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DOUBLING MEASURES WITH DIFFERENT BASES

BY

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Doubling measures on the real line appear in the study of quasiconformal mappings of the half-plane onto itself [BA] and harmonic measures of elliptic operators [CFMS]. Dyadic doubling measures are related to martingales. These measures have been completely characterized by Fefferman, Kenig and Pipher [FKP].

However, the null sets for these measures have been less thoroughly studied. A class of porous sets which are null for doubling measures was introduced by Martio [M], and was extended to a larger collection in [W]. It is hardly surprising that the class of doubling measures and the class of dyadic doubling measures are different, but much less trivial is the fact that the corresponding null sets are also different [W].

We can define doubling measures with different bases, thus extending the notion of dyadic doubling; and determine which classes have the same null-sets.

Let A be an integer greater than one. An A-*adic interval* is an interval having end points j/A^n and $(j+1)/A^n$ for some integers n and j. A measure μ on $\mathbb{R}^1$ is called an A-*adic doubling measure* if there exists a constant $\lambda(\mu)$ so that $\mu(I) \le \lambda(\mu)\mu(J)$ for every pair of A-adic intervals I and J of equal length whose union is contained in an A-adic interval of length $A|I|$. The number $\lambda(\mu)$ is called an A-*adic doubling constant* for μ.

Let $\mathcal{D}_A$ be the collection of all A-adic doubling measures on $\mathbb{R}^1$ and $\mathcal{N}_A$ consist of all sets in $\mathbb{R}^1$ having zero μ-measure for every $\mu \in \mathcal{D}_A$. Based on Kronecker's Theorem on irrational numbers, we prove the following:

THEOREM. *Let A and B be two integers greater than* 1. *Then $\mathcal{N}_A = \mathcal{N}_B$ if and only if $\log A/\log B$ is rational.*

A condition of this kind is not new. W. Schmidt [S] has proved that every number normal to base A is also normal to base B if and only if $\log A/\log B$ is rational.

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If $\log A/\log B$ is rational, then $A^a = B^b$ for some positive integers a and b. It is rather straightforward to check that $\mathcal{D}_A = \mathcal{D}_{A^a} = \mathcal{D}_{B^b} = \mathcal{D}_B$. Hence $\mathcal{N}_A = \mathcal{N}_B$.

It remains to prove the converse.

The following three lemmas have been proved in [W] for $A = 2$; the proof for any other A is similar.

LEMMA 1. *Let μ be an A-adic doubling measure on $\mathbb{R}^1$. Then there exist $\alpha, \beta \ge 1$, depending only on A and the doubling constant $\lambda(\mu)$, so that for any A-adic interval S and any subinterval T of S,*

$$\beta^{-1}(|T|/|S|)^{\alpha}\mu(S) \le \mu(T) \le \beta(|T|/|S|)^{1/\alpha}\mu(S).$$

Note that T need not be A-adic.

LEMMA 2. *Let S be any A-adic interval and μ and ν be two A-adic doubling measures on $\mathbb{R}^1$ satisfying $\mu(S) = \nu(S)$. Then the measure, equal to μ on S and equal to ν on $\mathbb{R}^1\backslash S$, is an A-adic doubling measure on $\mathbb{R}^1$ with a doubling constant bounded by the maximum of those for μ and ν.*

LEMMA 3. *Let A be an integer greater than 1, and a, ε and δ be numbers in $(0,1)$ satisfying $\varepsilon + \delta^a < 1/3$. Then there exists $\mu \in \mathcal{D}_A$ with a doubling constant $\lambda(\mu)$ bounded by a number $\Lambda(A, a)$, depending on A and a only, such that $\mu([0,1]) = 1$, $\mu([0,\varepsilon]) = \varepsilon$ and $\mu([1-\delta, 1]) = \delta^a$.*

For $x \in \mathbb{R}^1$, let $\|x\|$ denote the distance from x to the nearest integer.

LEMMA 4. *If $\|x\| > 1/10$, then the set $\{j : j$ is an integer satisfying $\|jx\| \le 1/20\}$ does not contain consecutive integers.*

LEMMA 5. *Assume that $\log A/\log B$ is irrational and that M, N and Q are numbers greater than or equal to 1. Then there exist integers $n > N$ and $q > Q$ so that*

$$M \le B^q A^{-n} \le 4AM \quad \textit{and} \quad \|B^q A^{-n}\| > 1/10.$$

P r o o f. Choose first $n_0 > N$ and $q_0 > Q$ so that

$$2M \le B^{q_0} A^{-n_0} \le 2AM.$$

Let $K = [B^{q_0} A^{-n_0}]$ and $\varepsilon = B^{q_0} A^{-n_0} - K$. If $1/10 < \varepsilon < 9/10$, then the lemma is true with $n = n_0$ and $q = q_0$.

Suppose $0 \le \varepsilon \le 1/10$. Because $\log B/\log A$ is irrational, $\{k \log B/\log A \pmod 1 : k$ is a positive integer$\}$ is dense in $[0,1]$ by a theorem of Kronecker [HW]. Therefore there exist positive integers n' and q' so that

$$(10K\log A)^{-1} \le q' \log B/\log A - n' \le (9K\log A)^{-1},$$

thus $1 + (10K)^{-1} \le B^{q'} A^{-n'} \le 1 + 2(9K)^{-1}$. Let $n = n_0 + n'$ and $q = q_0 + q'$, it is easy to check that $K + 1/10 < B^q A^{-n} < K + 1/2$ and $2M \le B^q A^{-n} \le 4AM$.

Suppose that $9/10 \le \varepsilon < 1$. Choose positive integers n'' and q'' so that

$$-(4(K+1)\log A)^{-1} \le q'' \log B/\log A - n'' \le -(5(K+1)\log A)^{-1};$$

and let $n = n_0 + n''$ and $q = q_0 + q''$.

In the remaining part of this note, we assume that $\log A/\log B$ *is irrational and construct a set* S *in* $\mathcal{N}_B \backslash \mathcal{N}_A$. Some of the ideas are adapted from [W].

Let $K_m = m^4 - 1$, and choose n_k and q_k inductively so that whenever $m \ge 1$ and $k \in [1 + K_m, K_{m+1}]$, we have $q_{k+1} > q_k$,

$$n_{k+1} > n_k + 20 + 8 \log_A m, \tag{1}$$

$$m^4 \le B^{q_k} A^{-n_k} \le 4Am^4, \tag{2}$$

and

$$\|B^{q_k} A^{-n_k}\| > 1/10. \tag{3}$$

This is possible in view of Lemma 5.

Given $m \ge 1$, denote by $\widetilde{m}$ the smallest integer power of A that is greater than m, i.e., $\widetilde{m} = A^{[\log_A m]+1}$. Let $k \in [1 + K_m, K_{m+1}]$ and j be integers, and define the A-adic intervals

$$L_{k,j} = \left[\frac{j}{A^{n_k}}, \frac{j+1}{A^{n_k}}\right], \qquad I_{k,j} = \left(\frac{j}{A^{n_k}}, \frac{j}{A^{n_k}} + \frac{1}{A^{n_k}\widetilde{m}^4}\right]$$
$$\text{and} \qquad J_{k,j} = \left[\frac{j+1}{A^{n_k}} - \frac{1}{A^{n_k}\widetilde{m}^8}, \frac{j+1}{A^{n_k}}\right];$$

note from (1) that

$$|I_{k,j}|/|J_{k,j'}| = \widetilde{m}^4 \ge A^4 \quad \text{and} \quad |J_{k,j}|/|L_{k+1,j'}| > A^2.$$

To construct S, we let $S_1^J = [0,1]$, select a group of mutually disjoint subintervals $J_{k,j}$ of S_1^J with k ranging from $1 + K_1$ to K_2, and call their union S_2^J. We again select a group of mutually disjoint subintervals $J_{k,j}$ of S_2^J with k ranging from $1 + K_2$ to K_3, and call their union S_3^J, etc. Finally, we let $S = \bigcap S_m^J$.

Define for each $k \ge 1$,

$$\mathcal{J}_k = \{j : j \text{ is an integer so that } \|jB^{q_k} A^{-n_k}\| > 1/20\}, \tag{4}$$

and note from (3) and Lemma 4 that the complement of $\mathcal{J}_k$ does not contain consecutive integers.

Let $\mathcal{J}_1'$ be the integers in $[1, A^{n_1} - 1]$,

$$\mathcal{C}_1^I = \{I_{1,j} : j \in \mathcal{J}_1 \cap \mathcal{J}_1'\}, \qquad \mathcal{C}_1^J = \{J_{1,j-1} : j \in \mathcal{J}_1 \cap \mathcal{J}_1'\},$$

T_1^I be the union of all intervals in $\mathcal{C}_1^I$, T_1^J be the union of all intervals in $\mathcal{C}_1^J$ and $T_1 = T_1^I \cup T_1^J$. Note that $\overline{S_1^J \backslash T_1}$ may be expressed as a union of A-adic intervals of length $|L_{2,j}|$.

After $\mathcal{J}'_k$, $\mathcal{C}^I_k$, $\mathcal{C}^J_k$, T^I_k, T^J_k and T_k are defined for some $k \in [1+K_1, K_2-1]$, let

$$\mathcal{J}'_{k+1} = \{j : J_{k+1,j-1} \cup I_{k+1,j} \subseteq S^J_1 \setminus T_k\},$$

$$\mathcal{C}^I_{k+1} = \mathcal{C}^I_k \cup \{I_{k+1,j} : j \in \mathcal{J}_{k+1} \cap \mathcal{J}'_{k+1}\}, \tag{5}$$

$$\mathcal{C}^J_{k+1} = \mathcal{C}^J_k \cup \{J_{k+1,j-1} : j \in \mathcal{J}_{k+1} \cap \mathcal{J}'_{k+1}\}, \tag{6}$$

$$T^I_{k+1} = \text{the union of intervals in } \mathcal{C}^I_{k+1}, \tag{7}$$

$$T^J_{k+1} = \text{the union of intervals in } \mathcal{C}^J_{k+1}, \tag{8}$$

and

$$T_{k+1} = T^I_{k+1} \cup T^J_{k+1}. \tag{9}$$

Note that all intervals in $\mathcal{C}^I_{K_2} \cup \mathcal{C}^J_{K_2}$ are mutually disjoint, and that each interval in $\mathcal{C}^J_{K_2}$ is adjacent to an interval in $\mathcal{C}^I_{K_2}$ and vice versa. Let

$$S^I_2 = T^I_{K_2} \quad \text{and} \quad S^J_2 = T^J_{K_2}.$$

We shall keep S^I_2 permanently out of S, and make the second stage construction inside S^J_2 only.

Let

$$\mathcal{J}'_{1+K_2} = \{j : J_{1+K_2,j-1} \cup I_{1+K_2,j} \subseteq S^J_2\},$$
$$\mathcal{C}^I_{1+K_2} = \{I_{1+K_2,j} : j \in \mathcal{J}_{1+K_2} \cap \mathcal{J}'_{1+K_2}\},$$
$$\mathcal{C}^J_{1+K_2} = \{J_{1+K_2,j-1} : j \in \mathcal{J}_{1+K_2} \cap \mathcal{J}'_{1+K_2}\},$$

$T^I_{1+K_2}$ be the union of intervals in $\mathcal{C}^I_{1+K_2}$, $T^J_{1+K_2}$ be the union of intervals in $\mathcal{C}^J_{1+K_2}$ and $T_{1+K_2} = T^I_{1+K_2} \cup T^J_{1+K_2}$.

After $\mathcal{J}'_k$, $\mathcal{C}^I_k$, $\mathcal{C}^J_k$, T^I_k, T^J_k and T_k are defined for some k in $[1+K_2, K_3-1]$, let

$$\mathcal{J}'_{k+1} = \{j : J_{k+1,j-1} \cup I_{k+1,j} \subseteq S^J_2 \setminus T_k\},$$

and define the sets $\mathcal{C}^I_{k+1}$, $\mathcal{C}^J_{k+1}$, T^I_{k+1}, T^J_{k+1} and T_{k+1} according to (5) through (9). Let

$$S^I_3 = T^I_{K_3} \quad \text{and} \quad S^J_3 = T^J_{K_3}.$$

Keep S^I_3 permanently in the complement of S and make the third stage construction inside S^J_3. Continue this process indefinitely and let

$$S = \bigcap_{m=1}^{\infty} S^J_m.$$

To prove that $S \in \mathcal{N}_B$, we let $m \geq 10$ and $J_{k,j-1} \in \mathcal{C}^J_{K_m}$; hence $k \in [1 + K_{m-1}, K_m]$ and $j \in \mathcal{J}_k \cap \mathcal{J}'_k$. Because of (4),

$$(p + 1/20)B^{-q_k} < jA^{-n_k} < (p + 19/20)B^{-q_k}$$

for an integer p. Recall that $J_{k,j-1}$ and $I_{k,j}$ share a common boundary point jA^{-n_k} and have lengths $A^{-n_k}\widetilde{m}^{-8}$ and $A^{-n_k}\widetilde{m}^{-4}$ respectively. Note from (2) that $|J_{k,j-1}| \le m^{-2}B^{-q_k}$, thus $J_{k,j-1}$ is contained in the B-adic interval $[pB^{-q_k},(p+1)B^{-q_k}]$. Note again from (2) that $|I_{k,j}| \ge A^{-4}B^{-q_k}$; since $A \ge 2$, we have $|I_{k,j} \cap [pB^{-q_k},(p+1)B^{-q_k}]| \ge A^{-5}B^{-q_k}$.

Let ν be any B-adic doubling measure. It follows from Lemma 1 that there exist α and β depending only on B and on a doubling constant for ν so that

$$\nu(J_{k,j-1}) \le \beta m^{-2/\alpha}\nu([pB^{-q_k},(p+1)B^{-q_k}]) \le \beta^2 m^{-2/\alpha}A^{5\alpha}\nu(I_{k,j}).$$

Because intervals in $\mathcal{C}^J_{K_m} \cup \mathcal{C}^I_{K_m}$ are mutually disjoint, summing over all $J_{k,j-1}$ in $\mathcal{C}^J_{K_m}$ we obtain

$$\nu(S^J_m) \le \beta^2 m^{-2/\alpha}A^{5\alpha}\nu([0,1]).$$

Therefore $\nu(S) = 0$ and S is a set in $\mathcal{N}_B$.

To prove that $S \notin \mathcal{N}_A$, we need to find a measure μ in $\mathcal{D}_A$ which carries a positive mass on S. In the reasoning below, scale invariant versions of Lemmas 2 and 3 are used respectively; all measures μ_k below are required to be periodic with period 1, and contained in $\mathcal{D}_A$ with doubling constants bounded above by the number $\Lambda(A,1/4)$ of the statement of Lemma 3.

Let μ_0 be the Lebesgue measure on $\mathbb{R}^1$. After μ_{k-1} has been constructed for some $k \in [1+K_m, K_{m+1}]$, we redistribute the mass in each $L_{k,j}$ to form a new measure μ_k in $\mathcal{D}_A$ with doubling constant again bounded by $\Lambda(A,1/4)$, which satisfies

$$\mu_k(L_{k,j}) = \mu_{k-1}(L_{k,j}), \qquad \mu_k(I_{k,j}) = \widetilde{m}^{-4}\mu_k(L_{k,j}),$$
$$\mu_k(J_{k,j}) = \widetilde{m}^{-2}\mu_k(L_{k,j}).$$

The existence of μ_k is guaranteed by Lemmas 2 and 3.

Let μ be the weak* limit of $\{\mu_{K_m}\}$. Clearly, μ is in $\mathcal{D}_A$ and has a doubling constant bounded by $\Lambda(A,1/4)$.

Because I's, J's and L's are A-adic intervals and $|L_{k+1,j}| < |J_{k,j'}| < |I_{k,j''}| < |L_{k,j'''}|$, any mass redistribution passing the kth step does not alter the total measures on $J_{k,j}, I_{k,j}$ and $L_{k,j}$. Therefore $\mu(J_{k,j}) = \mu_k(J_{k,j})$, $\mu(I_{k,j}) = \mu_k(I_{k,j})$ and $\mu(L_{k,j}) = \mu_k(L_{k,j})$; and for $k \in [1+K_m, K_{m+1}]$,

$$\mu(I_{k,j}) = \widetilde{m}^{-4}\mu(L_{k,j}) \quad \text{and} \quad \mu(J_{k,j}) = \widetilde{m}^{-2}\mu(L_{k,j}).$$

Let α and β be the numbers associated with μ as in Lemma 1. We claim that for each $m \ge 1$,

$$\mu(S^J_{m+1} \cup S^I_{m+1}) \ge (1-(1-m^{-2}\beta^{-1}A^{-4\alpha})^{K_{m+1}-K_m})\mu(S^J_m) \tag{10}$$

and

$$\mu(S^I_{m+1}) \le m^{-2}\beta A^{2\alpha}\mu(S^J_{m+1} \cup S^I_{m+1}). \tag{11}$$

Fix an $m \geq 1$ and note that S_m^J may be expressed as a union of A-adic intervals of length $A^{-n_{K_m}}\widetilde{m}^{-8}$, which in turn may be expressed as a union of A-adic intervals of length $A^{2-n_{1+K_m}}$. Let G be any A-adic interval contained in S_m^J of length $A^{2-n_{1+K_m}}$. Then G contains exactly A^2 intervals from $\{L_{1+K_m,j}\}$. Because $A^2 \geq 4$ and the complement of $\mathcal{J}_{1+K_m}$ does not contain consecutive integers, G must contain at least one pair of intervals $(J_{1+K_m,j'-1}, I_{1+K_m,j'})$ from $\mathcal{C}_{1+K_m}^J \times \mathcal{C}_{1+K_m}^I$. Denote these intervals by J' and I' respectively and the interval $L_{1+K_m,j'-1}$ by L'. Then by Lemma 1,

$$\begin{aligned}\mu(T_{1+K_m} \cap G) &\geq \mu(J' \cup I') \geq \mu(J') = \widetilde{m}^{-2}\mu(L') \\ &\geq \widetilde{m}^{-2}\beta^{-1}A^{-2\alpha}\mu(G) \geq m^{-2}\beta^{-1}A^{-4\alpha}\mu(G).\end{aligned}$$

Summing over all such G's, we have

$$\mu(T_{1+K_m}) \geq m^{-2}\beta^{-1}A^{-4\alpha}\mu(S_m^J).$$

Note that

$$\begin{aligned}\mu(T_{1+K_m}^I \cap G) &\leq \sum\nolimits' \mu(I_{1+K_m,j}) = \widetilde{m}^{-4}\sum\nolimits' \mu(L_{1+K_m,j}) \\ &= \widetilde{m}^{-4}\mu(G) \leq m^{-2}\beta A^{2\alpha}\mu(T_{1+K_m} \cap G),\end{aligned}$$

where $\sum'$ sums over all $I_{1+K_m,j}$ in G. Therefore

$$\mu(T_{1+K_m}^I) \leq m^{-2}\beta A^{2\alpha}\mu(T_{1+K_m}).$$

The set $\overline{S_m^J \backslash T_{1+K_m}}$ may be expressed as a union of A-adic intervals of length $|J_{1+K_m,j}|$, which in turn may be expressed as a union of A-adic intervals of length $A^2|L_{2+K_m,j}|$. Let G be any interval in $\overline{S_m^J \backslash T_{1+K_m}}$ of length $A^2|L_{2+K_m,j}|$. Repeating the argument of the last paragraph and using the fact that the complement of $\mathcal{J}_{2+K_m}$ does not contain consecutive integers, we obtain the inequalities

$$\mu(T_{2+K_m} \backslash T_{1+K_m}) \geq m^{-2}\beta^{-1}A^{-4\alpha}\mu(S_m^J \backslash T_{1+K_m})$$

and

$$\mu(T_{2+K_m}^I \backslash T_{1+K_m}) \leq m^{-2}\beta A^{2\alpha}\mu(T_{2+K_m} \backslash T_{1+K_m}).$$

Repeating the same reasoning in each $S_m^J \backslash T_k$, we conclude that for $k \in [1+K_m,\ K_{m+1}-1]$,

$$\mu(T_{k+1} \backslash T_k) \geq m^{-2}\beta^{-1}A^{-4\alpha}\mu(S_m^J \backslash T_k)$$

and

$$\mu(T_{k+1}^I \backslash T_k) \leq m^{-2}\beta A^{2\alpha}\mu(T_{k+1} \backslash T_k).$$

The estimates (10) and (11) follow from these inequalities.

Note from (11) that for $m > m_0 \equiv [\beta^{1/2}A^\alpha] + 1$,

$$\mu(S_{m+1}^J) \geq (1 - m^{-2}\beta A^{2\alpha})\mu(S_{m+1}^J \cup S_{m+1}^I) > 0.$$

Therefore by (10),

$$\mu(S) \geq \mu(S^J_{m_0}) \prod_{m>m_0} \left((1-m^{-2}\beta A^{2\alpha})\left(1-(1-m^{-2}\beta^{-1}A^{-4\alpha})^{K_{m+1}-K_m}\right)\right).$$

Since $K_m = m^4 - 1$, calculation shows that $\mu(S) > 0$. Therefore $\mathcal{S} \notin \mathcal{N}_A$.

This proves that $\mathcal{N}_A \neq \mathcal{N}_B$ when $\log A/\log B$ is irrational.

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