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THE ZIEGLER SPECTRUM OF A TAME HEREDITARY ALGEBRA

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1. Introduction. Let k be a field, and A be a connected finite-dimensional hereditary k-algebra of tame representation type. For a description of finite-dimensional hereditary k-algebras we refer to [DR2]. Let $\operatorname{Mod} A$ denote the category of all A-modules and $\operatorname{mod} A$ the full subcategory of A-modules of finite length. The global structure of the category of all A-modules is well-known (see [DR1] and [R1]).

Let $\operatorname{Com} A$ be a set of indecomposable algebraically compact A-modules, one from each isomorphism class. Let $\operatorname{ind} A$ be the subset of elements of $\operatorname{Com} A$ of finite length (since any finite length module is algebraically compact, $\operatorname{ind} A$ is just a complete set of indecomposable A-modules of finite length).

If $\mathcal{H}$ is a class of maps in $\operatorname{mod} A$, let $\mathcal{I}(\mathcal{H})$ be the full subcategory of all A-modules I with the following property: For any map $h : M \to M'$ in $\mathcal{H}$ and any map $f : M \to I$, there is $f' : M' \to I$ with $f'h = f$. A full subcategory of $\operatorname{Mod} A$ is said to be *definable* provided it is of the form $\mathcal{I}(\mathcal{H})$ for some class $\mathcal{H}$ of maps in $\operatorname{mod} A$; a full subcategory of $\operatorname{Mod} A$ is definable if and only if it is closed under products, direct limits and pure submodules (see [CB], 2.3 and Lemma 1 of 2.1; we will use the Trondheim survey of Crawley-Boevey [CB] as a general reference).

The subsets of $\operatorname{Com} A$ of the form $\mathcal{I}(\mathcal{H}) \cap \operatorname{Com} A$ are said to be *closed* (or *Ziegler closed*). It is obvious that the intersections of closed sets are again closed, thus any subset $\mathcal{X}$ of $\operatorname{Com} A$ has a closure, which we will denote by $\overline{\mathcal{X}}$ (it is the intersection of all closed sets containing $\mathcal{X}$). The set $\operatorname{Com} A$ together with its closed subsets is called the *Ziegler spectrum* of A.

The aim of this note is to provide a direct approach for determining all closed subsets of $\operatorname{Com} A$. The question has been investigated before by several authors. The case $\widetilde{D}_4$ was considered by Baur [B]. Parts of the answer can be found in an unpublished preprint of Prest [P1] and in the books of Prest [P2] and Jensen–Lenzing [JL]. The recent work of Geisler [G] has

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solved the problem in the case of a quiver algebra; his (rather tedious) proof is based on the inductive procedure of Donovan–Freislich [DF]. Prest [P3] has announced an independent approach dealing with the general case. The presentation given below was inspired by [G]; but to reduce the computations presented there, we use the structure theory of the category of A-modules.

The elements of $\mathrm{Com}\, A$ are known (see for example [CB], Theorem 3.1). First of all, there are the elements of $\mathrm{ind}\, A$, they are either preprojective, regular, or preinjective. There are countably many preprojectives and countably many preinjectives in $\mathrm{ind}\, A$. There is a set Ω which parametrizes the so-called simple regular modules (these are modules which are not necessarily simple, but they are simple objects in the subcategory of all "regular" modules— note that this is an abelian category, so that the notion of "simplicity" is defined). If k is finite, then Ω is countable, otherwise the cardinality of Ω is the same as the cardinality of k. For any simple regular module S and any $n \in \mathbb{N}_1$, there is a unique indecomposable module $S[n]$ which has a filtration of length n such that all the factors are simple regular and such that $S = S[1]$ occurs as a submodule of $S[n]$; in this way, one obtains all the regular modules in $\mathrm{ind}\, A$. The indecomposable regular modules may also be labelled dually: For any simple regular module S and any $n \in \mathbb{N}_1$, we denote by $[n]S$ the unique indecomposable regular module which has a filtration of length n with simple regular factors and which has S as a factor module.

For any simple regular module S, there is a chain of inclusions

$$S[1] \subset S[2] \subset S[3] \subset \ldots,$$

the union is indecomposable and is denoted by $S[\infty]$. Similarly, there is a chain of epimorphisms

$$[1]S \leftarrow [2]S \leftarrow [3]S \leftarrow \ldots,$$

the inverse limit is indecomposable and is denoted by $\widehat{S}$. A module of the form $S[\infty]$ is called a *Prüfer* module, a module of the form $\widehat{S}$ is called an *adic* module (the module $\widehat{S}$ itself is called the S-adic module, in the same way as one speaks of the p-adic integers). All the Prüfer modules and all the adic modules belong to $\mathrm{Com}\, A$. There is just one additional module in $\mathrm{Com}\, A$, the so-called *generic* module G. In the terminology of [R1], G is the unique indecomposable module which is torsionfree and divisible (a module X is said to be *torsionfree* provided that $\mathrm{Hom}(S, X) = 0$, and *divisible* provided that $\mathrm{Hom}(X, S) = 0$, for all $S \in \Omega$).

THEOREM. *A subset $\mathcal{X}$ of* $\mathrm{Com}\, A$ *is closed if and only if the following conditions are satisfied*:

(i) *If S is a simple regular A-module and if there are infinitely many finite length modules $X \in \mathcal{X}$ with* $\mathrm{Hom}(S, X) \neq 0$, *then $S[\infty]$ belongs to $\mathcal{X}$.*

(i*) *If S is a simple regular A-module and if there are infinitely many finite length modules $X \in \mathcal{X}$ with* $\operatorname{Hom}(X, S) \neq 0$, *then $\widehat{S}$ belongs to $\mathcal{X}$.*

(ii) *If there are infinitely many finite length modules in $\mathcal{X}$ or if there exists at least one module in $\mathcal{X}$ which is not of finite length*, *then the generic module belongs to $\mathcal{X}$.*

We see that the Ziegler closed subsets of $\operatorname{Com} A$ are related to the support of the functors $\operatorname{Hom}(S, -)$ and $\operatorname{Hom}(-, S)$, where S is simple regular. These functors have been considered before; in particular, we refer to the investigations of Geigle [Gg]. In [R2], the support $q(S)$ of the functor $\operatorname{Hom}(S, -)$ was studied in detail, in order to deal with one-point extensions. If one wants to combine similar cases, it was proposed in [R2] to consider equivalence classes called "patterns", and some of these patterns have been exhibited there. All the information needed in order to describe these patterns is contained in the tables of [DR1]; this will be recalled in Section 6.

In Section 2, we will construct Prüfer modules in the closure of a given subset $\mathcal{X}$ of $\operatorname{Com} A$. The dual situation of dealing with adic modules will be considered in Section 3; of course, one could also just refer to the so-called elementary duality introduced by Herzog [H] (see also [K]) in order to obtain dual assertions. Section 4 will show that a closed subset of $\operatorname{Com} A$ which does not contain the generic module has to be a finite subset of $\operatorname{ind} A$. Again, these arguments could be replaced by references to well-known facts: here one may use the compactness of the Ziegler spectrum. The three Sections 2–4 show that closed subsets of $\operatorname{Com} A$ have the properties (i), (i*) and (ii) stated in the Theorem. Section 5 will show that also the converse is true: subsets of $\operatorname{Com} A$ which have the properties (i), (i*) and (ii) are closed. Throughout the paper, $\mathcal{X}$ will be a subset of $\operatorname{Com} A$.

2. Which Prüfer modules belong to $\overline{\mathcal{X}}$? If $\mathcal{M}$ is a set of modules, we will denote the product $\prod_{M \in \mathcal{M}} M$ just by $\prod \mathcal{M}$; similarly, $\bigoplus \mathcal{M} = \bigoplus_{M \in \mathcal{M}} M$.

PROPOSITION 1. *Let $\mathcal{X}$ be an infinite subset of* $\operatorname{ind} A$. *Let S be a simple regular A-module and assume that* $\operatorname{Hom}(S, X) \neq 0$ *for all $X \in \mathcal{X}$.*

(a) *If all modules in $\mathcal{X}$ are regular*, *then $S[\infty]$ is the union of a chain of monomorphisms $X_1 \to X_2 \to X_3 \to \ldots$ with $X_i \in \mathcal{X}$.*

(b) *If all modules in $\mathcal{X}$ are preinjective*, *then $S[\infty]$ is a direct summand of the product $\prod \mathcal{X}$.*

Proof of (a). The only indecomposable regular modules X with $\operatorname{Hom}(S, X) \neq 0$ are (up to isomorphism) the modules $S[n]$ with $n \in \mathbb{N}$. There are the inclusion maps

$$S[1] \subset S[2] \subset S[3] \subset \ldots,$$

and the union is $S[\infty]$. The modules X_i in $\mathcal{X}$ are of the form $X_i = S[t_i]$ and we may assume that $i < j$ implies $t_i < t_j$. It follows that the direct limit of these modules X_i with respect to the inclusion maps is $S[\infty]$.

For the proof of (b), we will need the following lemma:

LEMMA. *Let S be simple regular, and $n \in \mathbb{N}$. Almost all preinjective modules X in* $\operatorname{ind} A$ *have the following property: The kernel of any non-injective map $f : S[n] \to X$ contains $S[1]$.*

Proof. We denote the length of a module M by $|M|$. Let $b = |S[n]|$. There are only finitely many preinjective modules X in $\operatorname{ind} A$ with $|X| \le b$. Any such module X has only finitely many successors. Let $\mathcal{Q}'$ be the set of all preinjective modules X in $\operatorname{ind} A$ which have the following property: if Y is an indecomposable preinjective module with $\operatorname{Hom}(Y, X) \neq 0$, then $|Y| > b$. Then almost all preinjective modules in $\operatorname{ind} A$ belong to $\mathcal{Q}'$. Assume now that a module $X \in \mathcal{Q}'$ is given and let $f : S[n] \to X$ be a map which is not injective. Let Y be an indecomposable direct summand of the image of f. Then $|Y| \le b$ shows that Y cannot be preinjective, thus Y has to be regular. This shows that the image of f is regular. Consequently, the kernel of f is regular. But $S[n]$ has just one regular submodule of regular length 1, namely $S[1]$. Thus, either $S[1]$ is contained in the kernel of f or else f is injective.

Proof of (b). For every module X in $\mathcal{X}$ we choose a non-zero map $f_X : S \to X$. Inductively, we obtain maps $f_X^{(n)} : S[n] \to X$ such that the restriction of $f_X^{(n)}$ to $S[n-1]$ is $f_X^{(n-1)}$. Namely, assume that $f_X^{(n-1)}$ has been constructed. Note that the cokernel C of the inclusion map $S[n-1] \to S[n]$ is regular, thus $\operatorname{Ext}^1(C, X) = 0$ due to the fact that X is preinjective. But the vanishing of $\operatorname{Ext}^1(C, X)$ implies that the map $f_X^{(n-1)}$ can be extended to $S[n]$. This yields the desired map $f^{(n)} : S[n] \to X$ with restriction to $S[n-1]$ being equal to $f_X^{(n-1)}$. We obtain in this way a map $f_X^{(\infty)} : S[\infty] \to X$ such that the restriction to $S[1]$ is f_X. Since f_X is non-zero, we see in particular that the restriction of $f_X^{(\infty)}$ to $S[1]$ is non-zero.

Consider now the map $g = (f_X^{(\infty)})_X : S[\infty] \to \prod \mathcal{X}$. We claim that g is injective and that the image of g intersects $\bigoplus \mathcal{X}$ in zero. Indeed, assume that some element $x \in S[\infty]$ is mapped under g to $\bigoplus \mathcal{X}$. Note that x belongs to some $S[n]$ with $n \in \mathbb{N}$. It follows that $f_X^{(\infty)}(x) = 0$ for almost all X. But $f_X^{(\infty)}(x) = f_X^{(n)}(x)$, and $f_X^{(n)}$ is injective for almost all n (since the kernel of $f_X^{(n)}$ does not contain $S[1]$). This yields a contradiction.

According to [R1, 3.7], the submodule $\bigoplus \mathcal{X}$ of $\prod \mathcal{X}$ is a direct summand and the corresponding complement Y has no non-zero preinjective direct summand. As we have seen, $S[\infty]$ embeds into Y. But $S[\infty]$ is injective in the category of all modules without non-zero preinjective direct summands [R1, 4.7]. This shows that $S[\infty]$ is a direct summand of Y and therefore of $\prod \mathcal{X}$. This completes the proof.

3. Which adic modules belong to $\overline{\mathcal{X}}$?

PROPOSITION 2. *Let $\mathcal{X}$ be an infinite subset of* $\operatorname{ind} A$. *Let S be a simple regular A-module and assume that* $\operatorname{Hom}(X, S) \neq 0$ *for all* $X \in \mathcal{X}$.

(a) *If all modules in $\mathcal{X}$ are regular, then $\widehat{S}$ is the inverse image of a chain of epimorphisms* $\ldots \to X_3 \to X_2 \to X_1$ *with* $X_i \in \mathcal{X}$.

(b) *If all modules in $\mathcal{X}$ are preprojective, then $\widehat{S}$ belongs to the closure $\overline{\mathcal{X}}$.*

REMARK. In (b) one may show that $\widehat{S}$ is actually the direct limit of a filtered set of homomorphisms between modules in $\mathcal{X}$. Since the k-dimension of $\widehat{S}$ is uncountable, one cannot obtain $\widehat{S}$ as the direct limit of a chain of maps in $\mathcal{X}$.

Proof of (a). The regular modules X with $\operatorname{Hom}(X, S) \neq 0$ are (up to isomorphism) the modules $[n]S$ with $n \in \mathbb{N}$. There is a chain of epimorphisms

$$[1]S \leftarrow [2]S \leftarrow [3]S \leftarrow \ldots,$$

and its inverse limit is $\widehat{S}$. The modules X_i in $\mathcal{X}$ are of the form $X_i = [t_i]S$ and we may assume that $i < j$ implies $t_i < t_j$. Thus we have inside $\mathcal{X}$ a chain of epimorphisms

$$[t_1]S \leftarrow [t_2]S \leftarrow [t_3]S \leftarrow \ldots,$$

with inverse limit $\widehat{S}$.

Before we start with the proof of (b), let us introduce the following notation: Given any k-space V, let $DV = \operatorname{Hom}(V, k)$ be the dual space. Of course, if V is a left A-module, then DV is a right A-module, or what is the same, a left A°-module, where A° is the opposite algebra. With A also A° is a finite-dimensional hereditary k-algebra of tame representation type. If M is a finite-dimensional indecomposable A-module, then DM is an indecomposable A°-module of the same dimension, and M is preprojective, or regular, or preinjective if and only if DM is preinjective, or regular, or preprojective, respectively. Also, if S is a simple regular A-module, then DS is a simple regular A°-module, and $D(S[\infty]) = \widehat{DS}$.

Proof of (b). By assumption, there is given an infinite set $\mathcal{X}$ of preprojective A-modules X with $\operatorname{Hom}(X, S) \neq 0$. Thus $\mathcal{Y} = \{DX \mid X \in \mathcal{X}\}$ is an infinite set of preinjective A°-modules Y with $\operatorname{Hom}(DS, Y) \neq 0$. According to Section 2, the module $(DS)[\infty]$ is a direct summand of $\prod \mathcal{Y}$. Dualizing, we see that $\widehat{S} = D((DS)[\infty])$ is a direct summand of $D(\prod \mathcal{Y})$.

The following lemma shows that the latter module belongs to $\overline{\mathcal{X}}$, thus also $\widehat{S}$ belongs to $\overline{\mathcal{X}}$.

LEMMA. *Let $\mathcal{H}$ be a class of maps in* $\operatorname{mod} A$. *Let $\mathcal{X}$ be a set of finite-dimensional A-modules in $\mathcal{I}(\mathcal{H})$. Let $D\mathcal{X}$ be the set of all modules DX with $X \in \mathcal{X}$. Then $D(\prod D\mathcal{X})$ belongs to $\mathcal{I}(\mathcal{H})$.*

Proof. Let Y be an A^{o}-module. The adjunction formula shows that the A-module DY belongs to $\mathcal{I}(\mathcal{H})$ if and only if all the k-linear maps $1_Y \otimes_A h$ with $h \in \mathcal{H}$ are injective.

Let $h : M \to M'$ be a map in $\mathcal{H}$. If X belongs to $\mathcal{X}$, we can apply this remark to $Y = DX$ and see that the map $1_{DX} \otimes_A h$ is injective. Since a product of injective maps is injective, it follows that the map

$$\prod_{X \in \mathcal{X}} 1_{DX} \otimes_A h : \prod_{X \in \mathcal{X}} DX \otimes_A M \to \prod_{X \in \mathcal{X}} DX \otimes_A M'$$

is injective. Since the modules M, M' are finite-dimensional, the tensor products $- \otimes_A M$ and $- \otimes_A M'$ commute with products, thus the displayed map is just $\big(\prod D\mathcal{X}\big) \otimes_A h$. Now we use the starting remark again, this time for $Y = \prod D\mathcal{X}$. It follows that DY belongs to $\mathcal{I}(\mathcal{H})$.

4. When does the generic module belong to $\overline{\mathcal{X}}$?

PROPOSITION 3. *Let S be simple regular. The generic module is a direct summand of any infinite power of $S[\infty]$.*

The proof is easy; see [R4]. There, it is shown that any infinite power of $S[\infty]$ is a direct sum of copies of $S[\infty]$ and of copies of the generic module.

Note that any non-zero endomorphism of an adic module is injective, and there are such endomorphisms which are not surjective.

PROPOSITION 4. *Let S be simple regular. Let $\phi : \widehat{S} \to \widehat{S}$ be any homomorphism which is injective but not surjective. The direct limit L of the chain of maps*

$$\widehat{S} \xrightarrow{\phi} \widehat{S} \xrightarrow{\phi} \widehat{S} \xrightarrow{\phi} \dots$$

is a direct sum of copies of the generic module.

Proof. We show that L is torsionfree and divisible and use [R1, 5.4]. As a union of torsionfree modules, L is torsionfree. There is precisely one simple regular module T with $\operatorname{Ext}^1(T, \widehat{S}) \neq 0$, namely $T = \tau^{-1}S$, where τ is the Auslander–Reiten translation, and the induced map $\operatorname{Ext}^1(T, \phi)$ is the zero map (note that T is isomorphic to a submodule of the cokernel of ϕ).

If X is a regular module in $\operatorname{ind} A$, then $X = S[n]$ for some simple regular module S and some $n \geq 1$. The module $S = S[1]$ is called the *regular socle* of X.

PROPOSITION 5. *Let $\mathcal{X}$ be an infinite set of regular modules of* $\operatorname{ind} A$, *with pairwise different regular socles. Then the module $\prod \mathcal{X} / \bigoplus \mathcal{X}$ is a direct sum of copies of G.*

Proof. We consider $Z(\mathcal{X}) = \prod \mathcal{X} / \bigoplus \mathcal{X}$. Note that for any cofinite subset $\mathcal{X}'$ of $\mathcal{X}$, the modules $Z(\mathcal{X})$ and $Z(\mathcal{X}')$ are isomorphic. Namely, if $X_1, \ldots, X_n$ are the modules of $\mathcal{X}$ which do not belong to $\mathcal{X}'$, then $\prod \mathcal{X} = \prod \mathcal{X}' \oplus \bigoplus_{i=1}^{n} X_i$, and similarly, $\bigoplus \mathcal{X} = \bigoplus \mathcal{X}' \oplus \bigoplus_{i=1}^{n} X_i$.

Let S be simple regular. There are only finitely many modules $X \in \mathcal{X}$ with $\operatorname{Hom}(T, X) \neq 0$, where T is in the τ-orbit of S. Let $\mathcal{X}'$ be obtained from $\mathcal{X}$ by removing these modules. Then all the groups $\operatorname{Hom}(S, \prod \mathcal{X}')$, $\operatorname{Ext}^1(S, \bigoplus \mathcal{X}')$ and $\operatorname{Ext}^1(S, \prod \mathcal{X}')$ are zero. The vanishing of these groups implies that $\operatorname{Hom}(S, Z(\mathcal{X}')) = 0 = \operatorname{Ext}^1(S, \prod \mathcal{X}')$ (using the long exact sequence $0 \to \bigoplus \mathcal{X}' \to \prod \mathcal{X}' \to Z(\mathcal{X}') \to 0$).

Thus $Z(\mathcal{X})$ is torsionfree and divisible, and therefore a direct sum of copies of the generic module G (see [R1, 5.4]).

COROLLARY. *Let $\mathcal{X}$ be an infinite subset of* $\operatorname{Com} A$. *Then the generic module G belongs to $\overline{\mathcal{X}}$.*

Proof. If $\mathcal{X}$ contains infinitely many preinjectives, then it contains almost all Prüfer modules. Dually, if it contains infinitely many preprojectives, then it contains almost all adic modules. If it contains infinitely many regular modules of the form $S[n]$ with fixed S, then it contains the Prüfer module $S[\infty]$ (as well as the adic module $\widehat{S}$). Propositions 3 and 4 show that in all these cases, $\overline{\mathcal{X}}$ contains the generic module G. Thus, we may assume that $\mathcal{X}$ contains infinitely many regular modules $S[n_S]$ with pairwise different modules S. According to Proposition 5, the module G is a direct summand of the module $L = \prod \mathcal{X} / \bigoplus \mathcal{X}$. However, L can be written as the direct limit of a chain of maps

$$Y_1 \to Y_2 \to Y_3 \to \ldots,$$

where the modules Y_i are products of elements in $\mathcal{X}$ (see the following lemma). Since any definable subcategory is closed under products, direct limits and direct summands, we conclude that G belongs to $\mathcal{X}$.

Here is the missing argument, its proof is straightforward:

LEMMA. *Let $\mathcal{X} = \{X_1, X_2, \ldots\}$ be an infinite sequence of modules. For $t \in \mathbb{N}$, let $\mathcal{X}_t = \{X_i \mid i \geq t\}$. Then there are canonical epimorphisms*

$$\prod \mathcal{X}_1 \to \prod \mathcal{X}_2 \to \prod \mathcal{X}_3 \to \ldots,$$

and the direct limit is $\prod \mathcal{X} / \bigoplus \mathcal{X}$.

REMARK. Proposition 5 has the following consequence: *Let $\mathcal{X}$ be an infinite set of regular modules of* $\operatorname{ind} A$, *with pairwise different regular socles.*

Then $\bigoplus \mathcal{X}$ is not a direct summand of $\prod \mathcal{X}$. Namely, the generic module G does not embed into $\prod \mathcal{X}$, since $\operatorname{Hom}(G, X) = 0$ for all modules $X \in \mathcal{X}$.

5. Proof of the Theorem. Consider a subset $\mathcal{X}$ of $\operatorname{Com} A$. First, let us assume that $\mathcal{X}$ is closed. If $\mathcal{X}$ is an infinite set, then we have seen in Section 4 that the generic module G belongs to $\mathcal{X}$. If $\mathcal{X}$ contains at least one Prüfer module or an adic module, then G is contained in $\mathcal{X}$ by Propositions 3 and 4. This shows that the condition (ii) is satisfied. Now, let S be a simple regular module. If there are infinitely many finite length modules $X \in \mathcal{X}$ with $\operatorname{Hom}(S, X) \neq 0$, then $S[\infty]$ belongs to $\mathcal{X}$ by Proposition 1. If there are infinitely many finite length modules $X \in \mathcal{X}$ with $\operatorname{Hom}(X, S) \neq 0$, then $\widehat{S}$ belongs to $\mathcal{X}$ by Proposition 2. This shows that the conditions (i) and (i*) are satisfied.

Conversely, assume that $\mathcal{X}$ satisfies the conditions (i), (i*) and (ii). We have to show that $\mathcal{X}$ is closed. We will use the following assertion: if $\mathcal{Y}$ is a closed subset of $\operatorname{Com} A$ and X is an element of $\operatorname{ind} A$, then $\mathcal{Y} \cup \{X\}$ is closed (this follows from the fact that the closed sets are those of a topology [CB, 2.5] and that for any element $X \in \operatorname{ind} A$, the set $\{X\}$ is closed [CB, 2.5]). In particular, if $\mathcal{X}$ is a finite subset of $\operatorname{ind} A$, then $\mathcal{X}$ is closed.

Thus, we may assume that $\mathcal{X}$ is not a finite subset of $\operatorname{ind} A$. Note that condition (ii) shows that the generic module belongs to $\mathcal{X}$. Also, one knows that the closure $\overline{\mathcal{X}}$ does not contain any additional finite-dimensional indecomposables [CB, Proposition 2.3]. Thus, it remains to consider the Prüfer modules and the adic modules. We show the following: if such a module belongs to $\overline{\mathcal{X}}$, then it belongs already to $\mathcal{X}$.

First, assume that the Prüfer module $S[\infty]$ does not belong to $\mathcal{X}$. According to condition (i), we see that there are only finitely many modules $X_1, \ldots, X_n$ in $\mathcal{X}$ with $\operatorname{Hom}(S, X_i) \neq 0$. Let $\mathcal{X}'$ be obtained from $\mathcal{X}$ by deleting these modules $X_1, \ldots, X_n$. The set $\mathcal{C}$ of all modules C in $\operatorname{Com} A$ such that $\operatorname{Hom}(S, C) = 0$ is a closed subset of $\operatorname{Com} A$, and by assumption, $\mathcal{X}'$ is contained in $\mathcal{C}$. We have

$$\mathcal{X} = \mathcal{X}' \cup \{X_1, \ldots, X_n\} \subseteq \mathcal{C} \cup \{X_1, \ldots, X_n\}.$$

The latter set is closed, but does not contain $S[\infty]$. Therefore, $S[\infty]$ is not contained in the closure of $\mathcal{X}$.

We argue in the same way for the adic module $\widehat{S}$. We assume that $\widehat{S}$ does not belong to $\mathcal{X}$. Then, according to condition (i*), there are only finitely many modules $X_1, \ldots, X_n$ in $\mathcal{X}$ with $\operatorname{Hom}(X_i, S) \neq 0$. Again, let $\mathcal{X}'$ be obtained from $\mathcal{X}$ by deleting these modules. The set $\mathcal{C}$ of all modules C in $\operatorname{Com} A$ such that $\operatorname{Hom}(C, S) = 0$ is a closed subset of $\operatorname{Com} A$, and by assumption, $\mathcal{X}'$ is contained in $\mathcal{C}$, whereas $\widehat{S}$ is not contained in $\mathcal{C} \cup \{X_1, \ldots, X_n\}$. This shows that $\widehat{S}$ is not contained in the closure of $\mathcal{X}$.

6. The patterns. Our description of the Ziegler spectrum refers to the support of the functors $\operatorname{Hom}(S,-)$ and $\operatorname{Hom}(-,S)$, where S is simple regular. In this final section, we are going to collect known results concerning these functors and to derive consequences for the Ziegler spectrum. We denote by $q(S)$ the set of modules $X \in \operatorname{ind} A$ with $\operatorname{Hom}(S,X) \neq 0$; similarly, let $p(S)$ be the set of modules $X \in \operatorname{ind} A$ with $\operatorname{Hom}(X,S) \neq 0$.

PROPOSITION 6. *Let S be a simple regular A-module. Let $\mathcal{X}$ be a closed subset of* $\operatorname{Com} A$. *If $\mathcal{X}$ contains infinitely many preinjective modules, then there is $t \in \mathbb{N}$ such that $\mathcal{X}$ contains the Prüfer module $T[\infty]$, where $T = \tau^t S$. If $\mathcal{X}$ contains infinitely many preprojective modules, then there is $t \in \mathbb{N}$ such that $\mathcal{X}$ contains the adic module $\widehat{T}$, where $T = \tau^t S$.*

Proof. The module S is τ-periodic, say with period $m = m(S)$. Let $S_i = \tau^i(S)$. Then $\bigcup_{1\le i\le m} q(S_i)$ contains all preinjective modules in $\operatorname{ind} A$. If $\mathcal{X}$ contains infinitely many preinjective modules, then one of the sets $\mathcal{X} \cap q(S_i)$ has to be infinite and then $S_i[\infty]$ has to belong to $\mathcal{X}$. The second assertion follows by duality.

COROLLARY. *Let $n(A)$ be the number of isomorphism classes of simple A-modules. Let $\mathcal{X}$ be a closed subset of* $\operatorname{Com} A$. *If $\mathcal{X}$ contains infinitely many preinjective modules, then all but at most $n(A)-2$ Prüfer modules belong to $\mathcal{X}$. If $\mathcal{X}$ contains infinitely many preprojective modules, then all but at most $n(A)-2$ adic modules belong to $\mathcal{X}$.*

Proof. This is an immediate consequence of Proposition 6, using [DR1], Theorem 4.1.

Let S be a simple regular A-module. The set $q(S)$ may be considered as the set of vertices of a quiver: given two modules X, Y in $q(S)$, we draw an arrow $X \to Y$ provided there is an irreducible map $f : X \to Y$ such that $\operatorname{Hom}(S,f) \neq 0$. These quivers (or more precisely, the equivalence classes of related vector space categories under a suitable equivalence relation) have been considered in [R2] since they are of interest when dealing with one-point extensions; they have been called "patterns". Actually, in the setting of [R2], it was necessary to mark also the dimension of the k-space $\operatorname{Hom}(S,X)$. On the other hand, only those patterns which yield tame one-point extensions have been exhibited there. As the Theorem shows, here we are only interested in the support of the patterns, thus we only have to keep track whether $\operatorname{Hom}(S,X)$ is zero or not.

The calculation of patterns has been described in Section 3.3 of [R2] in detail. We recall the main ideas: Let X belong to $q(S)$. Then X is either regular or preinjective. The regular modules in $q(S)$ always form a "ray": we deal with the modules $S[n]$, where $n \in \mathbb{N}_1$, and the corresponding part of the quiver $q(S)$ is a linearly oriented quiver of type A_∞, it looks as follows:

$$\circ \to \circ \to \circ \to \dots$$

Now assume that X is preinjective, say $X = \tau^t I(E)$, where $t \geq 0$ is an integer, and $I(E)$ is the injective envelope of the simple A-module E. Note that

$$\mathrm{Hom}(S, X) = \mathrm{Hom}(S, \tau^t I(E)) \simeq \mathrm{Hom}(\tau^{-t} S, I(E)).$$

Of course, given any A-module Y, we have $\mathrm{Hom}(Y, I(E)) \neq 0$ if and only if E occurs as a composition factor of Y. Thus, in order to decide whether $\mathrm{Hom}(S, X)$ is non-zero or zero, we only have to check whether E is a composition factor of $\tau^{-t} S$ or not. Thus, for a fixed simple regular module S, we have to display the composition factors of the τ-translates of S.

In case τS is isomorphic to S, all simple A-modules occur as composition factors of S, thus $\mathrm{Hom}(S, X) \neq 0$ for any indecomposable preinjective module X. Consider now the case where τS is not isomorphic to S; in this case S is said to be *non-homogeneous.* There are at most $n(A)+1$ non-homogeneous simple regular modules. Of course, these modules are τ-periodic (with period bounded by $n(A) - 1$). Note that we deal with the composition factors of just a finite number of modules. The tables in [DR1] provide all the composition factors of the non-homogeneous simple regular modules, thus they provide all the necessary information. For k an algebraically closed field, the same information is presented in the appendix of [R3], pages 363 and 364, and some of the corresponding quivers can be found in [R2], pages 253–255.

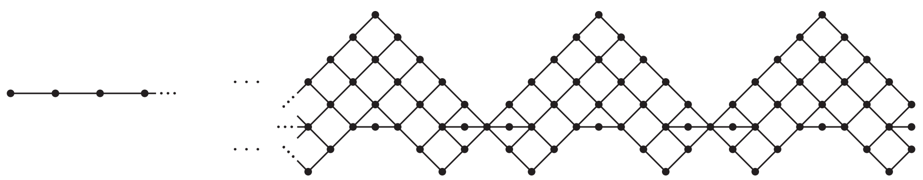
For example, consider the representations of the quiver of type $\widetilde{E}_8$ with the so-called subspace orientation:

$$\begin{array}{ccccccccccccccc} & & & & \circ & & & & & & & & & & \\ & & & & \downarrow & & & & & & & & & & \\ \circ & \to & \circ & \to & \circ & \leftarrow & \circ & \leftarrow & \circ & \leftarrow & \circ & \leftarrow & \circ & \leftarrow & \circ \end{array}$$

and the simple regular module S with dimension vector

$$\begin{array}{cccccccc} & & 1 & & & & & \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

The corresponding pattern $q(S)$ looks as follows (with arrows pointing from left to right):



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