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GENERALIZED COIL ENLARGEMENTS OF ALGEBRAS

BY

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Introduction. Let k be an algebraically closed field, and A be a basic and connected finite-dimensional k-algebra (associative, with identity). We are interested in the category $\operatorname{mod} A$ of finitely generated right A-modules. In [15] C. M. Ringel introduced the notion of a separating tubular family which exists, in particular, for all tame concealed algebras. Also in [15] C. M. Ringel introduced a notion of extension or coextension by branches using modules from a separating tubular family and he showed that this process preserves the existence of separating tubular families, so that the representation-infinite tilted algebras of Euclidean type and the tubular algebras also have such families. Separating tubular families may also occur in the module categories of wild algebras, for example for all wild canonical algebras [15].

In [2], [3] I. Assem and A. Skowroński introduced the notion of admissible operations which generalize that of branch extension or coextension. These operations allow one to define and describe particular components of the Auslander–Reiten quiver, called coils and multicoils, and further a class of algebras, called multicoil algebras. This class plays a fundamental role in the representation theory of polynomial growth strongly simply connected algebras established by A. Skowroński in [17]. One of the main purposes of the present paper is to introduce new admissible operations (ad 4) and (ad 4*), and a component obtained from a stable tube by a sequence of admissible operations in this larger sense will be called a generalized coil. We shall show that, for any generalized coil, there exists a triangular algebra (that is, an algebra having no oriented cycle in its ordinary quiver) having this generalized coil as a standard component of its Auslander–Reiten quiver.

In [5] I. Assem, A. Skowroński and B. Tomé generalized the notion of a separating tubular family as follows: a family of standard, pairwise orthogonal components $\mathcal{T} = (\mathcal{T}_i)_{i\in I}$ of the Auslander–Reiten quiver of A will be called a weakly separating family if the indecomposable modules not in $\mathcal{T}$

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split into two classes $\mathcal{P}$ and $\mathcal{Q}$ such that there is no non-zero morphism from $\mathcal{Q}$ to $\mathcal{P}$, from $\mathcal{Q}$ to $\mathcal{T}$, or from $\mathcal{T}$ to $\mathcal{P}$, while any non-zero morphism from $\mathcal{P}$ to $\mathcal{Q}$ factors through the additive closure of $\mathcal{T}$. They further defined a coil enlargement of an algebra A using modules from $\mathcal{T}$, described its module category and proved criteria for tameness of a coil enlargement of a tame concealed algebra.

Given a weakly separating family $\mathcal{T}$ in the module category $\operatorname{mod} A$, we say that an algebra B is a generalized coil enlargement of the algebra A using modules from $\mathcal{T}$ if B is obtained from A by an iteration of admissible operations of types (ad 1), (ad 2), (ad 3), (ad 4), (ad 1*), (ad 2*), (ad 3*), (ad 4*) performed either on a stable tube of $\mathcal{T}$, or on a generalized coil obtained from a stable tube of $\mathcal{T}$ by means of the operations done so far. We also define numerical invariants c_B^- and c_B^+ (see [5]) which count respectively the number of corays and rays inserted in the tubes of $\mathcal{T}$ by this sequence of admissible operations.

The aim of the present paper is to give a general description of the module category of a generalized coil enlargement of an algebra. If, in particular, A is a tame concealed algebra and $\mathcal{T}$ is its unique $\mathbb{P}_1(k)$-family of stable tubes, and B is a generalized coil enlargement of A using modules from $\mathcal{T}$, we obtain handy criteria allowing one to verify whether or not B is tame. Namely, B admits a convex subcategory B^- which is a tubular coextension of A and a convex subcategory B^+ which is a tubular extension of A. Then B is tame if and only if B^- and B^+ are tame, or if and only if the Tits form of B is weakly non-negative. Following [13] we also give some homological properties of generalized coil enlargements of tame concealed algebras.

In the last part of this paper we show how to iterate this process to obtain the tame generalized coil enlargements of a tame concealed algebra. We call these algebras tame iterated generalized coil enlargements, and we give a description of their module categories. Additionally, generalizing the definition given in [18] (see also [14]) we say that an algebra A has acceptable projectives if each indecomposable projective A-module lies either in a preprojective component without injective modules or in a standard generalized coil, and the standard generalized coils containing projectives are ordered with respect to homomorphisms. The main result of this part is a generalization of Theorem 4.3 from [18] stating that an algebra A with acceptable projectives is a tame iterated generalized coil enlargement of a tame concealed algebra if and only if A is tame, or if and only if the Tits form of A is weakly non-negative.

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1. Notation and preliminary definitions. Throughout this paper, k will denote a fixed algebraically closed field. An *algebra* A will always mean a basic, connected, associative finite-dimensional k-algebra with identity. Thus there exists a connected bound quiver (Q_A, I_A) and an isomorphism $A \cong kQ_A/I_A$. Equivalently, $A = kQ_A/I_A$ may be considered as a k-linear category, whose object class A_0 is the set of points of Q_A, and whose set of morphisms $A(x, y)$ from x to y is the quotient of the k-vector space $kQ_A(x, y)$ of all formal linear combinations of paths in Q_A from x to y by the subspace $I_A(x, y) = kQ_A(x, y) \cap I_A$ (see [8]). A full subcategory C of A is called *convex* (in A) if any path in A with source and target in C lies entirely in C.

By an A-*module* we mean a finitely generated right A-module. We denote by $\operatorname{mod} A$ the category of A-modules and by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ consisting of a complete set of representatives of the isomorphism classes of indecomposable A-modules. For a full subcategory C of $\operatorname{mod} A$, we denote by $\operatorname{add} C$ the additive full subcategory of $\operatorname{mod} A$ consisting of the direct sums of indecomposable direct summands of the objects in C. For two full subcategories C, C' of $\operatorname{mod} A$, the notation $\operatorname{Hom}_A(C, C') = 0$ means that $\operatorname{Hom}_A(M, M') = 0$ for all M in C and M' in C'.

Recall that the *Auslander–Reiten quiver* Γ_A of an algebra A is the translation quiver whose vertices are the members of $\operatorname{ind} A$, the arrows are representatives of the irreducible morphisms in $\operatorname{ind} A$ and the translation is the Auslander–Reiten translation $\tau_A = D\,\mathrm{Tr}$. Let Γ be a component of Γ_A. We denote by $\operatorname{ind} \Gamma$ the full subcategory of $\operatorname{mod} A$ whose objects are the vertices of Γ, and we say that Γ is *standard* if $\operatorname{ind} \Gamma$ is equivalent to the mesh-category $k(\Gamma)$ of Γ (see [15]).

Given a standard component Γ of Γ_A, and an indecomposable module X in Γ, the *support* $\mathcal{S}(X)$ of the functor $\operatorname{Hom}_A(X, -)|_\Gamma$ is the k-linear category defined as follows [4]. Let $\mathcal{H}_X$ denote the full subcategory of Γ consisting of the indecomposable modules M in Γ such that $\operatorname{Hom}_A(X, M) \neq 0$, and $\mathcal{I}_X$ denote the ideal of $\mathcal{H}_X$ consisting of the morphisms $f : M \to N$ (with M, N in $\mathcal{H}_X$) such that $\operatorname{Hom}_A(X, f) = 0$. We define $\mathcal{S}(X)$ to be the quotient category $\mathcal{H}_X/\mathcal{I}_X$. We usually identify the k-linear category $\mathcal{S}(X)$ with its quiver.

A translation quiver Γ is called a *tube* [10], [15] if it contains a cyclic path and if its underlying topological space is homeomorphic to $S^1 \times \mathbb{R}^+$. A tube has only two types of arrows: arrows pointing to infinity and arrows pointing to the mouth. Tubes containing neither projective vertices nor injective vertices are called *stable*. A stable tube is of the form $\mathbb{Z}A_\infty/(\tau^r)$, $r \geq 1$, and is said to be of *rank* r. Recall that a path $x_0 \to x_1 \to \ldots \to x_r$ in Γ is called *sectional* if $x_{i-2} \neq \tau x_i$ for each i, $2 \leq i \leq r$. If there exists a unique infinite sectional path in Γ starting at x (respectively, ending with x) it will be called a *ray* (respectively, a *coray*). It follows from [6] that the composition of morphisms lying on a sectional path in Γ_A is non-zero.

A *path* in $\operatorname{mod} A$ is a sequence of non-zero non-isomorphisms

$$X_0 \to X_1 \to \ldots \to X_r,$$

where the X_i are indecomposable. Such a path is called a *cycle* if $X_0 \cong X_r$. An indecomposable A-module X is called *directing* if it does not lie on any cycle in $\operatorname{mod} A$.

The *one-point extension* of an algebra A by an A-module X is the matrix algebra

$$A[X] = \begin{bmatrix} A & 0 \\ X & k \end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $A[X]$ contains Q_A as a convex subquiver and there is an additional (extension) point which is a source. The $A[X]$-modules are usually identified with the triples (V, M, φ), where V is a k-vector space, M an A-module and $\varphi : V \to \operatorname{Hom}_A(X, M)$ is a k-linear map. An $A[X]$-linear map $(V, M, \varphi) \to (V', M', \varphi')$ is then identified with a pair (f, g), where $f : V \to V'$ is k-linear, $g : M \to M'$ is A-linear and $\varphi' f = \operatorname{Hom}_A(X, g)\varphi$. One defines dually the *one-point coextension* $[X]A$ of A by X (see [15]).

Following [9], we say that an algebra A is *tame* if, for any dimension d, there exists a finite number of $k[X]$-A-bimodules M_i, $1 \le i \le n_d$, which are finitely generated and free as left $k[X]$-modules, and all but finitely many isomorphism classes of indecomposable A-modules of dimension d are of the form

$$k[X]/(X - \lambda) \otimes_{k[X]} M_i$$

for some $\lambda \in k$ and some i. Let $\mu_A(d)$ be the least number of bimodules M_i such that the above conditions for d are satisfied. Then A is called of *polynomial growth* (respectively, *linear growth, domestic*) if there is a positive integer m such that $\mu_A(d) \le d^m$ (respectively, $\mu_A(d) \le md$, $\mu_A(d) \le m$) for all $d \ge 1$ (see [16]).

For each vertex $x \in (Q_A)_0$, where $(Q_A)_0$ is the set of vertices of Q_A, we denote by S_x the corresponding simple A-module, and by P_x (respectively, I_x) the projective cover (respectively, the injective envelope) of S_x. The *dimension vector* of a module M is the vector

$$\underline{\dim}\, M = (\dim_k \operatorname{Hom}_A(P_x, M))_{x \in (Q_A)_0}.$$

The *support* $\operatorname{Supp}(d)$ of a vector $d = (d_x)_{x \in (Q_A)_0}$ is the full subcategory of A with the objects $\{x \in (Q_A)_0 \mid d_x \ne 0\}$. The *support* $\operatorname{Supp}(M)$ of a module M is the support of its dimension vector $\underline{\dim}\, M$. A module M is called *sincere* if its support is equal to A.

Recall that, if $A = kQ_A/I_A$, then the *Tits form* q_A of A is the integral quadratic form $q_A : \mathbb{Z}^n \to \mathbb{Z}, n = |(Q_A)_0|$, defined by

$$q_A(x) = \sum_{i\in(Q_A)_0} x_i^2 - \sum_{(i\to j)\in(Q_A)_1} x_i x_j + \sum_{i,j\in(Q_A)_0} r(i,j) x_i x_j,$$

where $r(i,j)$ is the cardinality of $\mathcal{R} \cap I(i,j)$ for a minimal set of generators $\mathcal{R} \subset \bigcup_{i,j\in(Q_A)_0} I(i,j)$ of the ideal I_A (see [7]). A quadratic form q_A is called *weakly non-negative* if $q_A(x) \geq 0$ whenever x has non-negative coordinates. We denote by $(-,-)_A$ the symmetric bilinear form associated with q_A.

Assume that $(Q_A)_0 = \{1,\ldots,n\}$. The *Cartan matrix* C_A of A is the $n \times n$ matrix whose ij-entry is $\dim_k \mathrm{Hom}_A(P_i, P_j)$. If the global dimension of A is finite (for instance, if A is triangular), then C_A is invertible and we can define the *Euler characteristic* on $\mathbb{Z}^{(Q_A)_0}$ by

$$\langle x, y\rangle_A = xC_A^{-t}y^t.$$

It has the following homological interpretation:

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_k \mathrm{Ext}_A^i(X,Y)$$

for any two A-modules X, Y. The *Euler form* χ_A of A is defined by $\chi_A(z) = \langle z, z\rangle_A$. If $\mathrm{gl.dim}\, A \leq 2$ then q_A and χ_A coincide [7].

2. Construction of standard components. In [2] I. Assem and A. Skowroński introduced admissible operations (ad 1), (ad 2), (ad 3), (ad 1*), (ad 2*), (ad 3*) (see also [3]). Among other things they described components of the Auslander–Reiten quiver, called coils. In this section, we shall introduce new admissible operations (ad 4), (ad 4*) and show that under reasonable assumptions, these preserve the standardness of components. Throughout this section, let A be an algebra, and Γ be a standard component of Γ_A.

(ad 4) Assume that $\mathcal{S}(X)$ consists of an infinite sectional path starting at X (then X is called an (ad 4)-*pivot*):

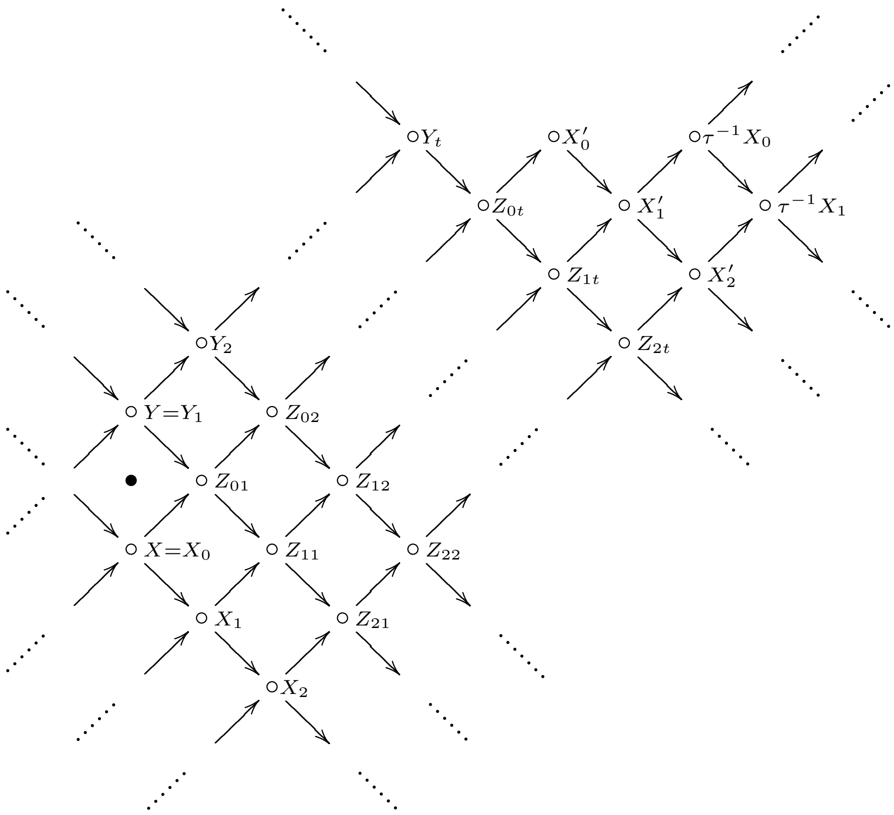
$$X = X_0 \to X_1 \to X_2 \to \ldots$$

Moreover, assume that $\mathrm{Supp}\,\mathrm{Hom}_A(Y,-)$ consists of a finite sectional path starting at Y:

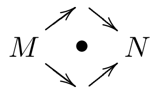
$$Y = Y_1 \to Y_2 \to \ldots \to Y_t$$

consisting of directing modules.

We define the modified algebra A' of A to be the one-point extension $A' = A[X \oplus Y]$, and the modified component Γ' of Γ to be



where



denotes that M is injective and N is projective, $Z_{ij} = \big(k, X_i \oplus Y_j, \binom{1}{1}\big)$ for $i \geq 0$, $1 \leq j \leq t$, $X'_i = (k, X_i, 1)$ and the morphisms are obvious ones. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = Y_{j-1}$ if $j \geq 2, P = Z_{01}$ is projective, $\tau' X'_0 = Y_t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ.

A finite sectional path $Y_1 \to Y_2 \to \ldots \to Y_t$ (occurring in (ad 4) and (ad 4^*)) consisting of arrows pointing to infinity (respectively, to the mouth) will be called a *finite ray* (respectively, a *finite coray*). The dual operation to (ad 4) will be denoted by (ad 4^*).

Note that a pivot X in (ad 4) (respectively, (ad 4^*)) is not necessarily injective (respectively, projective).

The integer $t \geq 1$ has the property that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \ldots$ in the inserted rectangle equals $t+1$. Just as for an admissible operation of type (ad 1), (ad 2), (ad 3), (ad 1*), (ad 2*) or (ad 3*) (see [5, 2.2]), we call t the *parameter* of the operation.

LEMMA 2.1. *In the case* (ad 4), *the component of $\Gamma_{A'}$ containing X* (*considered as an A'-module*) *is equal to Γ'. Further*, *if the subquiver of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$* (*if they exist*) *has the property that its connected component Γ^* containing X does not contain any of the $\tau_A^{-1} Y_{i-1}$, then Γ' is standard.*

Proof. The morphisms

$$Y_1 \to Y_2 \to \ldots \to Y_t \quad \text{and} \quad X_0 \to X_1 \to X_2 \to \ldots$$

in $\operatorname{mod} A$ remain irreducible in $\operatorname{mod} A'$ (see [3, 2.2]).

By construction P is the only indecomposable projective A'-module which is not an indecomposable projective A-module. Also, there are inclusion morphisms of X and Y as summands of $\operatorname{rad} P$, which are therefore irreducible in $\operatorname{mod} A'$. Moreover, the right minimal almost split morphisms ending at the X_i's and Y_i's in $\operatorname{mod} A$ remain so in $\operatorname{mod} A'$. Computing inductively Auslander–Reiten sequences, we prove, as in [3, 2.2], that Γ' is indeed the component of $\Gamma_{A'}$ containing X.

In our proof of the standardness of Γ' we must consider two cases. We present our proof in case when $\Gamma^* = \Gamma$, because the second case ($\Gamma^* \subset \Gamma$ and $\Gamma^* \neq \Gamma$) will follow by replacing Γ by Γ^*.

Let $\Phi : k(\Gamma) \to \operatorname{ind} \Gamma$ and $\Phi' : k(\Gamma') \to \operatorname{ind} \Gamma'$ denote the canonical functors. We want to show that Φ' is an equivalence, on the assumption that Φ is. Naturally Φ' is dense, so we must prove that it is full and faithful, that is, for all $M, N \in \operatorname{ind} \Gamma$, the functor Φ' induces an isomorphism $\operatorname{Hom}_{k(\Gamma')}(M,N) \xrightarrow{\sim} \operatorname{Hom}_{A'}(M,N)$.

Let $F : k(\Gamma) \to k(\Gamma')$ denote the k-linear embedding which is the identity on all objects and all arrows except arrows of the form $X_i \to \tau_A^{-1} X_{i-1}$, the image of which is the corresponding sectional path. Let $F' : \operatorname{ind} \Gamma \to \operatorname{ind} \Gamma'$ be the functor induced by F. We have a commutative diagram:

$$\begin{tikzcd}
k(\Gamma) \arrow[r, "F"] \arrow[d, "\Phi"] & k(\Gamma') \arrow[d, "\Phi'"] \\
\operatorname{ind} \Gamma \arrow[r, "F'"] & \operatorname{ind} \Gamma'
\end{tikzcd}$$

In particular, if $M, N \in \operatorname{ind} \Gamma$, then

$$\operatorname{Hom}_{k(\Gamma')}(M,N) = \operatorname{Hom}_{A'}(M,N).$$

If $M = Y_i$ (respectively, $N = Y_i$) and $\mathrm{Hom}_{A'}(M,N) \neq 0$, then $N = Z_{ij}$ (respectively, M is an A-module). Hence, if M or N is of the form Y_i, then Φ' induces the required isomorphism $\mathrm{Hom}_{k(\Gamma')}(M,N) \xrightarrow{\sim} \mathrm{Hom}_{A'}(M,N)$. We may thus assume that $M \neq Y_i$ and $N \neq Y_i$ for all $1 \leq i \leq t$.

Observe that the morphisms $Z_{ij} \to X_i'$ in $\operatorname{mod} A'$ induced by the corresponding sectional path in Γ' are surjective. Moreover, if $\tau_A^{-1}X_{i-1} \neq 0$, then the irreducible morphism $X_i \to \tau_A^{-1}X_{i-1}$ in $\operatorname{mod} A$ is surjective and hence so is the irreducible morphism $X_i' \to \tau_A^{-1}X_{i-1}$ in $\operatorname{mod} A'$.

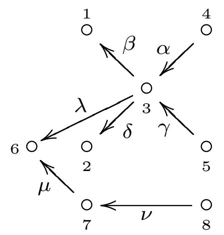
Let $M \notin \operatorname{ind}\Gamma$ and $N \in \operatorname{ind}\Gamma$. Then $M = Z_{ij}$ or $M = X_i'$ for some i,j. A non-zero morphism $f : M \to N$ in $\operatorname{mod} A'$ can always be written as $f = gh$, where $h : M \to \tau_A^{-1}X_{i-1}$ is induced by the corresponding sectional path in Γ'. The morphism h belongs to the image of Φ'. By commutativity of the above diagram we infer that the morphism g belongs to the image of the functor Φ', too. So Φ' induces a surjection $\mathrm{Hom}_{k(\Gamma')}(M,N) \to \mathrm{Hom}_{A'}(M,N)$. On the other hand, h is an epimorphism in $\operatorname{mod} A'$ (by the above observations) and F' is faithful. Consequently, the above surjection is an isomorphism.

Similarly, if $f : M \to N$ is non-zero morphism in $\operatorname{mod} A'$ with $M \in \operatorname{ind}\Gamma$ and $N \notin \operatorname{ind}\Gamma$, then f can be written as $f = uv$, for some $v : M \to X_i$ and $u : X_i \to N$ induced by the corresponding sectional paths. Since u is a monomorphism (now N is of the form Z_{ij} or X_i'), it follows from the commutativity of the above diagram that Φ' induces the required isomorphism $\mathrm{Hom}_{k(\Gamma')}(M,N) \xrightarrow{\sim} \mathrm{Hom}_{A'}(M,N)$.

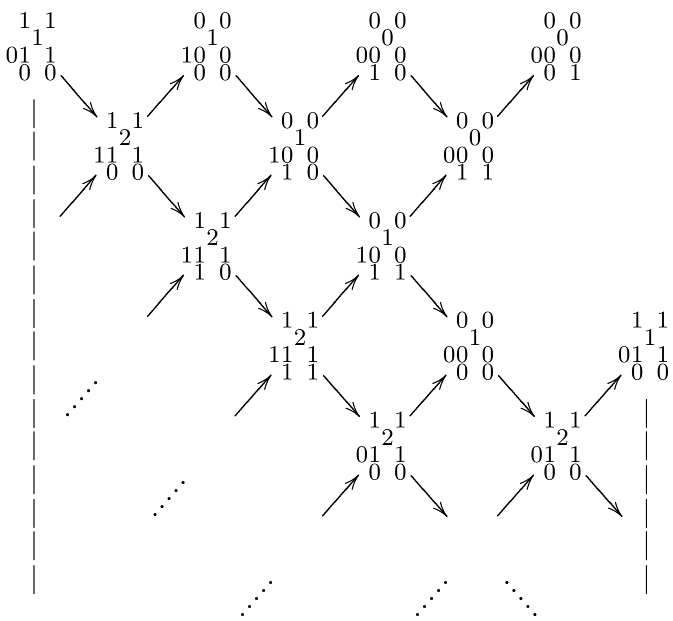
It remains to consider the case when $M, N \notin \operatorname{ind}\Gamma$. In this case, a non-zero morphism $f : M \to N$ in $\operatorname{mod} A'$ can be written as $f = pqr+s$, where $r : M \to \tau_A^{-1}X_{i-1}$ and $p : X_j \to N$ are induced by the corresponding sectional paths, $q : \tau_A^{-1}X_{i-1} \to X_j$ and s is zero or a composition of irreducible morphisms corresponding to arrows belonging to the support of the functor $\mathrm{Hom}_{k(\Gamma')}(Z_{01}, -)$. Since r, p and s belong to the image of Φ', and so does q (by the previous considerations), Φ' induces a surjection $\mathrm{Hom}_{k(\Gamma')}(M,N) \to \mathrm{Hom}_{A'}(M,N)$. Now s is non-zero in $\operatorname{mod} A'$ if and only if it is non-zero in $k(\Gamma')$. Similarly, since r is surjective and p is injective in $\operatorname{mod} A'$ and F is faithful, pqr is non-zero in $\operatorname{mod} A'$ if and only if it is non-zero in $k(\Gamma')$. So, any non-zero morphism $f : M \to N$ in $k(\Gamma')$ can be written as $f = pqr + s$ with r, q, p, s as above. Thus $\Phi'(f) = 0$ implies $0 \neq \Phi'(s) = -\Phi'(pqr)$. But s does not factor through modules in Γ, while q does. This contradiction shows that Φ' induces an isomorphism $\mathrm{Hom}_{k(\Gamma')}(M,N) \xrightarrow{\sim} \mathrm{Hom}_{A'}(M,N)$. The proof is now complete.

As we are going to show, a new admissible operation (ad 4) (or (ad 4*)) gives two possible shapes of the modified component Γ' depending on the position of the finite sectional path $Y_1 \to Y_2 \to \ldots \to Y_t$ in Γ.

EXAMPLE 2.2. Consider the algebra A given by the quiver

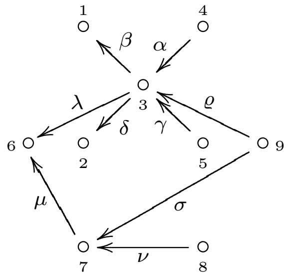


bound by $\alpha\lambda = 0, \gamma\lambda = 0$. The Auslander–Reiten quiver Γ_A has a standard component which is a tube of the form (see [4])

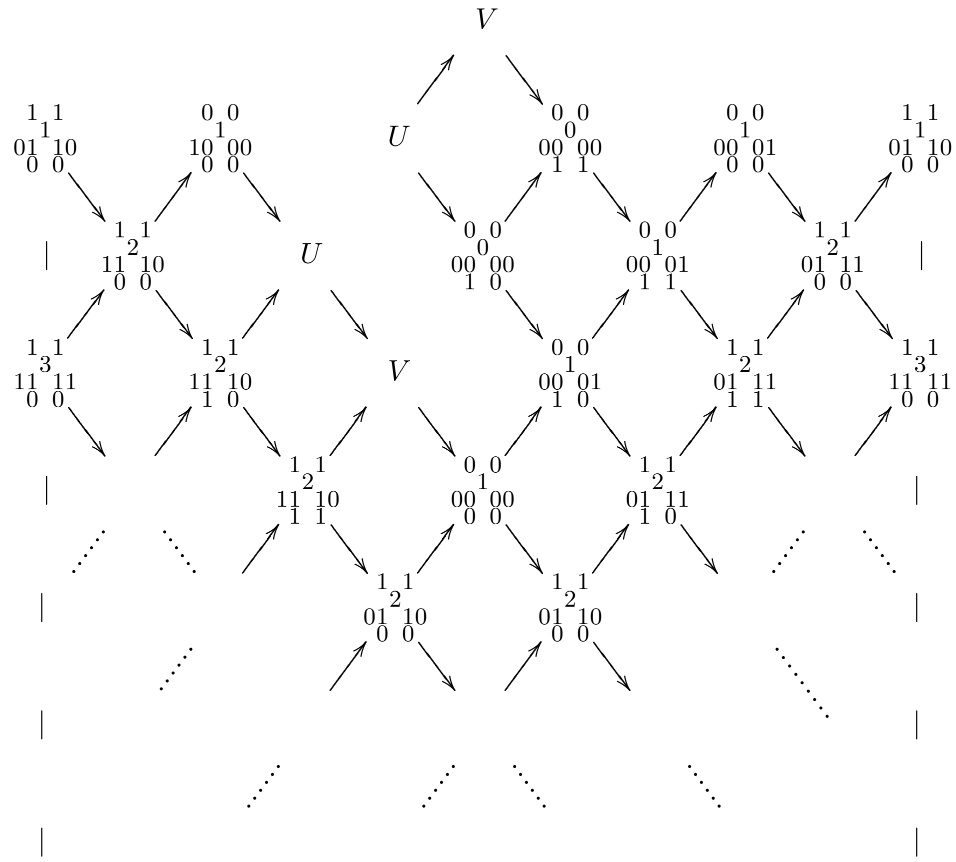


where the indecomposables are represented by their dimension vectors and one identifies along the vertical dotted lines to form the tube.

We can apply (ad 4) with pivot the idecomposable A-module X with dimension vector $\begin{smallmatrix} 0 & & 0\\ & 1 & \\ 00 & & 0\\ 0 & & 0\end{smallmatrix}$ and with a finite sectional path $Y_1 \to Y_2$, where Y_1, Y_2 are the A-modules with the dimension vectors respectively $\begin{smallmatrix} 0 & & 0\\ & 0 & \\ 00 & & 0\\ 1 & & 0\end{smallmatrix}$, $\begin{smallmatrix} 0 & & 0\\ & 0 & \\ 00 & & 0\\ 1 & & 1\end{smallmatrix}$, or with the dimension vectors $\begin{smallmatrix} 0 & & 0\\ & 0 & \\ 00 & & 0\\ 1 & & 1\end{smallmatrix}$ and $\begin{smallmatrix} 0 & & 0\\ & 0 & \\ 00 & & 0\\ 0 & & 1\end{smallmatrix}$. In the first case, the modified algebra $A_1 = A[X \oplus Y]$ is given by the quiver

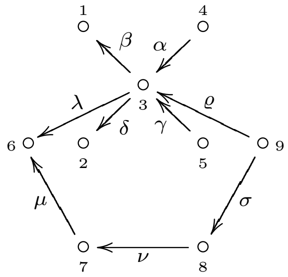


bound by $\alpha\lambda = 0, \gamma\lambda = 0, \varrho\lambda = 0, \sigma\mu = 0, \varrho\beta = 0, \varrho\delta = 0$. The Auslander–Reiten quiver Γ_{A_1} has a standard component which is the modified component Γ_1 of Γ, of the form

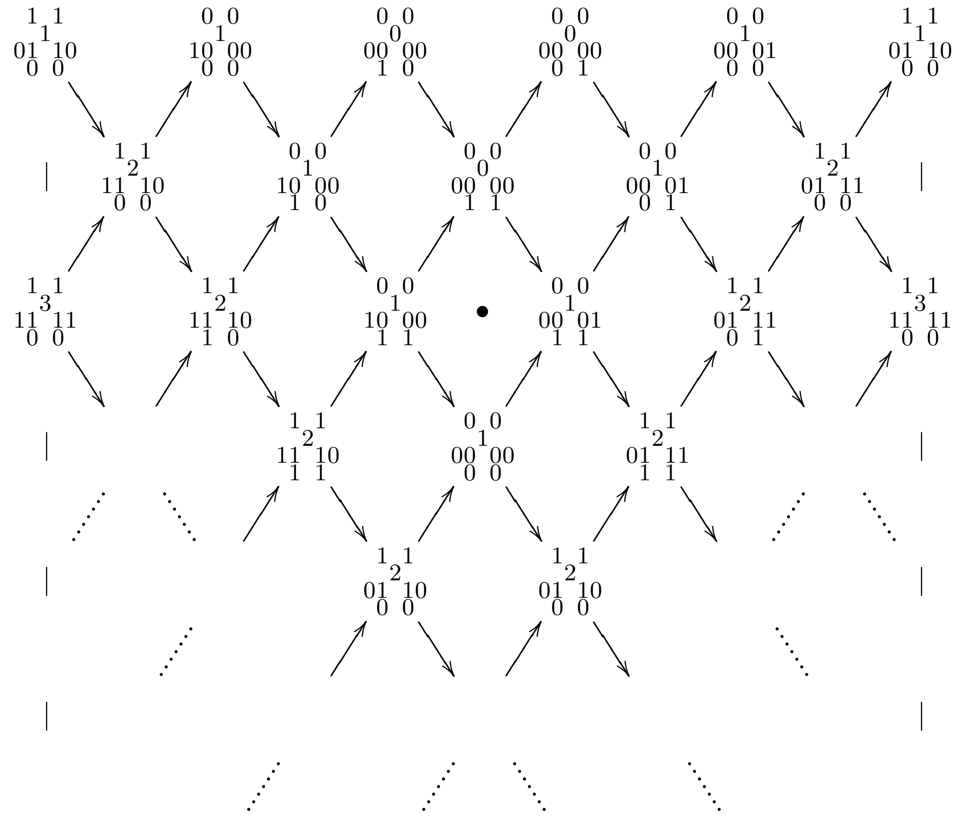


where $U = \begin{smallmatrix} 0\;0 \\ 1 \\ 10\;00 \\ 1\;0 \end{smallmatrix}$, $V = \begin{smallmatrix} 0\;0 \\ 1 \\ 10\;00 \\ 1\;1 \end{smallmatrix}$, and where we identify the two copies with dimension vector U and also the two copies with dimension vector V.

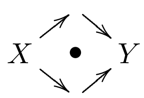
In the second case, the modified algebra $A_2 = A[X \oplus Y]$ is given by the quiver



bound by $\alpha\lambda = 0, \gamma\lambda = 0, \varrho\lambda = 0, \varrho\beta = 0, \varrho\delta = 0, \sigma\nu\mu = 0$. The Auslander–Reiten quiver Γ_{A_2} has a standard component which is the modified component Γ_2 of Γ, of the form



where



denotes that X is injective and Y is projective.

Let Γ be a component obtained from a stable tube $\mathcal{T}$ by an admissible operation of type (ad 1), (ad 2), (ad 3), (ad 1*), (ad 2*) or (ad 3*). It is known that in this case the fundamental group $\pi_1(\mathcal{T})$ does not change, namely $\pi_1(\Gamma) = \pi_1(\mathcal{T}) = \mathbb{Z}$. It is easily seen that if Γ' is a component obtained from $\mathcal{T}$ by an admissible operation of type (ad 4) or (ad 4*), then $\pi_1(\Gamma') = \mathbb{Z} \star \mathbb{Z}$ is the non-commutative, free group with two generators. As we can see in the above example the reason lies in the appearance of a hole and a Möbius strip on the periphery of the component Γ' or of a hole (depending on occurrence of a finite ray or a finite coray).

3. Weakly separating families of generalized coils. In this section, we recall the definition of weakly separating families which was introduced in [5]. We shall introduce generalized coil enlargements as a straightforward generalization of the definition of coil enlargements in [5].

DEFINITION 3.1. Let A be an algebra. A family $\mathcal{T} = (\mathcal{T}_i)_{i\in I}$ of components of Γ_A is called a *weakly separating family* in $\operatorname{mod} A$ if the idecomposable A-modules not in $\mathcal{T}$ split into two classes $\mathcal{P}$ and $\mathcal{Q}$ such that:

(i) The components $(\mathcal{T}_i)_{i\in I}$ are standard and pairwise orthogonal.
(ii) $\operatorname{Hom}_A(\mathcal{Q},\mathcal{P}) = \operatorname{Hom}_A(\mathcal{Q},\mathcal{T}) = \operatorname{Hom}_A(\mathcal{T},\mathcal{P}) = 0$.
(iii) Any morphism from $\mathcal{P}$ to $\mathcal{Q}$ factors through $\operatorname{add}\mathcal{T}$.

LEMMA 3.2. *Let A be an algebra, and $\mathcal{T}$ be a weakly separating family in* $\operatorname{mod} A$, *separating $\mathcal{P}$ from $\mathcal{Q}$. Then $\mathcal{P}$ and $\mathcal{Q}$ are uniquely determined by $\mathcal{T}$.*

Proof. See [5, 2.1].

DEFINITION 3.3. A translation quiver Γ is called a *generalized coil* if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ such that Γ_0 is a stable tube and, for each $0 \le i < m$, Γ_{i+1} is obtained from Γ_i by an admissible operation of type (ad 1), (ad 2), (ad 3), (ad 4) or (ad 1*), (ad 2*), (ad 3*), (ad 4*).

PROPOSITION 3.4. *Let Γ be a generalized coil. There exists a triangular algebra A such that Γ is a standard component of Γ_A.*

Proof. Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ be a sequence of translation quivers as in the Definition 3.3. Naturally, there exists a tame hereditary algebra B having the stable tube Γ_0 as a standard component. Inductively, we construct a sequence of algebras $B = A_0, A_1, \ldots, A_m = A$ such that A_{i+1} is obtained from A_i by the admissible operation of type (ad 1), (ad 2), (ad 3), (ad 4) or their duals (ad 1*), (ad 2*), (ad 3*), (ad 4*) with pivot in Γ_i such that the component of $\Gamma_{A_{i+1}}$ containing the pivot is Γ_{i+1}. It is easily seen that the condition for standardness in Lemma 2.1 is satisfied at each step. This shows that Γ is a standard component of Γ_A. The triangularity of the algebra A

follows from the fact that A is obtained from a tame hereditary algebra by a sequence of one-point extensions and coextensions.

Definition 3.5. Let A be an algebra, and $\mathcal{T}$ be a weakly separating family of stable tubes of Γ_A. An algebra B is called a *generalized coil enlargement* of A using modules from $\mathcal{T}$ if there is a finite sequence of algebras $A = A_0, A_1, \ldots, A_m = B$ such that, for each $0 \leq j < m$, A_{j+1} is obtained from A_j by an admissible operation of type (ad 1), (ad 2), (ad 3), (ad 4) or one of their duals with pivot either on a stable tube of $\mathcal{T}$ or on a generalized coil of Γ_{A_j}, obtained from a stable tube of $\mathcal{T}$ by means of the sequence of admissible operations done so far. The sequence $A = A_0, A_1, \ldots, A_m = B$ is then called an *admissible sequence*.

Definition 3.6. Let B be a generalized coil enlargement of A using modules from the weakly separating family $\mathcal{T} = (\mathcal{T}_i)_{i\in I}$ of stable tubes. The *generalized coil type* $c_B = (c_B^-, c_B^+)$ of B is a pair of functions $c_B^-, c_B^+ : I \to \mathbb{N}$ defined by induction on $0 \leq j < m$, where $A = A_0, A_1, \ldots, A_m = B$ is an admissible sequence.

(i) $c_A = c_0 = (c_0^-, c_0^+)$ is the pair of functions $c_0^- = c_0^+$ such that, for each $i \in I$, the common value of $c_0^-(i)$ and $c_0^+(i)$ is the rank of the stable tube $\mathcal{T}_i$.

(ii) Assume $c_{A_{j-1}} = c_{j-1} = (c_{j-1}^-, c_{j-1}^+)$ is known, and let t_j be the parameter of the admissible operation leading from A_{j-1} to A_j, then $c_{A_j} = c_j = (c_j^-, c_j^+)$ is the pair of functions defined by:

$$c_j^-(i) = \begin{cases} c_{j-1}^-(i) + t_j + 1 & \text{if the operation is (ad 1*), (ad 2*), (ad 3*) or} \\ & \text{(ad 4*) with pivot in the generalized coil of} \\ & \Gamma_{A_{j-1}} \text{ arising from } \mathcal{T}_i, \\ c_{j-1}^-(i) & \text{otherwise,} \end{cases}$$

and

$$c_j^+(i) = \begin{cases} c_{j-1}^+(i) + t_j + 1 & \text{if the operation is (ad 1), (ad 2), (ad 3) or} \\ & \text{(ad 4) with pivot in the generalized coil of} \\ & \Gamma_{A_{j-1}} \text{ arising from } \mathcal{T}_i, \\ c_{j-1}^+(i) & \text{otherwise.} \end{cases}$$

It is easy to see that the generalized coil type of a generalized coil enlargement B of A does not depend on the sequence of admissible operations leading from A to B since, for each $i \in I$, the integers $c_B^-(i)$ and $c_B^+(i)$ measure the rank of $\mathcal{T}_i$ plus, respectively, the total numbers of corays and rays inserted in $\mathcal{T}_i$ by the sequence of admissible operations.

Note that in Example 2.2 we have $c_A = ((2,2,5),(2,2,2))$, $c_{A_1} = c_{A_2} = ((2,2,5),(2,2,5))$, $c_B = ((2,2,5),(2,2,7))$.

Let B be a generalized coil enlargement of an algebra A having a weakly separating family of stable tubes. Its type $c_B = (c_B^-, c_B^+)$ is called *tame* if each of the sequences c_B^- and c_B^+ equals up to permutation one of the following: $(p, q), 1 \le p \le q, (2, 2, r), 2 \le r, (2, 3, 3), (2, 3, 4), (2, 3, 5)$ or $(3, 3, 3), (2, 4, 4),$ $(2, 3, 6), (2, 2, 2, 2)$.

LEMMA 3.7. *Let A be an algebra, Γ be a standard component of Γ_A and $X \in \Gamma$ be an* (ad 4) *or* (ad 4*)*-pivot. Let A' be the modified algebra and Γ' be the modified component. Any indecomposable A'-module whose restriction to A has an indecomposable direct summand of the form X_i, for some $i \ge 0$, belongs to Γ'.*

Proof. Similar to the proof of [5, 2.4].

LEMMA 3.8. *Let A be an algebra with a family $\mathcal{T}$ of generalized coils weakly separating $\mathcal{P}$ from $\mathcal{Q}$, Γ be a generalized coil in $\mathcal{T}$ and X be an* (ad 4)*-pivot in Γ. Let $A' = A[X \oplus Y]$, where e denotes the extension point. Let $\mathcal{P}', \mathcal{T}', \mathcal{Q}'$ be the classes in* $\operatorname{ind} A'$ *defined as follows*:

(i) $\mathcal{P}' = \mathcal{P}$.

(ii) *$\mathcal{T}'$ consists of all indecomposables $M_{A'}$ such that $M_e = 0$ and $M = M|_A$ is in $(\mathcal{T} \setminus \Gamma) \cup \Gamma^*$ (where Γ^* is as in Lemma* 2.1), *or $M_e \ne 0$ and $M|_A$ has an indecomposable direct summand of the form X_i, for some $i \ge 0$.*

(iii) *$\mathcal{Q}'$ consists of all indecomposables $M_{A'}$ such that $M_e = 0$ and $M = M|_A$ is in $\mathcal{Q} \cup (\Gamma \setminus \Gamma^*)$, or $M = (k, 0, 0)$, or $M_e \ne 0$ and indecomposable direct summands of $M|_A$ belong either to the set $\{Y_1, Y_2, \ldots, Y_t\}$ or to the support of* $\operatorname{Hom}_A(X, -)|_{\mathcal{Q}}$.

Then $\operatorname{ind} A' = \mathcal{P}' \vee \mathcal{T}' \vee \mathcal{Q}'$, *and $\mathcal{T}'$ separates weakly $\mathcal{P}'$ from $\mathcal{Q}'$.*

Proof. Similar to the proof of [5, 2.5 and 2.6], involving additionally Lemmas 2.1 and 3.7.

THEOREM 3.9. *Let A be an algebra with a family $\mathcal{T}$ of stable tubes weakly separating $\mathcal{P}$ from $\mathcal{Q}$, and let B be a generalized coil enlargement of A using modules from $\mathcal{T}$. Then* $\operatorname{mod} B$ *has a family $\mathcal{T}'$ of generalized coils*, *weakly separating $\mathcal{P}'$ from $\mathcal{Q}'$.*

Proof. Let $A = A_0, A_1, \ldots, A_m = B$ be an admissible sequence. We prove the statement by induction on $0 \le i \le m$. It holds for $i = 0$ by the hypothesis on A. Assume that it holds for some $0 \le i < m$. That it also holds for $i + 1$ follows from [5, 2.7], and from Lemma 3.8 and its dual.

4. Maximal branch enlargements inside a generalized coil enlargement. Let A be an algebra with a weakly separating family $\mathcal{T}$ of stable tubes and B be a generalized coil enlargement of A using modules from $\mathcal{T}$. By Theorem 3.9, $\operatorname{ind} B = \mathcal{P}' \vee \mathcal{T}' \vee \mathcal{Q}'$, where $\mathcal{T}'$ is a family of generalized coils weakly separating $\mathcal{P}'$ from $\mathcal{Q}'$. We want to describe the full subcategories $\mathcal{P}'$ and $\mathcal{Q}'$ of $\operatorname{ind} B$. For this purpose, we will show (similarly to [5]) that the admissible sequence leading from A to B can be replaced by another admissible sequence, which consists of a block of operations of type (ad 1*), followed by a block of operations of types (ad 1), (ad 2), (ad 3), (ad 4), and the dual fact.

LEMMA 4.1. *Let A be an algebra with a weakly separating family $\mathcal{T}$ of generalized coils, and A' be obtained from A by applying one of the following pairs of admissible operations*: (ad 4) *and* (ad 1*), (ad 4) *and* (ad 2*), (ad 4) *and* (ad 3*), (ad 4) *and* (ad 4*), (ad 3) *and* (ad 4*), (ad 2) *and* (ad 4*) *or* (ad 1) *and* (ad 4*) *using modules from $\mathcal{T}$. Suppose that*:

(i) *the pivot of the second operation belongs to no ray, or coray, inserted by the first*; *and*

(ii) *in case the second operation is of type* (ad 3) *or* (ad 3*) *and is applied first to A, the pivot of the first still belongs to the family of generalized coils obtained from $\mathcal{T}$.*

Then, denoting by A'' the algebra obtained from A by applying the two operations in reverse order, we have $A' \cong A''$.

Proof. Since the admissible operations (ad 1), (ad 2), (ad 3), (ad 4) and their duals consist of one-point extensions or coextensions, it is easily seen that both algebras have the same bound quiver.

LEMMA 4.2. *Let A be an algebra with a weakly separating family $\mathcal{T}$ of generalized coils, and X be an indecomposable in a generalized coil of $\mathcal{T}$ which is an* (ad 1) *and* (ad 1*)*-pivot. Let c be the root of a branch of length t, and let K, K' be the branches constructed as follows: K consists of a root a, the branch in c and an arrow $a \to c$, while K' consists of a root b, the branch in c and an arrow $c \to b$. Then $[X \oplus Y](A[X,K]) \cong ([K',X]A)[X \oplus Y]$, where $Y = Y_1$ is the first module which belongs to a finite sectional path* (*as in definition of* (ad 4) *and* (ad 4*)).

Proof. Let A_1 be an algebra with a weakly separating family $\mathcal{T}$ of generalized coils, and X be an indecomposable in a generalized coil Γ of $\mathcal{T}$ which is an (ad 4*)-pivot. We assume for the time being that A_1 was obtained from an algebra A by applying r consecutive operations of type (ad 1), the first of which had X as a pivot, and these operations built up a branch K in A_1 with points $a, a_1, \ldots, a_s$, thus $A_1 = A[X,K]$ and X is an indecomposable $A[X,K]$-module. Let $A_2 = [X \oplus Y]A_1$, where $Y = Y_1$ as in the definition of (ad 4*), and let b denote the coextension point of A_2. The bound quiver of A_2 is of the following form: the point a is a source of two arrows, one of them goes to Q_A, and the other goes to $a_1 \in K$. The point b is a target of two arrows, one of them comes from Q_A, and the other comes from $a_1 \in K$, with $A_2(a,b)$ one-dimensional. Let A' be the convex subcategory of A_2 consisting of all points except a. Then $A' \cong [K', X]A$, where K' is the branch with points $b, a_1, \ldots, a_s$ and $A_2 = A'[X \oplus Y]$.

Because we have two possibilities for choosing a finite sectional path, we must choose in (ad 4) and (ad 4*) the corresponding cases. For example, if we have executed operations of type (ad 1) and (ad 4*) and in the last one we have chosen a finite ray then in operation (ad 4) which will come after (ad 1*) we must choose a finite coray. The claim of the lemma follows from the shape of the bound quiver of A'.

From the above lemma we see that the sequence of operations of type (ad 1) that builds up K followed by (ad 4*) (with pivot X) can be replaced by the sequence of operations of type (ad 1*) that builds up K' followed by (ad 4) (with pivot X).

THEOREM 4.3. *Let A be an algebra with a weakly separating family $\mathcal{T}$ of stable tubes, and B be a generalized coil enlargement of A using modules from $\mathcal{T}$. Then*:

(i) *There is a unique maximal branch coextension B^- of A which is a convex subcategory of B, and c_B^- is the coextension type of B^-.*

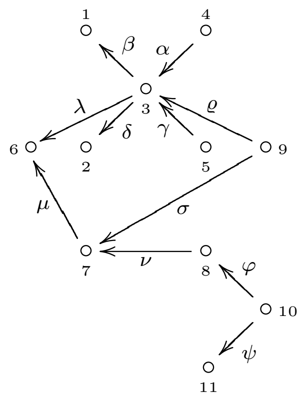
(ii) *There is a unique maximal branch extension B^+ of A which is a convex subcategory of B, and c_B^+ is the extension type of B^+.*

Proof. We will only prove (i), because the proof of (ii) is dual. We first prove that the admissible sequence leading from A to B can be replaced by another one consisting of block of operations of type (ad 1*) followed by a block of operations of type (ad 1), (ad 2), (ad 3), (ad 4). This is done by induction on the number n of operations in this admissible sequence. If $n = 0$, there is nothing to prove. Assume that $n > 0$, and let $A = A_0, A_1, \ldots, A_n = B$ be the corresponding sequence of algebras. We assume that the statement holds for A_{n-1}. If the nth operation is of type (ad 1), (ad 2), (ad 3) or (ad 4), there is nothing to show. If it is of type (ad 1*), (ad 2*) or (ad 3*) we are able, by Lemma 4.1 and [5, 3.5], to replace the given sequence by one of the required form. It remains to consider the case where the nth operation is of type (ad 4*). In the sequence there must be an operation of type (ad 1) that gives rise to the pivot X of (ad 4*). In this case we apply Lemma 4.1 as long as (ad 4*) will be after (ad 1) and then, using Lemma 4.2, replace these two operations by one of type (ad 1*) followed by one of type (ad 4). Using again Lemma 4.1 we are able to replace the given sequence by one of the required form.

Let now B^- be the branch coextension of A determined by the block of operations of type (ad 1*) in the new admissible sequence. Since the remaining block in the sequence consists of operations of types (ad 1), (ad 2), (ad 3), (ad 4), that is, one-point extensions or, in the case (ad 1), branch extensions, it is clear that B^- is a branch coextension of A maximal with respect to the property of being a convex subcategory of B. Furthermore, c_B^- is the coextension type of B^- because, if $\mathcal{T} = (\mathcal{T}_i)_{i\in I}$, then, for each $i \in I$, $c_B^-(i)$ equals the rank of $\mathcal{T}_i$ plus the number of corays inserted in $\mathcal{T}_i$ by the sequence of admissible operations of type (ad 1*).

The proof of uniqueness of B^- is identical as in [5, 3.5]. We shall repeat it here for the convenience of the reader. Let B^* be a branch coextension of A inside B. We first note that, by construction of B^-, all the coextension points of A inside B must belong to B^-. Now, if b is a point in B^*, it must belong to a coextension branch of A inside B, hence, since the root of this branch belongs to B^-, the point b itself must belong to B^- (by construction of the latter). This shows that B^* is contained in B^- and completes our proof.

Example 4.4. Let B be the algebra given by the quiver



bound by $\alpha\lambda = 0, \gamma\lambda = 0, \varrho\lambda = 0, \sigma\mu = 0, \varrho\beta = 0, \varrho\delta = 0, \varphi\nu\mu = 0$. Then the algebra B is obtained from A_1 by an admissible operation of type (ad 1) with pivot the indecomposable A_1-module with dimension vector $\begin{smallmatrix}0&&0\\&0&\\00&&00\\1&&1\end{smallmatrix}$, and with parameter $t = 1$. The algebra B^- coincides with the algebra A from Example 2.2. The algebra B^+ is given by the convex subcategory of B consisting of all the points except 6.

5. The module category of a generalized coil enlargement. We now complete the description of the module category of a generalized coil enlargement of an algebra having a weakly separating family of stable tubes. Let K be a branch in a (see [15]), and $A = kQ_A/I_A$ be any k-algebra and

$E \in \operatorname{mod} A$. Recall that the *branch extension* $A[E, K]$ by the branch K is constructed in the following way: to the one-point extension $A[E]$ with extension vertex w (that is, $\operatorname{rad} P_w = E$) we add the branch K by identifying the vertices a and w. If $E_1, \ldots, E_n \in \operatorname{mod} A$ and $K_1, \ldots, K_n$ is a set of branches, then the branch extension $A[E_i, K_i]_{i=1}^n$ is defined inductively as $A[E_i, K_i]_{i=1}^n = (A[E_i, K_i]_{i=1}^{n-1})[E_n, K_n]$. The concept of *branch coextension* is defined dually.

Following [15, 4.7] let

$$\mathcal{R}(K) = \{M \in \operatorname{ind} K \mid \langle l_K, \underline{\dim} M\rangle > 0\},$$
$$\mathcal{L}(K) = \{M \in \operatorname{ind} K \mid \langle \underline{\dim} M, l_K\rangle > 0\},$$

where K is a branch and l_K is the branch length function (see [15, 4.4]).

The main result of this section generalizes [5, 4.1].

THEOREM 5.1. *Let A be an algebra with a family $\mathcal{T} = (\mathcal{T}_i)_{i\in I}$ of stable tubes weakly separating $\mathcal{P}$ from $\mathcal{Q}$. Let B be a generalized coil enlargement of A using modules from $\mathcal{T}$, and $B^- = {}_{j=1}^{s}[K_j^*, E_j^*]A, B^+ = A[E_i, K_i]_{i-1}^{r}$. Let $\mathcal{P}'$ be the class of all indecomposable B-modules M such that either $M|_A$ is non-zero and in $\mathcal{P}$, or else* $\operatorname{Supp} M$ *is contained in some K_j^* and $M \in \mathcal{L}(K_j^*)$. Let $\mathcal{Q}'$ be the class of all indecomposable B-modules N such that either $N|_A$ is non-zero and in $\mathcal{Q}$, or else* $\operatorname{Supp} N$ *is contained in some K_i and $N \in \mathcal{R}(K_i)$. Then there exists a family $\mathcal{T}' = (\mathcal{T}_i')_{i\in I}$ of generalized coils in Γ_B such that* $\operatorname{ind} B = \mathcal{P}' \vee \mathcal{T}' \vee \mathcal{Q}'$, *$\mathcal{P}'$ consists of B^--modules, and $\mathcal{Q}'$ consists of B^+-modules.*

Proof. Following the proof of [5, 4.1], we have to use additionally two properties of the admissible operations of types (ad 4) and (ad 4*):

(i) The sequence of admissible operations leading from A to B can be replaced by a sequence consisting of a block of operations of type (ad 1*) followed by a block of operations of types (ad 1), (ad 2), (ad 3), (ad 4) (and its dual), a fact which follows from the proof of Theorem 4.3.

(ii) Theorem 3.9.

COROLLARY 5.2. *Let A be a tame concealed algebra and $\mathcal{T}$ be its unique $\mathbb{P}_1(k)$-family of stable tubes. Let B be a generalized coil enlargement of A using modules from $\mathcal{T}$. The following conditions are equivalent*:

(a) *B is tame*,
(b) *B^- and B^+ are tame*,
(c) *B is of polynomial growth*,
(d) *B is of linear growth*,
(e) *c_B is tame*,

(f) *The Tits form* q_B *of* B *is weakly non-negative.*

Moreover, B *is domestic if and only if both* B^- *and* B^+ *are tilted algebras of Euclidean type.*

Proof. (a)$\Rightarrow$(b). Clear, since B^- and B^+ are full convex subcategories of B.

(b)$\Rightarrow$(d). By [1, 2.3] and [11, 2.1], B^- and B^+ are both of linear growth. Applying Theorem 5.1, B itself is of linear growth.

(c)$\Rightarrow$(a). Trivial.

(d)$\Rightarrow$(c). Trivial.

(a)$\Rightarrow$(f). Follows from [12, 1.3].

(f)$\Rightarrow$(e). Because B^- and B^+ are full convex subcategories of B, each of the Tits forms q_{B^-} and q_{B^+} is weakly non-negative, and by [14, 3.3], c_B is tame.

(e)$\Rightarrow$(b). This follows from [15, 4.9, (2), and 5.2, (4)].

The last assertion follows from [4, 2.3], and [15, 4.9, (1)].

To end this section we describe some homological properties of generalized coil enlargements of tame concealed algebras. Analogous facts about coil enlargements of tame concealed algebras have been proved by J. A. de la Peña and A. Skowroński in [13] (Proposition 1.2, Corollaries 1.3 and 1.4). We formulate the relevant facts without proofs, because the proofs from [13] can be easily extended to the case of a generalized coil enlargement. The most important ingredient in these proofs is the existence of both a unique maximal tubular extension B^+ of A and unique maximal tubular coextension B^- of A (which follows from Theorem 4.3).

As we have shown, for a generalized coil enlargement B of A, the Auslander–Reiten quiver Γ_B of B contains a family $\mathcal{T}' = (\mathcal{T}'_\lambda)_{\lambda\in\mathbb{P}_1(k)}$ of generalized coils obtained from the family $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda\in\mathbb{P}_1(k)}$ of stable tubes of Γ_A by the corresponding sequence of admissible operations. If B is tame, we say that B is a *generalized coil algebra*.

Proposition 5.3. *Let* B *be a generalized coil enlargement of a tame concealed algebra* A *and* X *be an indecomposable* B*-module lying in a generalized coil* $\mathcal{T}'_\lambda$ *of* $\mathcal{T}'$. *Then*:

(i) $\mathrm{pd}_B\, X \le 2$ *and* $\mathrm{id}_B\, X \le 2$.

(ii) $\mathrm{Ext}^r_B(X,X) = 0$ *for* $r \ge 2$.

Corollary 5.4. *Let* B *be a generalized coil enlargement of a tame concealed algebra* A. *Then* $\mathrm{gl.dim}\, B \le 3$ *and for any indecomposable* B*-module* X, *either* $\mathrm{pd}_B\, X \le 2$ *or* $\mathrm{id}_B\, X \le 2$.

COROLLARY 5.5. *Let B be a generalized coil algebra and X be an indecomposable B-module. Then* $\mathrm{Ext}^r_B(X,X)=0$ *for any* $r\geq 2$.

6. Construction of the tame iterated generalized coil enlargements. In [18] B. Tomé described algebras obtained by iteration of the process given in [5] for defining the tame coil enlargements of a tame concealed algebra, and called the resulting class of algebras iterated coil enlargements. She also gave a complete description of their module categories.

In this section we show how to iterate the procedure described in Section 3 of this paper, in the spirit of [18] (compare also with [14]), in order to obtain the tame algebras. We call these algebras tame iterated generalized coil enlargements, and we give a description of their module categories.

Recall that if A is a domestic tubular extension of the tame concealed algebra, then its module category may be described as follows: $\mathrm{mod}\,A=\mathcal{P}\vee\mathcal{T}\vee\mathcal{Q}$, where $\mathcal{P}$ is a preprojective component, $\mathcal{Q}$ is a preinjective component and $\mathcal{T}$ is a tubular $\mathbb{P}_1(k)$-family separating $\mathcal{P}$ from $\mathcal{Q}$ (see [15, 4.9]).

If A is a tubular algebra, then we know from [15, 5.2] that A is non-domestic of polynomial growth (see [16, 3.6]) and

$$\mathrm{ind}\,A=\mathcal{P}_0\vee\mathcal{T}_0\vee\bigvee_{\gamma\in\mathbb{Q}^+}\mathcal{T}_\gamma\vee\mathcal{T}_\infty\vee\mathcal{Q}_\infty,$$

where $\mathcal{P}_0$ is a semi-regular preprojective component, $\mathcal{Q}_\infty$ is a semi-regular preinjective component, $\mathcal{T}_0$ is a $\mathbb{P}_1(k)$-family of ray tubes separating $\mathcal{P}_0$ from $\bigvee_{\gamma\in\mathbb{Q}^+}\mathcal{T}_\gamma\vee\mathcal{T}_\infty\vee\mathcal{Q}_\infty$, $\mathcal{T}_\infty$ is a $\mathbb{P}_1(k)$-family of coray tubes separating $\mathcal{P}_0\vee\mathcal{T}_0\vee\bigvee_{\gamma\in\mathbb{Q}^+}\mathcal{T}_\gamma$ from $\mathcal{Q}_\infty$ (because A is also a cotubular algebra), and each $\mathcal{T}_\gamma,\gamma\in\mathbb{Q}_+$, where $\mathbb{Q}_+$ is the set of all positive rationals, is a $\mathbb{P}_1(k)$-family of stable tubes separating $\mathcal{P}_0\vee\mathcal{T}_0\vee\bigvee_{\delta<\gamma}\mathcal{T}_\delta$ from $\bigvee_{\gamma<\delta}\mathcal{T}_\delta\vee\mathcal{T}_\infty\vee\mathcal{Q}_\infty$.

Domestic tubular extensions and coextensions and tubular algebras are obtained from a tame concealed algebra by performing a sequence of admissible operations (ad 1) or (ad 1*) in the stable tubes of its separating tubular family. We call these algebras 0-*tame iterated generalized coil enlargements*.

Let Λ_0 be a branch coextension of a tame concealed algebra A_0, and assume that Λ_0 is domestic or tubular. Then $\mathrm{ind}\,\Lambda_0=\mathcal{P}_0\vee\mathcal{T}_0\vee\mathcal{Q}_0$, where $\mathcal{P}_0$ is the preprojective component of Γ_{Λ_0}, $\mathcal{Q}_0$ is the preinjective component of Γ_{Λ_0}, and $\mathcal{T}_0$ is a tubular family separating $\mathcal{P}_0$ from $\mathcal{Q}_0$. Using admissible operations of types (ad 1), (ad 2), (ad 3), (ad 4), we insert projectives in the coinserted and stable tubes of $\mathcal{T}_0$. We obtain a generalized coil enlargement Λ_1 of A_0 with $\Lambda_1^-=\Lambda_0$. If Λ_1^+ is tame, we call Λ_1 a 1-*tame iterated generalized coil enlargement.* By Theorem 5.1, $\mathrm{ind}\,\Lambda_1=\mathcal{P}_0\vee\mathcal{T}_0'\vee\mathcal{Q}_0'$, where $\mathcal{T}_0'$ is the weakly separating family of the generalized coil obtained from $\mathcal{T}_0$, and $\mathcal{Q}_0'$ consists of Λ_1^+-modules.

If Λ_1^+ is domestic, then $\mathcal{Q}_0'$ is the preinjective component of $\Gamma_{\Lambda_1^+}$ and the process stops.

If Λ_1^+ is tubular, then Λ_1^+ is a branch coextension of a tame concealed algebra A_1, and we can write

$$\operatorname{ind}\Lambda_1^+ = \mathcal{P}_0^1 \vee \mathcal{T}_0^1 \vee \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}_\gamma^1 \vee \mathcal{T}_\infty^1 \vee \mathcal{Q}_\infty^1,$$

where $\mathcal{Q}_\infty^1$ is the preinjective component of Γ_{A_1}, and $\mathcal{T}_\infty^1$ is the separating tubular family of $\operatorname{mod}\Lambda_1^+$ that is obtained from the family of stable tubes of $\operatorname{mod} A_1$ by coray insertions. Then $\mathcal{Q}_0' = \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}_\gamma^1 \vee \mathcal{T}_\infty^1 \vee \mathcal{Q}_\infty^1$, and

$$\operatorname{ind}\Lambda_1 = \mathcal{P}_0 \vee \mathcal{T}_0' \vee \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}_\gamma^1 \vee \mathcal{T}_\infty^1 \vee \mathcal{Q}_\infty^1.$$

LEMMA 6.1. *With the notation introduced above*:

(i) $\mathcal{T}_\infty^1$ *is a tubular family separating* $\mathcal{P}_0 \vee \mathcal{T}_0' \vee \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}_\gamma^1$ *from* $\mathcal{Q}_\infty^1$.

(ii) *For each* $\gamma\in\mathbb{Q}^+, \mathcal{T}_\gamma^1$ *is a tubular family separating* $\mathcal{P}_0 \vee \mathcal{T}_0' \vee \bigvee_{\delta<\gamma} \mathcal{T}_\delta^1$ *from* $\bigvee_{\gamma<\delta} \mathcal{T}_\delta^1 \vee \mathcal{T}_\infty^1 \vee \mathcal{Q}_\infty^1$.

Proof. Analogous to the proof of [18, 3.1].

Let $\mathcal{P}_1 = \mathcal{P}_0 \vee \mathcal{T}_0' \vee \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}_\gamma^1, \mathcal{T}_1 = \mathcal{T}_\infty^1$ and $\mathcal{Q}_1 = \mathcal{Q}_\infty^1$. Then we can write $\operatorname{ind}\Lambda_1 = \mathcal{P}_1 \vee \mathcal{T}_1 \vee \mathcal{Q}_1$, where $\mathcal{T}_1$ is a separating tubular family in $\operatorname{mod}\Lambda_1$ consisting of coinserted and stable tubes, and $\mathcal{Q}_1$ is the preinjective component of Γ_{Λ_1}.

Now we can iterate the process as follows: using admissible operations of types (ad 1), (ad 2), (ad 3), (ad 4), we insert projectives in the tubes of $\mathcal{T}_1$. We obtain a generalized coil enlargement Λ^2 of A_1 with $(\Lambda^2)^- = \Lambda_1^+$. If $(\Lambda^2)^+$ is tame, we call the algebra Λ_2 obtained from Λ_1 by inserting projectives in the tubes of $\mathcal{T}_1$ a 2-*tame iterated generalized coil enlargement*.

By Theorem 3.9 we know that $\operatorname{ind}\Lambda_2 = \mathcal{P}_1 \vee \mathcal{T}_1' \vee \mathcal{Q}_1'$, where $\mathcal{T}_1'$ is the weakly separating family of generalized coils obtained from $\mathcal{T}_1$. We want to describe $\mathcal{Q}_1'$. As before, using Theorem 5.1 we have $\operatorname{ind}\Lambda^2 = \mathcal{P}^2 \vee \mathcal{T}^2 \vee \mathcal{Q}^2$, where $\mathcal{T}^2 = \mathcal{T}_1'$ and $\mathcal{Q}^2$ consists of $(\Lambda^2)^+$-modules.

LEMMA 6.2. *With the above notation*, $\mathcal{Q}_1' = \mathcal{Q}^2$.

Proof. As in [18, 3.2].

If $(\Lambda^2)^+$ is domestic, then $\mathcal{Q}_1'$ is the preinjective component of $\Gamma_{(\Lambda^2)^+}$ and the process stops.

If $(\Lambda^2)^+$ is tubular, then it is a branch coextension of a tame concealed algebra A_2, and we can write

$$\operatorname{ind}(\Lambda^2)^+ = \mathcal{P}_0^2 \vee \mathcal{T}_0^2 \vee \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}_\gamma^2 \vee \mathcal{T}_\infty^2 \vee \mathcal{Q}_\infty^2,$$

where $\mathcal{Q}^2_\infty$ is the preinjective component of Γ_{A_2}, and $\mathcal{T}^2_\infty$ is the separating tubular family of $\operatorname{mod}(\Lambda^2)^+$ that is obtained from the family of stable tubes of $\operatorname{mod} A_2$ by coray insertions. Then

$$\mathcal{Q}'_1 = \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}^2_\gamma \vee \mathcal{T}^2_\infty \vee \mathcal{Q}^2_\infty,$$

and defining $\mathcal{P}_2 = \mathcal{P}_1 \vee \mathcal{T}'_1 \vee \bigvee_{\gamma\in\mathbb{Q}^+} \mathcal{T}^2_\gamma$, $\mathcal{T}_2 = \mathcal{T}^2_\infty$ and $\mathcal{Q}_2 = \mathcal{Q}^2_\infty$, we can write $\operatorname{ind}\Lambda_2 = \mathcal{P}_2 \vee \mathcal{T}_2 \vee \mathcal{Q}_2$, where $\mathcal{T}_2$ is a separating tubular family in $\operatorname{mod}\Lambda_2$ consisting of coinserted and stable tubes, and $\mathcal{Q}_2$ is the preinjective component of Γ_{Λ_2}. Now we can iterate the process once more.

By induction, we define the *n-tame iterated generalized coil enlargements* of a tame concealed algebra.

Let A be a tame iterated generalized coil enlargement. From the description of $\operatorname{ind} A$, given above, we immediately obtain the following facts.

PROPOSITION 6.3. *If A is a tame iterated generalized coil enlargement of a tame concealed algebra, then*

(i) *A is of polynomial growth.*

(ii) *q_A is weakly non-negative.*

Proof. (i) follows from [18, 3.3] and Corollary 5.2, (ii) follows from (i) and [12, 1.3].

EXAMPLE 6.4. In this example, Λ_n is an n-tame iterated generalized coil enlargement of a tame concealed algebra. Λ_0 is given by the quiver

$$\begin{tikzcd}
 & & \circ \arrow[d, "\lambda"] & & \\
 & & \circ \arrow[d, "\mu"] & & \\
\circ \arrow[r, "\alpha"] & \circ \arrow[r, "\beta"] & \circ \arrow[d, "\varepsilon"] & \circ \arrow[l, "\gamma"'] & \circ \arrow[l, "\delta"'] \\
 & & \circ & \circ \arrow[l, "\xi"'] & \circ \arrow[l, "\eta"']
\end{tikzcd}$$

bound by $\beta\varepsilon = 0, \lambda\mu\varepsilon = 0$; Λ_1 is given by the quiver

$$\begin{tikzcd}
 & & \circ \arrow[d, "\lambda"] & & & \\
 & & \circ \arrow[d, "\mu"] & & \circ \arrow[ll, "\varrho"'] \arrow[r, "\omega"] \arrow[d, "\sigma"] & \circ \arrow[dd, "\eta"] \\
\circ \arrow[r, "\alpha"] & \circ \arrow[r, "\beta"] & \circ \arrow[d, "\varepsilon"] & \circ \arrow[l, "\gamma"'] & \circ \arrow[l, "\delta"'] & \\
 & & \circ & & & \circ \arrow[lll, "\xi"']
\end{tikzcd}$$

bound by $\beta\varepsilon = 0, \lambda\mu\varepsilon = 0, \varrho\mu = \sigma\delta\gamma, \omega\eta\xi = 0$; Λ_2 is given by the quiver

$$\begin{tikzcd}
 & & \circ \arrow[d, "\chi"] \arrow[dddll, "\varphi"'] \arrow[ddrr, "\psi"] & & & \\
 & & \circ \arrow[d, "\lambda"] & & & \\
 & & \circ \arrow[d, "\mu"] & & \circ \arrow[ll, "\varrho"'] \arrow[r, "\omega"] \arrow[d, "\sigma"] & \circ \arrow[dd, "\eta"] \\
\circ \arrow[r, "\alpha"] & \circ \arrow[r, "\beta"] & \circ \arrow[d, "\varepsilon"] & \circ \arrow[l, "\gamma"'] & \circ \arrow[l, "\delta"'] & \\
 & & \circ & & & \circ \arrow[lll, "\xi"']
\end{tikzcd}$$

bound by $\beta\varepsilon = 0, \lambda\mu\varepsilon = 0, \varrho\mu = \sigma\delta\gamma, \omega\eta\xi = 0, \psi\omega = 0, \psi\varrho = \chi\lambda, \chi\lambda\mu = \varphi\alpha\beta$.

7. The main theorem. In this section we generalize the definition of acceptable projectives given in [18]. We show that an algebra A having acceptable projectives is triangular and consequently, by [7] the Tits form q_A of A is defined. The main result of this section is the generalization of Theorem 4.3 from [18].

DEFINITION 7.1. Let A be a finite-dimensional, basic and connected k-algebra. An algebra A has *acceptable projectives* if the Auslander–Reiten quiver Γ_A of A has components $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_r$ with the following properties:

(i) Any indecomposable projective A-module lies in $\mathcal{P}$ or in some $\mathcal{C}_i$.
(ii) $\mathcal{P}$ is a preprojective component without injective modules.
(iii) Each $\mathcal{C}_i$ is a standard generalized coil.
(iv) If $\mathrm{Hom}_A(\mathcal{C}_i, \mathcal{C}_j) \neq 0$, then $i \leq j$.

Observe that tame iterated generalized coil enlargements of tame concealed algebras have acceptable projectives.

LEMMA 7.2. *If an algebra A has acceptable projectives, then A is triangular.*

Proof. Assume that A is not triangular. Let $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_r$ be as in the above definition. Then there exists a cycle in $\mathrm{mod}\, A$ consisting of indecomposable projective modules none of which lies in $\mathcal{P}$, for otherwise $\mathcal{P}$ would contain a cycle. Hence the indecomposable projective modules in the cycle lie in the standard generalized coils $\mathcal{C}_1, \ldots, \mathcal{C}_r$. From Definition 7.1(iv), they all lie in one standard generalized coil $\mathcal{C}_i$. Thus $\mathcal{C}_i$ contains a cycle of projectives and we obtain a contradiction with Proposition 3.4.

Assume that an algebra A has acceptable projectives and let $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_r$ be as in Definition 7.1. Consider the standard generalized coil $\mathcal{C}_r$. From [3, 4.5] we know that if $\mathcal{C}$ is a coil then the mesh-category $k(\mathcal{C})$ has no oriented cycle of projectives. Therefore, if there exists a cycle in the mesh-category $k(\mathcal{C}_r)$, then the projective P which is generated by step (ad 4) is equal to some projective P' which was in $k(\mathcal{C}_r)$ before applying (ad 4), which is impossible. Analogously we see that the admissible operations performed after the step (ad 4) have not created an oriented cycle of projectives.

Consequently, the mesh-category $k(\mathcal{C}_r)$ has no oriented cycle of projectives, there is a projective P in $\mathcal{C}_r$ such that P is a sink in the full subcategory of $k(\mathcal{C}_r)$ consisting of projectives, that is, the support of $\mathrm{Hom}_{k(\mathcal{C}_r)}(P,-)$ contains no projective. In comparison to [18, 4.2] we have to consider an additional case.

Let $P = P_a$ be as in Section 2, and $A' = A/Ae_aA$. Denote by $\mathcal{R}$ the set of the vertices $X'_i, i \geq 0$ and $Z_{ij}, 1 \leq j \leq t$, of a mesh-complete translation subquiver $\mathcal{C}_r$ (compare the figure in the description of (ad 4) in Section 2).

Let $\mathcal{C}'_r$ be the translation quiver obtained from $\mathcal{C}_r$ by deleting $\mathcal{R}$ and replacing the sectional paths $X_i \to Z_{ij} \to \ldots \to X'_i \to \tau_A^{-1}X_{i-1}$ (if they exist) by the respective compositions $X_i \to \tau_A^{-1}X_{i-1}$.

PROPOSITION 7.3. *With the notation introduced above, we have*:

(i) $A = A'[X \oplus Y]$, *where X is an indecomposable direct summand of* $\operatorname{rad} P$, *Y is a directing module and* $\operatorname{rad} P = X \oplus Y$.

(ii) *A' has acceptable projectives and $\mathcal{C}'_r$ is a standard generalized coil of $\Gamma_{A'}$.*

Proof. (i) Since $P = P_a$ is a sink in the full subcategory of $\operatorname{ind} A$ consisting of projectives, the vertex a is a source in Q_A. Hence $A = A'[X \oplus Y]$, where X is the indecomposable direct summand of $\operatorname{rad} P$ that belongs to $\operatorname{mod} A'$, Y is a directing module such that $\operatorname{rad} P = X \oplus Y$.

(ii) Since $\mathcal{C}_r$ is a generalized coil, so is $\mathcal{C}'_r$. Standardness of $\mathcal{C}'_r$ follows from that of $\mathcal{C}_r$ (see [18, 4.2] or [3, Lemma 5.3]). Because $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_{r-1}, \mathcal{C}'_r$ are the components of $\Gamma_{A'}$ where the projectives lie, we see that A' has acceptable projectives.

THEOREM 7.4. *Let A be an algebra with acceptable projectives. Then the following conditions are equivalent*:

(i) *A is a tame iterated generalized coil enlargement of a tame concealed algebra.*

(ii) *A is tame.*

(iii) *q_A is weakly non-negative.*

Proof. (i)$\Rightarrow$(ii) is Proposition 6.3.

(ii)$\Rightarrow$(iii) follows from [12].

(iii)$\Rightarrow$(i). Let $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_r$ be the components of Γ_A where the projectives lie, with $\mathcal{P}$ preprojective without injective modules, and $\mathcal{C}_1, \ldots, \mathcal{C}_r$ standard generalized coils such that $\mathrm{Hom}_A(\mathcal{C}_i, \mathcal{C}_j) \neq 0$ implies $i \leq j$. Let $A', \mathcal{C}'_r$ and

$P = P_a$ be as in Proposition 7.3 and in [18, 4.2, Proposition]. Then A' has acceptable projectives, and $\mathcal{P}, \mathcal{C}_1, \ldots, \mathcal{C}_{r-1}, \mathcal{C}'_r$ (if it still has projectives) are the components of $\Gamma_{A'}$ where the projectives lie. We proceed by induction on the number p of projectives in the standard generalized coils $\mathcal{C}_1, \ldots, \mathcal{C}_r$.

If $p = 0$, then $\mathcal{P}$ is a preprojective component. By [14, 1.3] and [15, 4.9], A is a domestic tubular coextension of a tame concealed algebra, that is, a 0-tame iterated generalized coil enlargement.

Let $p > 0$. Since A' is convex in A, $q_{A'}$ is weakly non-negative. By induction hypothesis, A' is an n-tame iterated generalized coil enlargement. Thus, $A' = \Lambda_n$, where Λ_n is obtained from an $(n-1)$-tame iterated generalized coil enlargement Λ_{n-1} by inserting projectives using admissible operations of types (ad 1), (ad 2), (ad 3), or (ad 4) in the last separating tubular family $\mathcal{T}_{n-1}$ of $\operatorname{mod} \Lambda_{n-1}$ (we may assume $n \geq 1$).

Using the notation introduced in Section 6, we see that if $\operatorname{mod} \Lambda_{n-1} = \mathcal{P}_{n-1} \vee \mathcal{T}_{n-1} \vee \mathcal{Q}_{n-1}$ then $\operatorname{mod} \Lambda_n = \mathcal{P}_{n-1} \vee \mathcal{T}'_{n-1} \vee \mathcal{Q}'_{n-1}$, where $\mathcal{T}'_{n-1}$ is the last weakly separating family of generalized coils containing projectives in $\operatorname{mod} \Lambda_n$. Hence $\mathcal{C}'_r$ belongs to $\mathcal{T}'_{n-1} \vee \mathcal{Q}'_{n-1}$. Also, there is a generalized coil enlargement Λ^n of a tame concealed algebra A_{n-1} such that $\operatorname{mod} \Lambda^n = \mathcal{P}^n \vee \mathcal{T}'_{n-1} \vee \mathcal{Q}'_{n-1}$, and the branch extension $(\Lambda^n)^+$ of A_{n-1} is either domestic or tubular.

If $(\Lambda^n)^+$ is domestic, then $\mathcal{Q}'_{n-1}$ is the preinjective component of $\Gamma_{(\Lambda^n)^+}$ and $\mathcal{C}'_r$ belongs to $\mathcal{T}'_{n-1}$. By performing the admissible operation on $\mathcal{C}'_r$ to obtain A from A', we get another generalized coil enlargement of A_{n-1} which, being convex in A, has weakly non-negative Tits form. By Corollary 5.2, it is tame and therefore A is also an n-tame iterated generalized coil enlargement.

If $(\Lambda^n)^+$ is a tubular algebra, then it is a branch coextension of a tame concealed algebra A_n, and

$$\mathcal{Q}'_{n-1} = \bigvee_{\gamma \in \mathbb{Q}^+} \mathcal{T}^n_\gamma \vee \mathcal{T}^n_\infty \vee \mathcal{Q}^n_\infty,$$

where $\mathcal{Q}^n_\infty$ is the preinjective component of Γ_{A_n}, and $\mathcal{T}^n_\infty$ is obtained from the separating tubular family of $\operatorname{mod} A_n$ by coray insertions. Then $\operatorname{mod} \Lambda_n = \mathcal{P}_n \vee \mathcal{T}_n \vee \mathcal{Q}_n$, where $\mathcal{P}_n = \mathcal{P}_{n-1} \vee \mathcal{T}'_{n-1} \vee \bigvee_{\gamma \in \mathbb{Q}^+} \mathcal{T}^n_\gamma$, $\mathcal{T}_n = \mathcal{T}^n_\infty$ and $\mathcal{Q}_n = \mathcal{Q}^n_\infty$. In this case $\mathcal{C}'_r$ must belong to $\mathcal{T}_n$, otherwise we can construct a vector x with non-negative coordinates such that $q_A(x) < 0$. Indeed, if X is the indecomposable direct summand of $\operatorname{rad} P$ that lies in $\operatorname{mod} A'$ and $X \in \mathcal{T}'_{n-1}$, then $\Lambda_n[X]$ is also a generalized coil enlargement of A_{n-1} which, being convex in A, has weakly non-negative Tits form and, by Corollary 5.2, is tame, so A is an n-tame iterated generalized coil enlargement. This contradicts the fact that $(\Lambda^n)^+$ is a tubular algebra (because by construction of the tame iterated generalized coil enlargement of a tame concealed algebra we obtain in this step an $(n+1)$-tame iterated generalized coil enlargement). Therefore

$X \in \mathcal{Q}'_{n-1}$. If $X \notin \mathcal{T}_n$, then by Lemma 6.1, there exist $\gamma \in \mathbb{Q}^+$ and a module $Y \in \mathcal{T}^n_\gamma$ such that $q_{(\Lambda^n)^+}(\underline{\dim}\, Y) = 0$ and $\operatorname{Hom}_{(\Lambda^n)^+}(X, Y) \neq 0$. Since $(\Lambda^n)^+[X]$ is convex in A and gl.dim$(\Lambda^n)^+[X] \le 3$, we get, for $e_a = \underline{\dim}\, S_a$,

$$q_A(2\,\underline{\dim}\, Y + e_a) = q_{(\Lambda^n)^+[X]}(2\,\underline{\dim}\, Y + e_a) = 2(\underline{\dim}\, Y, e_a)_{(\Lambda^n)^+[X]} + 1 < 0$$

because we have

$$\begin{aligned}(\underline{\dim}\, Y, e_a)_{(\Lambda^n)^+[X]} &= \langle \underline{\dim}\, Y, e_a\rangle + \langle e_a, \underline{\dim}\, Y\rangle \\ &= \langle \underline{\dim}\, Y, \underline{\dim}\, I_a\rangle + \langle \underline{\dim}\, P_a - \underline{\dim}\, X, \underline{\dim}\, Y\rangle \\ &= -\langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle = -\dim \operatorname{Hom}_{(\Lambda^n)^+}(X, Y) < 0.\end{aligned}$$

Hence we obtain a generalized coil enlargement Λ^{n+1} of A_n which, being convex in A, has a weakly non-negative Tits form. By Corollary 5.2, Λ^{n+1} is tame. Therefore A is an $(n+1)$-tame iterated generalized coil enlargement of a tame concealed algebra.

COROLLARY 7.5. *Let A be an algebra with acceptable projectives which has a sincere indecomposable module. Then the following conditions are equivalent*:

(i) *A is either a* 0*-tame iterated or a* 1*-tame iterated generalized coil enlargement.*

(ii) *A is tame.*

(iii) *q_A is weakly non-negative.*

Proof. Follows from Theorem 7.4 and the fact that 0-tame iterated and 1-tame iterated generalized coil enlargements have indecomposable sincere modules.

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