with $d_F(\xi) = 1$, and put $r_n = r_0/2^n$. For every n choose $\xi_n \in \{[c_1, \ldots, c_m] : m \le n\}$ so that the sequence ξ_n converges to ξ and $[\xi_n, \xi_n + r_n] \subset (\xi - 1/2^{n-1}, \xi + 1/2^{n-1})$. Since $m([\xi_n, \xi_n + r_n])/2^{n-2} = r_0/4$, the subsets $[\xi_n, \xi_n + r_n]$ shrink to ξ nicely. For fixed r_0, there exists

$\varepsilon > 0$ such that $\varepsilon < h_0(r_0) < 1-\varepsilon$. If not, the metric density at 0 must exist. Hence

$$\frac{m(F\cap[\xi_n,\xi_n+r_n])}{r_n} = h_{\xi_n}(r_n) = h_0(r_n) \quad \text{or} \quad 1-h_0(r_n) < 1-\varepsilon$$

for all n. Since the metric density of F at ξ is 1, this is a contradiction. ■

LEMMA 4. *If* $[0,b] = F \triangle T^{-1}F$ *with* $b = [b_1,\ldots,b_q]$ *a dyadic number, then*

(i) $h_0(r/2^{2n}) = h_0(r)$ *and* $h_0(r/2^{2n-1}) = 1-h_0(r)$ *for all* $n\in\mathbb{N}$ *and* $0<r\le\min\{2b,1\}$.

(ii) *Put* $r_0 = 1/2^q$. *Then for* $t=[c_1,\ldots,c_l]$, $h_t(r/2^{2n}) = h_t(r)$, $h_t(r/2^{2n-1}) = 1-h_t(r)$ *and* $h_t = h_0$ *or* $1-h_0$ *for all* $n\in\mathbb{N}$ *and* $0<r\le r_0/2^l$.

(iii) $[0,b]\ne F\triangle T^{-1}F$ *for every measurable set* F.

Proof. (i) Take r with $0<r\le\min\{2b,1\}$. Since $(F\triangle T^{-1}F)\cap[0,r/2]=[0,r/2]$, we have $m(F\cap[0,r/2]) = r/2-m(T^{-1}F\cap[0,r/2]) = r/2-m\big(\frac{1}{2}F\cap[0,r/2]\big) = r/2-\frac{1}{2}m(F\cap[0,r])$ and $h_0(r/2)=1-h_0(r)$. Hence $h_0(r/2^{2n}) = 1-h_0(r/2^{2n-1}) = \ldots = h_0(r)$ and $h_0(r/2^{2n-1}) = 1-h_0(r/2^{2n-2}) = \ldots = h_0(r)$.

(ii) Put $E=[0,b]$. Then $h_0(r/2^{2n}) = h_0(r)$ and $h_0(r/2^{2n-1}) = 1-h_0(r)$ for $n\in\mathbb{N}$ and $0<r\le r_0$ by (i) and $E\cap[1/2,1/2+r_0/2]=\emptyset$ or $E\cap[1/2,1/2+r_0/2]=[1/2,1/2+r_0/2]$. Now proceed as in Lemma 2.

(iii) Take ξ, r_0, r_n and ξ_n as in the proof of Lemma 3. Then

$$d_F(\xi) = \lim_{n\to\infty} h_{\xi_n}(r_n) = \lim_{n\to\infty} h_0(r_n) \quad\text{or}\quad 1-h_0(r_n) = 1-h_0(r_0) \quad\text{or}\quad h_0(r_0)$$

by (ii). If $d_F(\xi) = 1-h_0(r_0)$, then $h_0(r_0)=0$ and $[0,b]\ne F\triangle T^{-1}F$ for this F. The other cases are similarly proved. ■

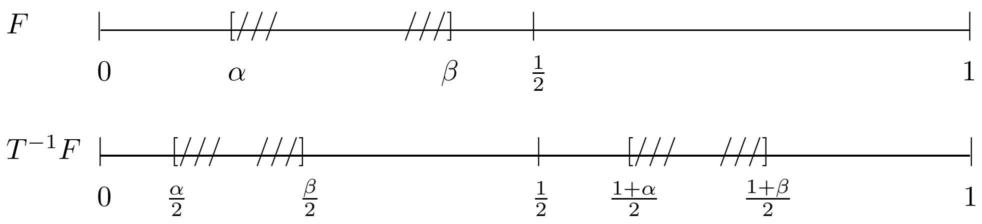
REMARK. For the case $[a,1] = F\triangle T^{-1}F$ with $a=[a_1,\ldots,a_p]$ a dyadic number, we consider the left metric density for F and $g_t(r)$. Then we have the same conclusion as in Lemma 4. For example, put $r_0=1/2^p$. Then for $t=[c_1,\ldots,c_l]$, $g_t(r/2^{2n}) = g_t(r)$, $g_t(r/2^{2n-1}) = 1-g_t(r)$ and either $g_t=g_1$ or $1-g_1$ for all $n\in\mathbb{N}$ and $0<r\le r_0/2^l$. Hence we may assume that if $[a,b]=F\triangle T^{-1}F$ for some F with a,b dyadic, then either F or F^{c} contains an interval of the form $[0,r]$ or $[r,1]$ for some $0<r<1$.

3. Main result. We say that $[\alpha,\beta]$ is the *optimal bounding interval* for F if $F\subset[\alpha,\beta]$ modulo measure zero sets and α is the infimum of points at which F has a positive metric density, while β is the supremum of points at which F has a positive right metric density. From now on, if K is connected and $m(S)=0$, then we regard $K\backslash S$ as being connected, and if E is an interval, then we regard $E\backslash S$ as an interval.

THEOREM 1. *Let T be the transformation defined by $x \mapsto 2x \pmod 1$ on $[0,1)$. Let a and b be dyadic numbers. Then $\exp(\pi i \chi_{[a,b]})$ is a coboundary if and only if $a = 1/4$ and $b = 3/4$.*

Proof. Recall that there exists a measurable set F such that neither F nor its complement contain any interval of positive length [5]. But if E is an interval with dyadic endpoints, then F or F^c contains an interval of the form $[0,r]$ or $[s,1]$, $r > 0$, $s < 1$, and $E = F^c \triangle T^{-1}F^c$. Hence we may assume that F^c contains an interval of the form $[0,r]$ or $[s,1]$, $r > 0$, $s < 1$. For this F, the optimal bounding interval $[\alpha,\beta]$ has either $\alpha > 0$ or $\beta < 1$.

CASE 1. Assume that $F \subset [\alpha,\beta]$, $0 \le \alpha < \beta < 1/2$ and $[\alpha,\beta]$ is the optimal bounding interval for F.



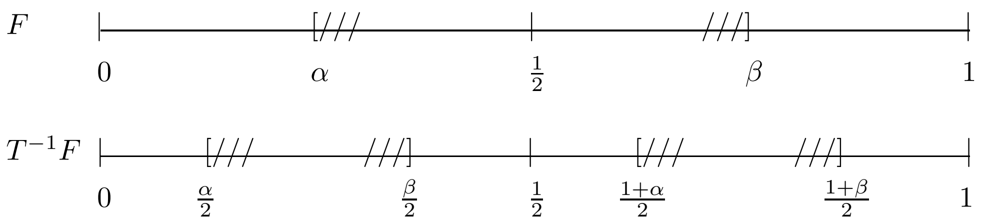
Then $m(F \cap [\beta/2,\beta]) > 0$, $m(T^{-1}F \cap [\beta/2,\beta]) = 0$ and

$$m\left(T^{-1}F \cap \left[\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right]\right) > 0, \qquad m\left(F \cap \left[\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right]\right) = 0.$$

But in $E = F \triangle T^{-1}F$, $m(E \cap [\beta/2,\beta]) > 0$, $m(E \cap [\beta,1/2]) = 0$, and $m(E \cap [1/2,(1+\beta)/2]) > 0$. So this reduces to the assumption that E is an interval. If $F \triangle T^{-1}F$ is an interval, then $\alpha = 0$, $\beta = 1/2$ and $F = [0,1/2]$. In this case $E = [1/4,3/4]$.

CASE 2. If $F \subset [\alpha,\beta]$ where $1/2 < \alpha < \beta \le 1$, and $[\alpha,\beta]$ is the optimal bounding interval for F, then as in Case 1, if $F \triangle T^{-1}F$ is an interval, then $\alpha = 1/2$, $\beta = 1$ and $F = [1/2,1]$. In this case $E = [1/4,3/4]$.

CASE 3. If $F \subset [\alpha,\beta]$ where $0 < \alpha < 1/2 < \beta < 1$, and $[\alpha,\beta]$ is the optimal bounding interval for F, then there are three possibilities.

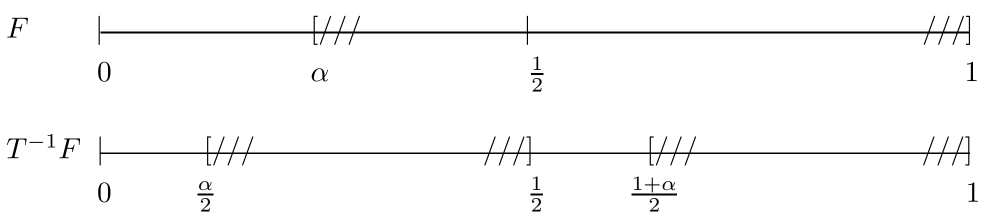


If $\beta/2 < \alpha$, then $F \cap \frac{1}{2}F = \emptyset$, $m(T^{-1}F \cap [\alpha/2, \beta/2]) > 0$, $m(T^{-1}F \cap [\beta/2, \alpha]) = 0$, $m(F \cap [\beta/2, \alpha]) = 0$, and $m(F \cap [\alpha, 1/2]) > 0$. This contradicts the fact that $E = F \vartriangle T^{-1}F$ is an interval.

If $\beta/2 = \alpha$ then $m\big(F \cap \frac{1}{2}F\big) = 0$, $m(T^{-1}F \cap [\alpha/2, \beta/2]) > 0$, $m(F \cap [\alpha, 1/2]) > 0$, $m(F \cap [\beta, 1 + \beta/2]) = 0$ and $m(T^{-1}F \cap [\beta, (1+\beta)/2]) = 0$. Thus for E to be an interval, F must contain the interval $[\alpha, 2\alpha]$. This is due to the fact that the measure of $\frac{1}{2}F$ is half that of F. Since $2\alpha = \beta$ and $[\alpha, \beta]$ is the optimal bounding interval for F by assumption, $F = [\alpha, \beta]$. Furthermore, $(1+\alpha)/2 = \beta$. If not, we have a contradiction to the fact that E is an interval. Thus $F = [\alpha, \beta] = [1/3, 2/3]$. In this case $F \vartriangle T^{-1}F = E = [1/6, 5/6]$. But this is not an interval with dyadic endpoints.

If $\beta/2 > \alpha$, then $m(F \cap [\alpha/2, \alpha]) = 0$, $m(F \cap [\beta, (1+\beta)/2]) = 0$ and $m\big(\frac{1}{2}F \cap [\alpha/2, \alpha]\big) > 0$, $m\big(\big(\frac{1}{2}F + \frac{1}{2}\big) \cap [\beta, (1+\beta)/2]\big) > 0$. Hence for E to be an interval, F must contain the intervals $[\alpha, 2\alpha]$ and $[2\beta - 1, \beta]$. Since $m((F \vartriangle T^{-1}F) \cap [\alpha/2, \alpha]) > 0$ and $m((F \vartriangle T^{-1}F) \cap [\beta, (1+\beta)/2]) > 0$ since F contains the interval $[\alpha, 2\alpha]$, and since $[\alpha, \beta]$ is the optimal bounding interval for F, for $F \vartriangle T^{-1}F$ to be connected, we must have $m(F \cap [2\alpha, 4\alpha]) = 0$. By similar reasons, F must contain the interval $[4\alpha, 8\alpha]$. By induction we see that F contains the interval $[2^{2(n-1)}\alpha, 2^{2n}\alpha]$ for n such that $2^{2n}\alpha < 1$, and does not contain the interval $[2^{2n}\alpha, 2^{2(n+1)}\alpha]$ for n such that $2^{2(n+1)}\alpha < 1$. Furthermore, $m(T^{-1}F \cap [\beta/2, (1+\alpha)/2]) = 0$. For $F \vartriangle T^{-1}F$ to be connected, F must contain the interval $[\beta/2, (1+\alpha)/2]$ and $T^{-1}F \cap [(1+\alpha)/2, 1/2 + \alpha] = [(1+\alpha)/2, 1/2 + \alpha]$. Thus we obtain the following equalities: $2^n\alpha = \beta - 1/2$, $2^{n+1}\alpha = \beta/2$, $2^{n+2}\alpha = (1+\alpha)/2$, and $2^{n+3}\alpha = \alpha + 1/2$ for some n. Hence $2^{n+3}\alpha = 1 + \alpha = \alpha + 1/2$, which is a contradiction. So if F is bounded by the pair (α, β) then $E = F \vartriangle T^{-1}F$ cannot be connected.

Case 4. If $F \subset [\alpha, 1]$ where $0 < \alpha < 1/2$, and $[\alpha, 1]$ is the optimal bounding interval for F, then we know that F is a disjoint union of $[\alpha_i, \beta_i]$, i.e., $F = \bigcup_{i=1}^{n}[\alpha_i, \beta_i]$ with $\alpha_1 = \alpha$, and $\beta_n = 1$ as in Case 3. Let $\alpha_n = \beta$. If $\beta \le 1/2$ then $F^{c} \subset [0, \beta]$. This is the situation of Case 1. So we assume that $\beta > 1/2$. In other words, $F = \bigcup_{i=1}^{n-1}[\alpha_i, \beta_i] \cup [\beta, 1]$. Then by a similar argument to Case 3, we see that there is no F such that $F \vartriangle T^{-1}F$ is an interval.



CASE 5. If $F \subset [0,\beta]$ with $1/2 < \beta < 1$, and $[0,\beta]$ is the optimal bounding interval for F, then by a similar argument to Case 4, there is no F for which $F \triangle T^{-1}F$ is an interval. ■

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