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CHAIN CONDITIONS IN MODULAR LATTICES

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We give an analogue for complete modular lattices of the result of Goodearl who proved that an arbitrary module M over an arbitrary ring satisfies the ascending chain condition on essential submodules if and only if $M/\mathrm{Soc}(M)$ is Noetherian, where $\mathrm{Soc}(M)$ denotes the socle of M. Goodearl's Theorem can be extended for certain complete modular lattices to any dual Krull dimension.

Let R be a ring with identity and let M be a unital right R-module. Recall that a submodule K of M is *essential* provided $K \cap L \neq 0$ for every non-zero submodule L of M. The socle $\mathrm{Soc}(M)$ of M is the sum of all simple submodules of M, or 0 if M has no simple submodules. It is well known that $\mathrm{Soc}(M)$ is the intersection of all essential submodules of M (see, for example [5, Prop. 9.7]). A well known theorem of Goodearl [6, Prop. 3.6] asserts that the module M satisfies the ascending chain condition on essential submodules if and only if the R-module $M/\mathrm{Soc}(M)$ is Noetherian.

There is a dual result. A submodule N of M is *superfluous* if $N+L \neq M$ for every proper submodule L of M. The *radical* $\mathrm{Rad}(M)$ of M is defined to be the intersection of all maximal submodules of M, or M if M has no maximal submodules. It is well known that $\mathrm{Rad}(M)$ is the sum of all superfluous submodules of M (see, for example, [5, Prop. 9.13]). In [4] it is proved that $\mathrm{Rad}(M)$ is Artinian if and only if M satisfies the descending chain condition on superfluous submodules.

It is natural to ask whether these dual results have dual proofs. In order to investigate this question Alkhazzi [3] looked at the corresponding results in the context of modular lattices, his philosophy being that any proof of the analogue of Goodearl's result for modular lattices gives immediately the dual theorem by passing to the opposite lattice. Alkhazzi was unable to find such a proof. In fact, we suspect that no such proof exists but have

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been unable to find a counter-example. We do give an example of a modular lattice L with least element 0 such that if E denotes the set of essential elements of L then E is Noetherian, $0 = \bigwedge E$, but L is not Noetherian. However, this lattice L is not complete. In addition, we give necessary and sufficient conditions for a complete modular lattice L with $s = \bigwedge E$ to have the property that the sublattice $[s, 1]$ is Noetherian, where again E denotes the set of essential elements of L.

There is another motivation for studying these questions in the context of modular lattices, namely the papers of Albu and Smith [1], [2] where they show that the lattice viewpoint is both natural and helpful for the Hopkins–Levitzki Theorem and more particularly for its various generalizations in terms of torsion theories. After the Hopkins–Levitzki Theorem, Goodearl's Theorem is a natural candidate for such treatment.

1. Independent sets. Throughout this paper L will denote a modular lattice with least element 0 and greatest element 1, i.e. $0 \le a \le 1$ for all $a \in L$. Let x and y be elements in L with $x \le y$. Then we define $y/x = \{a \in L : x \le a \le y\}$. Recall that because L is modular, $(a \vee b)/b \simeq a/(a \wedge b)$, for all a, b in L. This fact will be used repeatedly in the sequel.

A finite non-empty subset $\{x_1, \ldots, x_n\}$ of L is called *join-independent* if

$$x_i \neq 0 \quad \text{and} \quad x_i \wedge (x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n) = 0$$

for all $1 \le i \le n$. An arbitrary non-empty subset X of L is called *join-independent* if every finite non-empty subset of X is join-independent.

LEMMA 1.1 (see [10, Prop. 1.5.1 and 1.5.2]). *Let L be a modular lattice. The following statements are equivalent for a finite non-empty subset $X = \{x_1, \ldots, x_n\}$ of non-zero elements.*

(i) *X is join-independent.*
(ii) *$x_i \wedge (x_1 \vee \ldots \vee x_{i-1}) = 0$ for all $2 \le i \le n$.*
(iii) *$(\bigvee A) \wedge (\bigvee B) = 0$ for all disjoint non-empty subsets A and B of X.*

COROLLARY 1.2. *Let L be a modular lattice. Let a be a non-zero element and let X be a non-empty subset of L. Then $X \cup \{a\}$ is join-independent if and only if X is join-independent and $a \wedge (\bigvee S) = 0$ for every finite subset S of X.*

Proof. By Lemma 1.1. ∎

An element e in a lattice L is called *essential* if $e \wedge a \neq 0$ for every non-zero element a in L. The set of essential elements of L will be denoted by $\mathrm{E}(L)$. Note the following properties of essential elements.

LEMMA 1.3. *Let $\{b_1, \ldots, b_n\}$ be a finite subset of a modular lattice and let $a_i \in \mathrm{E}(b_i/0)$ $(1 \le i \le n)$. Then*

(i) $a_1 \wedge \ldots \wedge a_n \in \mathrm{E}((b_1 \wedge \ldots \wedge b_n)/0)$, *and*
(ii) *if* $\{b_1, \ldots, b_n\}$ *is join-independent then*

$$a_1 \vee \ldots \vee a_n \in \mathrm{E}((b_1 \vee \ldots\ldots \vee b_n)/0).$$

Moreover, *if* $a \leq b \leq c$ *in* L, $a \in \mathrm{E}(b/0)$ *and* $b \in \mathrm{E}(c/0)$ *then* $a \in \mathrm{E}(c/0)$.

Proof. By [8, Lemma 2 and Corollary 4]. ∎

An element u of a lattice L is called *uniform* if $u \neq 0$ and $a \wedge b \neq 0$ for all non-zero elements a and b in $u/0$, i.e. every non-zero element of $u/0$ is essential. For any set X, the cardinality of X will be denoted by $|X|$. The next result can be found in [8, Theorem 5].

LEMMA 1.4. *Suppose that a modular lattice* L *does not contain an infinite join-independent subset. Then there exists a positive integer* m *and uniform elements* u_i $(1 \leq i \leq m)$ *such that the set* $\{u_1, \ldots, u_m\}$ *is join-independent and the element* $u_1 \vee \ldots \vee u_m$ *is essential in* L. *Moreover*, $|X| \leq m$ *for any join-independent subset* X *of* L.

Let L be a lattice with no infinite join-independent sets. If m and n are positive integers and $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$ sets of uniform elements of L with $u_1 \vee \ldots \vee u_m$ and $v_1 \vee \ldots \vee v_n$ both essential then Lemma 1.4 gives $m = n$. The integer n will be called the *Goldie* (or *uniform*) *dimension of* L and we say that L has *finite Goldie dimension*. The Goldie dimension of L will be denoted by $u(L)$. If L does not have finite Goldie dimension then L contains an infinite join-dependent subset and we write $u(L) = \infty$.

LEMMA 1.5 (see [10, Lemma 1.6.4]). *Let* L *be a modular lattice and let* a *be an element of* L *such that* $a/0$ *and* $1/a$ *both have finite Goldie dimension. Then* L *has finite Goldie dimension.*

2. Semi-essential elements. Let L be a modular lattice (with least element 0 and greatest element 1). An element e of L is called *semi-essential* (*in* L) if $e \neq 0$ and e does not belong to an infinite independent subset X of L. (This idea can be found in [7], although the term "semi-essential" is not used.) We now list some properties of semi-essential elements.

LEMMA 2.1. *Let* L *be a modular lattice.*

(i) *Every essential element of* L *is semi-essential.*
(ii) *If* e *is semi-essential in* L *and* $e \leq f \in L$ *then* f *is semi-essential in* L.
(iii) *If* e *is a semi-essential element of* L *then* $e \wedge a$ *is a semi-essential element of* $a/0$ *for each* a *in* L.
(iv) *Let* e *be any semi-essential element of* L *and let* $b \in L$ *with* $e \wedge b = 0$. *Then* $b/0$ *has finite Goldie dimension.*

(v) *Let c be any element of L such that $1/c$ has finite Goldie dimension. Then c is semi-essential.*

Proof. (i) Clear.

(ii) By Corollary 1.2.

(iii) By [7, Prop. 3(ii)].

(iv) Suppose that $b \neq 0$. Let X be any join-independent subset of $b/0$. By Corollary 1.2, $X \cup \{e\}$ is a join-independent subset of L. Thus X is finite. It follows that $b/0$ has finite Goldie dimension.

(v) Let $S = \{x_1, \ldots, x_n\}$ be any finite subset of L such that $S \cup \{c\}$ is join-independent. Consider the subset $T = \{x_i \vee c : 1 \leq i \leq n\}$ of $1/c$. For each $i \geq 2$,

$$\begin{aligned}(x_i \vee c) \wedge [(x_1 \vee c) \vee \ldots \vee (x_{i-1} \vee c)] &= (x_i \vee c) \wedge (c \vee x_1 \vee \ldots \vee x_{i-1}) \\ &= c \vee [(x_i \vee c) \wedge (x_1 \vee \ldots \vee x_{i-1})] \\ &= c \vee 0 = c,\end{aligned}$$

by Lemma 1.1. Thus T is join-independent in $1/c$. It follows that n is bounded above by the Goldie dimension of $1/c$. Thus c is semi-essential. ∎

It will be convenient to consider 0 as a "special" semi-essential element. We say that 0 is *semi-essential* if L has finite Goldie dimension.

The converse of Lemma 2.1(v) is false in general. For example, if L is the lattice of $\mathbb{Z}$-submodules of the $\mathbb{Z}$-module $\mathbb{Q}$ and e denotes the submodule $\mathbb{Z}$ then $e \in \mathrm{E}(L)$ but $1/e$ does not have finite Goldie dimension.

LEMMA 2.2 (see [7, Theorem 1]). *Let L be a modular lattice and let e be any semi-essential element which is not essential. Then there exists a positive integer n and uniform elements u_i $(1 \leq i \leq n)$ of L such that $\{e, u_1, \ldots, u_n\}$ is join-independent and $e \vee u_1 \vee \ldots \vee u_n$ is essential in L.*

LEMMA 2.3 (see [7, Theorem 1]). *Let L be a modular lattice and let e be any semi-essential element which is not essential. Let m be a positive integer and let u_i $(1 \leq i \leq m)$ be uniform elements of L such that $\{e, u_1, \ldots, u_m\}$ is a join-independent set with $e \vee u_1 \vee \ldots \vee u_m$ essential in L. Then $|X| \leq m$ for any subset X of L such that $X \cup \{e\}$ is join-independent.*

Let e be any semi-essential element of L. We define $u_e(L) = 0$ if $e \in \mathrm{E}(L)$. Otherwise, we define $u_e(L) = n$ where n is the positive integer such that $\{e, u_1, \ldots, u_n\}$ (or $\{u_1, \ldots, u_n\}$, if $e = 0$) is join-independent for uniform elements u_i $(1 \leq i \leq n)$ of L with $e \vee u_1 \vee \ldots \vee u_m$ essential in L. By Lemma 2.3, $u_e(L)$ is well defined. We call $u_e(L)$ the *Goldie* (or *uniform*) *dimension of L relative to e*. If a is any element of L which is not semi-essential then we set $u_a(L) = \infty$.

THEOREM 2.4. *Let L be a modular lattice. Then the following statements are equivalent for an element e of L.*

(i) *The element e is semi-essential.*

(ii) *There exists a in L such that $e \wedge a = 0$, $e \vee a$ is essential in L and $a/0$ has finite Goldie dimension.*

(iii) *For every ascending chain $a_1 \leq a_2 \leq \ldots$ in $1/e$ there exists a positive integer n such that a_i is essential in $a_{i+1}/0$ for all $i \geq n$.*

In this case $u_e(L) = u(a/0)$.

Proof. (i)$\Rightarrow$(ii). By Lemma 2.2.

(ii)$\Rightarrow$(iii). Let $a_1 \leq a_2 \leq \ldots$ be any ascending chain in $1/e$. Consider the ascending chain $a \wedge a_1 \leq a \wedge a_2 \leq \ldots$ in $a/0$. Since $a/0$ has finite Goldie dimension it follows that there exists a positive integer n such that $a\wedge a_i i \in \mathrm{E}((a\wedge a_{i+1})/0)$ for all $i \geq n$. Let $i \geq n$. Now $e\wedge (a\wedge a_{i+1}) = 0$, so that Lemma 1.3 gives that $e \vee (a \wedge a_i) \in \mathrm{E}(e \vee (a \wedge a_{i+1})/0)$. But $e \vee (a \wedge a_{i+1}) = (e \vee a) \wedge a_{i+1} \in \mathrm{E}(a_{i+1}/0)$, again by Lemma 1.3. Since $e \vee (a \wedge a_i) \leq a_i$ it follows that $a_i \in \mathrm{E}(a_{i+1}/0)$. Thus $a_i \in \mathrm{E}(a_{i+1}/0)$ for all $i \geq n$.

(iii)$\Rightarrow$(i). Suppose that e is not semi-essential. Then there exists an infinite subset X of L such that $\{e\} \cup X$ is join-independent. Let $x_i \in X$ for all $i \geq 1$. Then $e \leq e \vee x_1 \leq e \vee x_1 \vee x_2 \leq \ldots$ and $e \vee x_1 \vee \ldots \vee x_j$ is not essential in $e \vee x_1 \vee \ldots \vee x_{j+1}$, for all $j \geq 1$. It follows that (iii) implies (i).

The last part follows by Lemmas 1.5 and 2.3. ∎

3. Chain conditions on essential elements. Again L is a modular lattice with least element 0 and greatest element 1. It is clear that L satisfies the ascending chain condition on essential elements (i.e. the sublattice $\mathrm{E}(L)$ is Noetherian) if and only if $1/e$ is Noetherian for every essential element e in L. In this case $\mathrm{E}(L)$ has Krull dimension [9, Corollaire 6] and $1/e$ has finite Goldie dimension for every essential element e in L. It is perhaps worth pointing out here that, in general, lattices with Krull dimension do not have finite Goldie dimension. Let $L(\mathbb{Z})$ denote the lattice of ideals of the ring $\mathbb{Z}$ of integers. Let $L^\circ(\mathbb{Z})$ denote the opposite lattice of $L(\mathbb{Z})$. Then $L^\circ(\mathbb{Z})$ is a complete modular lattice with Krull dimension 0 (i.e. $L^\circ(\mathbb{Z})$ is Artinian) and dual Krull dimension 1, but $L^\circ(\mathbb{Z})$ does not have finite Goldie dimension. For example, the set X of maximal ideals of $\mathbb{Z}$ is an infinite join-independent subset of $L^\circ(\mathbb{Z})$. It is proved in [1, Remarks 1.4(2)] that *an Artinian lattice L has finite Goldie dimension if and only if for each $a \in L$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in \mathrm{E}(L)$.*

The next result gives a further characterization of semi-essential elements of L in a particular case.

THEOREM 3.1. *Let L be a modular lattice and let a be an element of L such that $1/e$ has finite Goldie dimension for every $e \in \mathrm{E}(L)$ with $a \le e$. Then a is semi-essential if and only if $1/a$ has finite Goldie dimension.*

Proof. The sufficiency is proved in Lemma 2.1(v). Conversely, suppose that a is semi-essential. By Theorem 2.4 there exists $b \in L$ such that $a \wedge b = 0$, $a \vee b \in \mathrm{E}(L)$ and $b/0$ has finite Goldie dimension. Now $(a \vee b)/a \simeq b/0$. Thus $1/(a \vee b)$ and $(a \vee b)/a$ both have finite Goldie dimension. It follows that $1/a$ has finite dimension (Lemma 1.5). ■

Let L be a modular lattice with Krull dimension. Then L has dual Krull dimension, i.e. the opposite lattice L° has Krull dimension (see [9, Corollaire 6]). In this case we shall denote the Krull and dual Krull dimensions of L by $k(L)$ and $k^0(L)$, respectively. Next note that $\mathrm{E}(L)$ has Krull dimension if and only if $1/e$ has Krull dimension for all $e \in \mathrm{E}(L)$. For if $\mathrm{E}(L)$ has Krull dimension then so does the sublattice $1/e$ for any $e \in \mathrm{E}(L)$ by [9, Prop. 2]. On the other hand, if $e \in \mathrm{E}(L)$ and $1/e$ has Krull dimension then $1/e$ has dual Krull dimension, by [9, Corollaire 6]. Clearly $\mathrm{E}(L)$ has dual Krull dimension. By [9, Corollaire 6] again, $\mathrm{E}(L)$ has Krull dimension.

COROLLARY 3.2. *Let L be a modular lattice and let a be a semi-essential element of L such that $1/e$ has Krull dimension for all $e \in \mathrm{E}(L)$ with $a \le e$. Then $1/a$ has Krull dimension. In this case*,

$$k(1/a) \le 1 + \sup\{k(1/e) : a \le e \in \mathrm{E}(L)\},$$
$$k^0(1/a) \le \sup\{k^0(1/e) : a \le e \in \mathrm{E}(L)\}.$$

Proof. If a is essential then $1/a$ has Krull dimension, by hypothesis. If a is not essential then by Lemma 2.2 there exists a positive integer n and uniform elements u_i $(1 \le i \le n)$ such that the set $\{a, u_1, \ldots, u_n\}$ is join-independent and $b = a \vee u_1 \vee \ldots \vee u_n \in \mathrm{E}(L)$. By hypothesis, $1/b$ has Krull dimension.

Let $1 \le i \le n$. Let $0 \ne c \le u_i$. Let $d = a \vee u_1 \vee \ldots u_{i-1} \vee c \vee u_{i+1} \vee \ldots \vee u_n$. Then $a \le d$, $d \in \mathrm{E}(L)$ (Lemma 1.3) and $u_i/c \simeq b/d$. Thus u_i/c has (dual) Krull dimension for all $0 \ne c \le u_i$. It follows that $u_i/0$ has Krull dimension. Thus $u_i/0$ has Krull dimension for all $1 \le i \le n$ and hence b/a has Krull dimension, because $b/a \simeq (u_1 \vee \ldots \vee u_n)/0$. Since $1/b$ has Krull dimension, it follows that $1/a$ has Krull dimension by [10, Prop. 3.2.1].

The rest of the proof clearly follows. ■

Note that the hypothesis of Corollary 3.2 is satisfied if $\mathrm{E}(L)$ has Krull dimension. Note too that the converse of Corollary 3.2 is false in general, i.e. if b is an element of a lattice L such that $1/b$ has Krull dimension then it does not follow that b is semi-essential. Consider again the opposite lattice

$L^\circ(\mathbb{Z})$ of $L(\mathbb{Z})$. Note that $L^\circ(\mathbb{Z})$ is an Artinian lattice. For each maximal ideal m in $\mathbb{Z}$, $1/m$ has Krull dimension but m is not semi-essential.

Combining Theorem 3.1 and Corollary 3.2 we have at once:

COROLLARY 3.3. *Let L be a modular lattice and let a be an element of L such that $1/a$ has finite Goldie dimension and $1/e$ has Krull dimension for all $e \in \mathrm{E}(L)$ with $a \le e$. Then $1/a$ has Krull dimension.*

The next result can be found in [11, Prop. 3.1].

COROLLARY 3.4. *Let L be a modular lattice and let $a \in L$. Then $1/a$ is Noetherian if and only if a is semi-essential and $1/e$ is Noetherian for all $e \in \mathrm{E}(L)$ with $a \le e$.*

Proof. By Lemma 2.1, Theorem 3.1 and Corollary 3.2. ∎

4. Complete lattices. Throughout this section all lattices will be complete modular lattices. We shall investigate whether there is an analogue of Goodearl's Theorem mentioned in the introduction. In terms of lattices we have the following question: *Let L be a complete modular lattice such that $\mathrm{E}(L)$ is Noetherian and let $s = \bigwedge \mathrm{E}(L)$; is $1/s$ Noetherian*? Goodearl's Theorem [6, Prop. 3.6] asserts that this question has an affirmative answer for the complete modular lattice of submodules of an arbitrary module over an arbitrary ring. The next result is [11, Lemma 1.3], but we give its proof for completeness.

LEMMA 4.1. *Let L be a complete modular lattice and let X be a maximal independent subset of L. Suppose that $y(x) \in \mathrm{E}(x/0)$ for each $x \in X$. Let $Y = \{y(x) : x \in X\}$. Then $\bigvee Y \in \mathrm{E}(L)$.*

Proof. Suppose not. Let $0 \ne a \in L$ such that $a \wedge (\bigvee Y) = 0$. Then $a \wedge y(x) = 0$ for all $x \in X$, and hence $a \notin X$. Thus $X \cup \{a\}$ is not independent. By Corollary 1.2, there exists a finite subset F of X such that $a \wedge (\bigvee F) \ne 0$. Let $Y' = \{y(x) : x \in F\}$. Then $a \wedge (\bigvee Y') \ne 0$ (Lemma 1.3). But $a \wedge (\bigvee Y') \le a \wedge (\bigvee Y) = 0$, a contradiction. ∎

Let L be a (complete modular) lattice. A non-empty subset X of L will be called *completely join-independent* if $x \wedge (\bigvee (X \setminus \{x\})) = 0$ for each x in X. Clearly, completely join-independent sets are join-independent and finite join-independent sets are completely join-independent. However, let $L^\circ(\mathbb{Z})$ denote the opposite lattice of the lattice of ideals of the ring $\mathbb{Z}$ of integers. Let X denote the set of maximal ideals of $\mathbb{Z}$. Then in $L^\circ(\mathbb{Z})$, the set X is join-independent but not completely join-independent.

Recall that a lattice L is called *upper continuous* if

$$a \wedge (\textstyle\bigvee_I b_i) = \textstyle\bigvee_I (a \wedge b_i),$$

for any $a \in L$, any index set I and chain $\{b_i : i \in I\}$ in L. The lattice L will be called *weakly upper continuous* if $a \wedge (\bigvee_I b_i) = 0$ for any $a \in L$ and chain $\{b_i : i \in I\}$ in L such that $a \wedge b_i = 0$ for all $i \in I$. A lattice L is upper continuous if and only if $1/a$ is weakly upper continuous for all a in L. In particular, upper continuous lattices are weakly upper continuous but the converse is not true as the following example of Alkhazzi [3, Example 5.4.2] shows.

EXAMPLE 4.2. *Let L be the subset of $\mathbb{R}^2$ consisting of the points $(0,0)$, $(1,1)$, and (x,y) for all $0< x< 1$, $0< y< 1$. For any (a,b) and (c,d) in L, we define $(a,b) \le (c,d)$ if $a \le c$ and $b \le d$. Then L is a complete modular lattice with Goldie dimension 1 which is weakly upper continuous but not upper continuous.*

Proof. It can easily be checked that L is a complete modular lattice with greatest element $(1,1)$ and least element $(0,0)$. Moreover, $u(L) = 1$ so that L is clearly weakly upper continuous. Let $a = (3/4, 1/4)$ and let $C = \{(1/2, y) : 0 < y < 1\}$. Then C is a chain in L with $\bigvee C = (1,1)$ and so $a \wedge (\bigvee C) = a = (3/4, 1/4)$. However, $a \wedge (1/2, y) = (3/4, 1/4) \wedge (1/2, y) = (1/2, 1/4)$ for all $1/4 \le y \le 1$, so that $\bigvee\{a \wedge c : c \in C\} = (1/2, 1/4)$. Thus L is not upper continuous. ∎

LEMMA 4.3. *Let L be a weakly upper continuous modular lattice. Then any join-independent subset of L is completely join-independent.*

Proof. Let X be any join-independent subset of L with $|X| \ge 2$. Let $x \in X$ and let $X' = X \setminus \{x\}$. Let

$$\Omega = \{Y \subseteq X' : (\textstyle\bigvee Y) \wedge (\bigvee F) = 0 \text{ for all finite subsets } F \text{ of } X \setminus Y\}.$$

Because $|X| \ge 2$, Ω is non-empty. Let $\{C_i : i \in I\}$ be any chain in Ω and let $C = \bigcup_I C_i$. Then C is a subset of X'. Let $c_i = \bigvee C_i$ for each $i \in I$ and let $c = \bigvee C = \bigvee_I c_i$. Let F be any finite subset of $X \setminus C$. For each $i \in I$, $F \subseteq X \setminus C_i$ and hence $(\bigvee F) \wedge c_i = 0$. Now $\{c_i : i \in I\}$ is a chain in L, so we have $(\bigvee F) \wedge c = 0$. Thus C belongs to Ω. By Zorn's Lemma, Ω has a maximal member P (say).

Now $P \subseteq X'$. Suppose that $P \ne X'$. Let $y \in X' \setminus P$. Let $p = \bigvee P$. Let G be any finite subset of $X \setminus (P \cup \{y\})$. Then $y \wedge (\bigvee G) = 0$ and $p \wedge (y \vee (\bigvee G)) = 0$. By Lemma 1.1, $(\bigvee G) \wedge (p \vee y) = 0$. But $p \vee y = \bigvee (P \cup \{y\})$. Thus $P \cup \{y\}$ belongs to Ω, contradicting the choice of P. Therefore $P = X'$. It follows that $x \wedge (\bigvee X') = 0$, as required. ∎

There exist complete modular lattices which, although not weakly upper continuous, have finite Goldie dimension and hence every join-independent subset is completely join-independent, as the following example shows.

EXAMPLE 4.4. *For each positive integer $n \geq 2$ there exists an Artinian complete modular lattice with Goldie dimension n which is not weakly upper continuous.*

Proof. Let R be a semilocal Noetherian commutative domain with n maximal ideals. Let $L(R)$ denote the lattice of ideals of R and let $L^\circ(R)$ denote the opposite lattice of $L(R)$. Then $L^\circ(R)$ is an Artinian complete modular lattice. Since every proper ideal of R is contained in a maximal ideal, the maximal ideals are atoms in the lattice $L^\circ(R)$ and hence $u(L^\circ(R)) = n$. For any distinct maximal ideals p and q in R, $p+q^k = R$ for each positive integer k but $p+(\bigcap_{k\geq 1} q^k) = p$ by [13, p. 216, Cor. 1]. Thus in $L^\circ(R)$, $\{q^k : k \geq 1\}$ is a chain such that $p \wedge q^k = 0$ for all $k \geq 1$, but $p \wedge (\bigvee\{q^k : k \geq 1\}) = p$. Thus $L^\circ(R)$ is not weakly upper continuous. ∎

In Example 4.4, if in addition the ring R is a Dedekind domain (i.e. a principal ideal domain in this case [13, p. 278, Theorem 16]) then the lattice $L(R)$ is distributive (see, for example, [13, pp. 279–280]), and hence so too is $L^\circ(R)$. In particular, if $p_1, \ldots, p_n$ are n distinct primes in $\mathbb{Z}$ then the localization R of $\mathbb{Z}$ with respect to the set $\{p_1, \ldots, p_n\}$ is a commutative principal ideal domain with precisely n maximal ideals.

LEMMA 4.5. *Let L be a complete modular lattice. Let X be a completely join-independent subset of L. For each x in X, let $y(x) \in x/0$ with $y(x) \neq x$. Let $y = \bigvee\{y(x) : x \in X\}$. Then $\{x \vee y : x \in X\}$ is a completely join-independent subset of $1/y$.*

Proof. Let $x \in X$ and let $X' = X \setminus \{x\}$. Let $y' = \bigvee\{y(z) : z \in X'\}$. Then

$$x \wedge y = x \wedge (y(x) \vee y') = y(x) \vee (x \wedge y') = y(x) \neq x,$$

because $x \wedge y' \leq x \wedge (\bigvee X') = 0$. Thus $x \vee y \neq y$. Let $w = \bigvee X'$. Then

$$\begin{aligned}(x \vee y) \wedge (\textstyle\bigvee\{z \vee y : z \in X'\}) &= (x \vee y) \wedge (w \vee y) = y \vee (x \wedge (w \vee y)) \\ &= y \vee (y(x) \vee (x \wedge (w \vee y'))) \\ &= y \vee (x \wedge w) = y.\end{aligned}$$

Thus $x \vee y \neq y$ and $(x \vee y) \wedge (\bigvee\{z \vee y : z \in X'\}) = y$ for all x in X. Thus $\{x \vee y : x \in X\}$ is a completely join-independent subset of $1/y$. ∎

As we have already seen there exist complete modular lattices which have Krull dimension but which do not have finite Goldie dimension. However, we have the following result for completely join-independent sets.

THEOREM 4.6. *Let L be a complete modular lattice with Krull dimension. Then any completely join-independent subset of L is finite.*

Proof. Suppose the result is false. Let $\alpha \geq -1$ be the least ordinal such that there exists a complete modular lattice L such that L has Krull dimension α and L contains an infinite completely join-independent subset. Then $\alpha \geq 0$. Let X be an infinite completely join-independent subset of L. Let $\{x_n : n \geq 1\}$ be a countably infinite subset of X. Let $y_1 = x_1 \vee x_2 \vee x_3 \vee \ldots$, $y_2 = x_2 \vee x_4 \vee x_6 \vee \ldots$, $y_3 = x_4 \vee x_8 \vee x_{12} \vee \ldots$, and so on. Then $y_1 \geq y_2 \geq y_3 \geq \ldots$ There exists a positive integer n such that $k(y_n/y_{n+1}) < k(L)$. Now there exist elements z_i $(i \geq 1)$ in X such that $y_n = z_1 \vee z_2 \vee z_3 \vee \ldots$ and $y_{n+1} = z_2 \vee z_4 \vee z_6 \vee \ldots$ Note that $z_{2n+2} < z_{2n+1} \vee z_{2n+2}$ for all $n \geq 0$. By Lemma 4.5, $\{z_{2i+1} \vee z_{2i+2} \vee y_{n+1} : i \geq 0\}$ is an infinite completely join-independent subset of y_n/y_{n+1}, a contradiction. ■

We shall say that an element a of a lattice L has *property* $(*)$ if for every non-empty subset X of L such that the set $\{a\} \cup X$ (X in case $a = 0$) is join-independent and for every x in X, we have

$$x \wedge [a \vee (\bigvee(X \setminus \{x\}))] = 0.$$

Note that 0 has property $(*)$ if and only if every join-independent subset of L is completely join-independent. This is typical, as the following results shows.

LEMMA 4.7. *Let L be a complete modular lattice. Then an element a in L has property $(*)$ if and only if for every non-empty subset X of L such that the set $\{a\} \cup X$ is join-independent the set $\{x \vee a : x \in X\}$ is a completely join-independent subset of $1/a$.*

Proof. The necessity is clear. Conversely, suppose that $\{x \vee a : x \in X\}$ is completely join-independent in $1/a$. Let $x \in X$ and let $X' = X \setminus \{x\}$. Then $x \wedge [a \vee (\bigvee X')] \leq (x \vee a) \wedge [\bigvee\{x' \vee a : x' \in X'\}] = a$, and thus $x \wedge [a \vee (\bigvee X')] \leq x \wedge a = 0$. Thus a has property $(*)$. ■

It is clear that for any lattice L any semi-essential element a has property $(*)$. In certain circumstances, the converse is also true as the following result shows.

THEOREM 4.8. *Let L be a complete modular lattice. Let $s = \bigwedge \mathrm{E}(L)$ and let $a \in 1/s$ such that $1/e$ has finite Goldie dimension for every essential element e in L with $a \leq e$. Then the following statements are equivalent for a.*

(i) *a is semi-essential.*

(ii) *Every join-independent subset of L containing a is completely join-independent.*

(iii) *a has property $(*)$.*

In Theorem 4.8 if $a = 0$ then by "every join-independent subset of L containing a" we mean "every join-independent subset of L."

Proof. (i)⇒(ii)⇒(iii). Clear.

(iii)$\Rightarrow$(i). Let X be a maximal join-independent subset of L containing a. Let $Z = X \setminus \{a\}$. Let $x \in Z$. Then $x \wedge a = 0$ and hence $x \neq x \wedge s$. There exists $e \in \mathrm{E}(L)$ such that $x \neq x \wedge e$. Let $y(x) = x \wedge e$. Note that $y(x) \in \mathrm{E}(x/0)$ by Lemma 1.3. Let $Y = \{y(x) : x \in Z\}$. By Lemma 4.1, $t = a \vee (\bigvee Y) \in \mathrm{E}(L)$. Now $\{x \vee a : x \in Z\}$ is a completely join-independent subset of $1/a$ by Lemma 4.7 and hence $\{x \vee t : x \in Z\}$ is a (completely) join-independent subset of $1/t$ by Lemma 4.5. Hence Z is finite. It follows that a is semi-essential. ∎

Combining Theorem 4.8 with Corollary 3.4 we have at once:

Corollary 4.9. *Let L be a complete modular lattice, let $s = \bigwedge \mathrm{E}(L)$ and let $a \in 1/s$. Then $1/a$ is Noetherian if and only if a has property $(*)$ and $1/e$ is Noetherian for every $e \in \mathrm{E}(L)$ with $a \leq e$.* ∎

As special cases of Theorem 4.8 and Corollary 4.9, we have the following:

Corollary 4.10. *Let L be a complete modular lattice such that every join-independent subset is completely join-independent. Let $s = \bigwedge \mathrm{E}(L)$. Then*

(i) *$1/e$ has finite Goldie dimension for every essential element e in L if and only if $1/a$ has finite Goldie dimension for every $a \in 1/s$, and*

(ii) *$1/e$ is Noetherian for every essential element e in L if and only if $1/a$ is Noetherian for every $a \in 1/s$.*

Corollary 4.10 holds for (weakly) upper continuous lattices L by Lemma 4.3 and so is a lattice version of [6, Prop. 3.6] and [12, Theorem 2.1]. We now want to prove an analogue of Corollary 4.10 for distributive lattices L. Note that the opposite lattice of the lattice of ideals of $\mathbb{Z}$ is a distributive lattice for which not every join-independent subset is completely join-independent. However, for certain distributive lattices L, we can show that every element a of L has property $(*)$. First we note a result of Puczyłowski [11, Lemma 3.5].

Lemma 4.11. *Let L be a complete distributive lattice and let X be any join-independent subset of L. Let $a \in L$. Then $\{x \vee a : x \in X,\ x \not\leq a\}$ is a join-independent subset of $1/a$.*

Lemma 4.12. *Let L be a complete distributive lattice and let B be a non-empty subset of L such that $1/b$ has finite Goldie dimension for every $b \in B$. Let $c = \bigwedge B$. Then every element a of $1/c$ has property $(*)$.*

Proof. Let $a \in 1/c$. Let X be any subset of L such that $X \cup \{a\}$ is join-independent. Let $x \in X$ and let $X' = X \setminus \{x\}$. Let $b \in B$. By Lemma

4.11, the set $X'' = \{y \in X' : y \not\leq b\}$ is finite, so $b \vee (\bigvee X') = b \vee (\bigvee X'')$ and

$$\begin{aligned} x \wedge (a \vee (\textstyle\bigvee X')) &= (x \wedge a) \vee (x \wedge (\textstyle\bigvee X')) \leq x \wedge (b \vee (\textstyle\bigvee X'')) \\ &= (x \wedge b) \vee (x \wedge (\textstyle\bigvee X'')) = x \wedge b \leq b. \end{aligned}$$

Thus

$$x \wedge (a \vee (\textstyle\bigvee X')) \leq \textstyle\bigwedge B = c \leq a.$$

Now $x \wedge (a \vee (\bigvee X')) \leq x \wedge a = 0$. Thus a has property $(*)$. ∎

Note that in Corollary 4.12 we can weaken the hypothesis "$1/b$ has finite Goldie dimension" to "every join-independent subset of $1/b$ is completely join-independent" for every $b \in B$. Combining Theorems 3.1 and 4.8 and Corollaries 4.9 and 4.12 we have the following analogue of Corollary 4.10.

THEOREM 4.13. *Let L be complete distributive lattice and let $s = \bigwedge \mathrm{E}(L)$. Then*

(i) *$1/e$ has finite Goldie dimension for every $e \in \mathrm{E}(L)$ if and only if $1/a$ has finite Goldie dimension for every $a \in 1/s$, and*

(ii) *$1/e$ is Noetherian for every $e \in \mathrm{E}(L)$ if and only if $1/a$ is Noetherian for every $a \in 1/s$.*

Theorem 4.13(ii) is proved by Puczyłowski [11, Theorem 3.7] in case $a = s$, but his proof is completely different.

Let L be a lattice and let $s = \bigwedge \mathrm{E}(L)$. If $1/s$ has Krull dimension then so does $1/e$ for any $e \in \mathrm{E}(L)$. Now we show that the converse holds in certain situations.

THEOREM 4.14. *Let L be a complete modular lattice such that $\mathrm{E}(L)$ has Krull dimension. Let $s = \bigwedge \mathrm{E}(L)$ and suppose that s has property $(*)$. Then $1/s$ has Krull dimension. Moreover,*

$$k(1/s) \leq 1 + \sup\{k(1/e) : e \in \mathrm{E}(L)\},$$
$$k^\circ(1/s) = \sup\{k^\circ(1/e) : e \in \mathrm{E}(L)\}.$$

Proof. Let X be a maximal join-independent subset of L containing s. Let $X' = X \setminus \{s\}$. For each x in X' there exists $e \in \mathrm{E}(L)$ such that $x \neq x \wedge e$ and we set $y(x) = x \wedge e$. Then $x \neq y(x) \in \mathrm{E}(x/0)$ for all $x \in X'$. Let $y = \bigvee\{y(x) : x \in X'\}$. Then $t = y \vee s \in \mathrm{E}(L)$ (Lemma 4.1). Now $\{x \vee t : x \in X'\}$ is a completely join-independent subset of $1/t$ by Lemmas 4.5 and 4.7. By Theorem 4.6, $\{x \vee t : x \in X'\}$ is finite. Thus X is finite. Hence s is semi-essential and the result follows by Corollary 3.2. ∎

COROLLARY 4.15. *Let L be a complete modular lattice such that every independent subset is completely independent. Let $s = \bigwedge \mathrm{E}(L)$. Then $1/s$ has Krull dimension if and only if $\mathrm{E}(L)$ has Krull dimension.*

Proof. By Theorem 4.14. ∎

In particular, Corollary 4.15 applies to (weakly) upper continuous lattices. Note also that Theorem 4.14 applies to distributive lattices L such that $1/e$ has Krull dimension and finite Goldie dimension for every $e \in \mathrm{E}(L)$.

5. An example. In this section we shall give an example of a non-complete modular lattice L such that $\mathrm{E}(L)$ is Noetherian, $0 = \bigwedge \mathrm{E}(L)$ but L is not Noetherian. We do not know if there exists a complete modular lattice with these properties.

Let $\mathbb{Z}$ denote the ring of integers and let M be the free $\mathbb{Z}$-module of countably infinite rank. Each element of M is a countable sequence of integers with at most a finite number non-zero. For each $n \geq 1$, let e_n denote the element of M with nth component 1 and all other components 0 and let $L_n = \sum_{i \geq n} \mathbb{Z}e_i$. Let $L(M)$ denote the lattice of submodules of M. Let L denote the collection of submodules N of M such that either N is finitely generated or $N = K + L_n$ for some $n \geq 1$ and some finitely generated submodule K.

LEMMA 5.1. *With the above notation*, *L is a sublattice of $L(M)$. In particular*, *L is a modular lattice with least element* 0 *and greatest element M.*

P r o o f. Let X and Y belong to L. If X or Y is finitely generated then so too is $X \wedge Y = X \cap Y$. Also $X \vee Y = X + Y$ is finitely generated or has the form $K + L_n$ for some finitely generated submodule K and positive integer n. Suppose that neither X nor Y is finitely generated. Then there exist finitely generated submodules K_1 and K_2 of M and a positive integer n such that $X = K_1 \oplus L_n$ and $Y = K_2 \oplus L_n$. Clearly, $X \wedge Y = (K_1 \cap K_2) \oplus L_n$ and $X \vee Y = (K_1 + K_2) \oplus L_n$, so that $X \wedge Y \in L$ and $X \vee Y \in L$. The last part follows at once.

LEMMA 5.2. *With the above notation*, *$E \in \mathrm{E}(L)$ if and only if there exists a non-zero ideal A of $\mathbb{Z}$ and a positive integer n such that $E \supseteq Ae_1 \oplus \ldots \oplus Ae_n \oplus L_{n+1}$.*

P r o o f. If $E \supseteq Ae_1 \oplus \ldots \oplus Ae_n \oplus L_{n+1}$ for some non-zero ideal A and positive integer n then E is essential in $L(M)$ and hence $E \in \mathrm{E}(L)$. Conversely, suppose that E is essential in L. Clearly, E is not finitely generated. There exist an integer $n \geq 0$ and a finitely generated submodule K of $\mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$ such that $E = K \oplus L_{n+1}$. Since $E \in \mathrm{E}(L)$ it follows that $E \cap \mathbb{Z}e_i \in \mathrm{E}(\mathbb{Z}e_i)$ for each $1 \leq i \leq n$. Thus $Ae_1 \oplus \ldots \oplus Ae_n \subseteq K$ for some non-zero ideal A of $\mathbb{Z}$. ∎

THEOREM 5.3. *With the above notation*, *L is a modular lattice such that $\mathrm{E}(L)$ is Noetherian*, $0 = \bigwedge \mathrm{E}(L)$, *and L is not Noetherian.*

P r o o f. By Lemmas 5.1 and 5.2, L is a modular lattice with least element 0 and greatest element M such that $\mathrm{E}(L)$ is Noetherian. Now let $E_1 =$

$\mathbb{Z}2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$, $E_2 = \mathbb{Z}4 \oplus \mathbb{Z}2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$, $E_3 = \mathbb{Z}8 \oplus \mathbb{Z}4 \oplus \mathbb{Z}2 \oplus \mathbb{Z} \oplus \ldots$, and so on. Clearly each $E_n \in \mathrm{E}(L)$ (Lemma 5.2) and $\bigwedge_n E_n = \bigcap_n E_n = 0$. Finally, it is clear that L is not Noetherian. ■

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