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PERIODIC Lip^{α} *FUNCTIONS WITH* Lip^{β} *DIFFERENCE FUNCTIONS*

BY

TAMÁS KELETI (BUDAPEST)

1. Introduction. In [7] the following notion was introduced: Let $\mathbb{G}$ be either the additive subgroup of the reals $\mathbb{R}$ or the circle group $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of functions on $\mathbb{G}$ with $\mathcal{F}\supset\mathcal{G}$. We denote by $\mathfrak{H}(\mathcal{F},\mathcal{G})$ the class of those subsets H of $\mathbb{G}$ for which a function $f\in\mathcal{F}$ can have difference functions $\Delta_h f(x)=f(x+h)-f(x)$ in $\mathcal{G}$ for every $h\in H$ without f belonging to $\mathcal{G}$. That is,

$$\mathfrak{H}(\mathcal{F},\mathcal{G})=\{H\subset\mathbb{G}:\exists f\in\mathcal{F}\setminus\mathcal{G}\ \Delta_h f\in\mathcal{G}\ \forall h\in H\}.$$

We denote by Lip^{α} the class of functions f on $\mathbb{T}$ for which there exists an $L>0$ such that $|f(x)-f(y)|\le L|x-y|^{\alpha}$ for every $x,y\in\mathbb{T}$. (Sometimes we identify $\mathbb{T}$ with $[0,1)$. If $a\in\mathbb{T}$ then by $|a|$ we mean $\min(a,1-a)$.)

It was proved in [7] (Theorem 4.10) that for $0<\alpha<\beta\le 1$ we have $\mathfrak{H}(\mathrm{Lip}^{\alpha},\mathrm{Lip}^{\beta})\subset\mathfrak{F}_{\sigma}$, where $\mathfrak{F}_{\sigma}$ denotes the family of those subsets of $\mathbb{T}$ that can be covered by a proper F_{σ} subgroup of $\mathbb{T}$. Generalizing a result of M. Balcerzak, Z. Buczolich and M. Laczkovich [1], it was also proved in [7] (Theorem 5.3) that equality holds if $\beta=1$. In this paper we investigate the case when $0<\alpha<\beta<1$.

A set $H\subset\mathbb{T}$ is called a *pseudo-Dirichlet set* if there exists an increasing sequence (q_n) of integers and a sequence (ε_n) converging to zero such that for any $x\in H$ there exists an $n_0(x)$ such that $|\sin q_n\pi x|<\varepsilon_n$ if $n\ge n_0(x)$. We denote the family of pseudo-Dirichlet sets by $p\mathfrak{D}$.

The pseudo-Dirichlet sets were considered by N. Bary [2] who proved that they are contained in the class of N_0-sets. Clearly $p\mathfrak{D}$ contains all the Dirichlet sets; in fact, pseudo-Dirichlet sets are the countable increasing unions of Dirichlet sets (see [3]). These notions together with several other notions of thinness in harmonic analysis are discussed e.g. in [4] and [5].

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In this paper we prove that

$$p\mathfrak{D} \subset \mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta),$$

and that all the $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$ classes are the same for any $0 < \alpha < \beta < 1$.

2. $p\mathfrak{D} \subset \mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$**.** We will use the following well known lemma (see e.g. [2], Chapter XI, p. 691, Theorem 2):

LEMMA 2.1. *If* $0 < \gamma < 1$ *and* (q_n) *is an increasing sequence of integers such that* $q_{n+1}/q_n > \lambda$ *for a suitable* $\lambda > 1$, *then for any sequence* (a_n) *of complex numbers*,

$$|a_n| = O(1/q_n^\gamma) \Leftrightarrow g(x) = \sum_{n=1}^{\infty} a_n e^{2\pi i q_n x} \in \mathrm{Lip}^\gamma .$$

THEOREM 2.2. *For any* $0 < \alpha < \beta < 1$,

$$p\mathfrak{D} \subset \mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta).$$

That is, *for any* $0 < \alpha < \beta < 1$ *and for any pseudo-Dirichlet set* H, *there exists a* Lip^α *function* f *such that* $\Delta_h f$ *is* Lip^β *for any* $h \in H$ *but* f *is not* Lip^β.

Proof. Let H be a pseudo-Dirichlet set. Take a sequence $q_1 < q_2 < \ldots$ and a sequence $\varepsilon_n \to 0$ witnessing the pseudo-Dirichlet property of H. Selecting a suitable subsequence, we may assume that $q_{n+1} > 2q_n$ for every $n \in \mathbb{N}$. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{q_n^\beta \delta_n} e^{2\pi i q_n x}, \quad \text{where} \quad \delta_n = \max(\varepsilon_n, 1/q_n^{\beta-\alpha}).$$

Since $1/(q_n^\beta \delta_n) \le 1/q_n^\alpha$ and $q_{n+1}/q_n > 2$ $(n \in \mathbb{N})$, we can apply Lemma 2.1 to obtain $f \in \mathrm{Lip}^\alpha$. On the other hand, since $\delta_n \to 0$, we have $1/(q_n^\beta \delta_n) \ne O(1/q_n^\beta)$, so Lemma 2.1 implies that $f \notin \mathrm{Lip}^\beta$.

For $h \in H$,

$$\Delta_h f(x) = \sum_{n=1}^{\infty} \frac{1}{q_n^\beta \delta_n} (e^{2\pi i q_n h} - 1) e^{2\pi i q_n x}$$

and

$$\left| \frac{1}{q_n^\beta \delta_n} (e^{2\pi i q_n h} - 1) \right| = \frac{1}{q_n^\beta \delta_n} 2 |\sin \pi q_n h| \le \frac{1}{q_n^\beta \delta_n} 2\varepsilon_n \le \frac{2}{q_n^\beta},$$

therefore, by Lemma 2.1, $\Delta_h f \in \mathrm{Lip}^\beta$. ∎

Combining the previous theorem with the result of [7] mentioned in the introduction, we get the following:

COROLLARY 2.3. *For any* $0 < \alpha < \beta < 1$,

$$p\mathfrak{D} \subset \mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta) \subset \mathfrak{F}_\sigma.$$

3. The classes $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$ are the same for $0 < \alpha < \beta < 1$. The following easy lemmas were obtained in [7]:

LEMMA 3.1 (Monotonicity Lemma). *If $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{G}$ then $\mathfrak{H}(\mathcal{F}_1, \mathcal{G}) \supset \mathfrak{H}(\mathcal{F}_2, \mathcal{G})$.*

LEMMA 3.2. *If $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3$ and $\mathfrak{H}(\mathcal{F}_1, \mathcal{F}_2) \subset \mathfrak{H}(\mathcal{F}_2, \mathcal{F}_3)$ then $\mathfrak{H}(\mathcal{F}_1, \mathcal{F}_3) = \mathfrak{H}(\mathcal{F}_2, \mathcal{F}_3)$.*

We will need the notion of fractional integration. There are several different notions of fractional integrals (see e.g. the monograph [8]); here we use the so-called Weyl fractional integral which is defined in the following way (see [8], p. 263, or [9], Vol. II, p. 133):

Let f be an integrable function on $\mathbb{T}$ and suppose that $\int_{\mathbb{T}} f = 0$. Then, for any $\gamma > 0$, let

$$I_\gamma[f](x) = \int_{\mathbb{T}} f(t)\Psi_\gamma(x-t)\,dt, \tag{1}$$

where

$$\Psi_\gamma(t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi int}}{(2\pi in)^\gamma}. \tag{2}$$

It is known (see e.g. [9]) that the series in (2) converges everywhere on $\mathbb{T}\setminus\{0\}$ and the integral in (1) exists almost everywhere. (If $f \sim \sum c_n e^{2\pi inx}$ then $I_\gamma[f] \sim \sum c_n e^{2\pi inx}/(2\pi in)^\gamma$.)

Since the operator I_γ is defined by a convolution it commutes with the translation operator and it is linear; that is,

$$I_\gamma[f(y+h)](x) = I_\gamma[f(y)](x+h), \tag{3}$$

and

$$I_\gamma[cf + dg] = cI_\gamma[f] + dI_\gamma[g]. \tag{4}$$

Denote by Lip_0^λ the class of Lip^λ functions with integral 0 (over $\mathbb{T}$). It is also well known (see e.g. [8], p. 275) that if $\gamma, \lambda > 0$ and $\gamma + \lambda < 1$, then I_γ is a bijection (actually, an isomorphism) between the classes Lip_0^λ and $\mathrm{Lip}_0^{\lambda+\gamma}$; that is,

$$I_\gamma :\ \mathrm{Lip}_0^\lambda \leftrightarrow \mathrm{Lip}_0^{\lambda+\gamma} \qquad (\lambda + \gamma < 1). \tag{5}$$

THEOREM 3.3. *For any $0 < \alpha_1 < \beta_1 < 1$ and $0 < \alpha_2 < \beta_2 < 1$,*

$$\mathfrak{H}(\mathrm{Lip}^{\alpha_1}, \mathrm{Lip}^{\beta_1}) = \mathfrak{H}(\mathrm{Lip}^{\alpha_2}, \mathrm{Lip}^{\beta_2}).$$

P r o o f. First we prove that if $0<\alpha<\beta$ and $\beta+\gamma<1$ then

$$\mathfrak{H}(\mathrm{Lip}^{\alpha+\gamma},\mathrm{Lip}^{\beta+\gamma})=\mathfrak{H}(\mathrm{Lip}^{\alpha},\mathrm{Lip}^{\beta}). \tag{6}$$

Indeed, (3) and (4) implies that the operator I_γ also commutes with the difference operator Δ_h; that is,

$$\Delta_h I_\gamma[f]=I_\gamma[\Delta_h f]. \tag{7}$$

It follows from (5) and (7) that if $f_0\in\mathrm{Lip}_0^\alpha\setminus\mathrm{Lip}_0^\beta$ and $\Delta_h f_0\in\mathrm{Lip}_0^\beta$ for every $h\in H$ then $I_\gamma[f_0]\in\mathrm{Lip}_0^{\alpha+\gamma}\setminus\mathrm{Lip}_0^{\beta+\gamma}$ and $\Delta_h I_\gamma[f_0]\in\mathrm{Lip}_0^{\beta+\gamma}$ for every $h\in H$. Furthermore, if $g_0\in\mathrm{Lip}_0^{\alpha+\gamma}\setminus\mathrm{Lip}_0^{\beta+\gamma}$ and $\Delta_h g_0\in\mathrm{Lip}_0^{\beta+\gamma}$ for every $h\in H$ then $I_\gamma^{-1}[g_0]\in\mathrm{Lip}_0^\alpha\setminus\mathrm{Lip}_0^\beta$ and $\Delta_h I_\gamma^{-1}[g_0]\in\mathrm{Lip}_0^\beta$ for every $h\in H$. Therefore if the function $f:\mathbb{T}\to\mathbb{R}$ witnesses that $H\in\mathfrak{H}(\mathrm{Lip}^\alpha,\mathrm{Lip}^\beta)$ then $I_\gamma[f_0]$, where $f_0=f-\int_{\mathbb{T}}f$, witnesses that $H\in\mathfrak{H}(\mathrm{Lip}^{\alpha+\gamma},\mathrm{Lip}^{\beta+\gamma})$; furthermore, if the function $g:\mathbb{T}\to\mathbb{R}$ witnesses that $H\in\mathfrak{H}(\mathrm{Lip}^{\alpha+\gamma},\mathrm{Lip}^{\beta+\gamma})$ then $I_\gamma^{-1}[g_0]$, where $g_0(x)=g-\int_{\mathbb{T}}g$, witnesses that $H\in\mathfrak{H}(\mathrm{Lip}^\alpha,\mathrm{Lip}^\beta)$.

Now we prove that for any $0<\eta<\delta<\beta<1$,

$$\mathfrak{H}(\mathrm{Lip}^{\beta-\delta},\mathrm{Lip}^{\beta})=\mathfrak{H}(\mathrm{Lip}^{\beta-\eta},\mathrm{Lip}^{\beta}). \tag{8}$$

Indeed, by (6), $\mathfrak{H}(\mathrm{Lip}^{\beta-\delta},\mathrm{Lip}^{\beta-\delta/2})=\mathfrak{H}(\mathrm{Lip}^{\beta-\delta/2},\mathrm{Lip}^{\beta})$, which implies, by Lemma 3.2, that $\mathfrak{H}(\mathrm{Lip}^{\beta-\delta},\mathrm{Lip}^{\beta})=\mathfrak{H}(\mathrm{Lip}^{\beta-\delta/2},\mathrm{Lip}^{\beta})$. Thus we also have $\mathfrak{H}(\mathrm{Lip}^{\beta-\delta},\mathrm{Lip}^{\beta})=\mathfrak{H}(\mathrm{Lip}^{\beta-\delta/2^k},\mathrm{Lip}^{\beta})$ for any $k\in\mathbb{N}$. Then, by the Monotonicity Lemma, $\mathfrak{H}(\mathrm{Lip}^{\beta-\delta},\mathrm{Lip}^{\beta})=\mathfrak{H}(\mathrm{Lip}^{\beta-\eta},\mathrm{Lip}^{\beta})$.

Finally, supposing that $\beta_1\le\beta_2$ and applying (6) and (8), we get

$$\mathfrak{H}(\mathrm{Lip}^{\alpha_1},\mathrm{Lip}^{\beta_1})=\mathfrak{H}(\mathrm{Lip}^{\alpha_1+\beta_2-\beta_1},\mathrm{Lip}^{\beta_2})=\mathfrak{H}(\mathrm{Lip}^{\alpha_2},\mathrm{Lip}^{\beta_2}). \ \blacksquare$$

REMARK 3.4. Unfortunately this proof does not work if β_1 or β_2 equals 1. Namely, (5) is not true for $\lambda+\gamma=1$. In this case I_γ is a bijection between $\mathrm{Lip}_0^{1-\gamma}$ and Λ_{*0}, the class of Zygmund functions on $\mathbb{T}$ with zero integral.

(A function f is *Zygmund* if for any x and h, $|f(x+h)-2f(x)+f(x-h)|\le Ch$. The class of Zygmund functions is denoted by Λ_*. It is known (see e.g. [9], Vol. I, 43–44, and Vol. II, p. 138) that

$$\mathrm{Lip}^1(\mathbb{R})\subset\Lambda_*\subset\mathrm{Lip}^\alpha(\mathbb{R})\ \forall 0<\alpha<1,$$

and $\Lambda_*\ne\mathrm{Lip}^1(\mathbb{R})$.)

Therefore with this method we can only prove that

$$\mathfrak{H}(\mathrm{Lip}^{\alpha},\Lambda_*(\mathbb{T}))=\mathfrak{H}(\mathrm{Lip}^{\alpha_1},\mathrm{Lip}^{\beta_1})$$

for any $0<\alpha<1$ and $0<\alpha_1<\beta_1<1$.

However, if, for a fixed $0<\alpha<1$, one could find a linear operator I that commutes with the translation operator and is a bijection between Lip^1 and Lip^α (or between Lip^1 and Λ_*) then Theorem 3.3 would remain true for $\beta_1=1$. This would give a complete answer to our question since, as we mentioned in the introduction, it was proved in [7] that $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^1) = \mathfrak{F}_\sigma$, so the existence of such an operator would imply that $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta) = \mathfrak{F}_\sigma$ for any $0 < \alpha < \beta \leq 1$.

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Mathematical Institute of the
Hungarian Academy of Sciences
P.O. Box 127
1364 Budapest, Hungary
E-mail: elek@cs.elte.hu

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