A NOTE ON THE DIOPHANTINE EQUATION $(\binom{k}{2}) - 1 = q^n + 1$

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In this note we prove that the equation $(\binom{k}{2}) - 1 = q^n + 1$, $q \geq 2$, $n \geq 3$, has only finitely many positive integer solutions $(k, q, n)$. Moreover, all solutions $(k, q, n)$ satisfy $k < 10^{10^{182}}$, $q < 10^{10^{165}}$, and $n < 2 \cdot 10^{17}$.

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. The solutions $(k, q, n)$ of the equation

$$(1) \quad \left(\binom{k}{2}\right) - 1 = q^n + 1, \quad k, q, n \in \mathbb{N}, \quad q \geq 2, \quad n \geq 3,$$

are connected with some questions in coding theory. In this respect, Alter [1] proved that (1) has no solution $(k, q, n)$ with $q = 8$. Recently, Hering [3] found out that all solutions $(k, q, n)$ of (1) satisfy $3 \mid n$ or $q$ is a prime power with $q < 47$. In this note, we prove a general result as follows.

**Theorem.** The equation (1) has only finitely many solutions $(k, q, n)$. Moreover, all solutions $(k, q, n)$ satisfy $k < 10^{10^{182}}$, $q < 10^{10^{165}}$, and $n < 2 \cdot 10^{17}$.

The proof of the Theorem depends on the following lemmas.

Let $\alpha$ be an algebraic number of degree $r$ with conjugates $\sigma_1 \alpha, \ldots, \sigma_r \alpha$ and minimal polynomial

$$a_0 x^r + a_1 x^{r-1} + \ldots + a_r = a_0 \prod_{i=1}^{r} (x - \sigma_i \alpha) \in \mathbb{Z}[x], \quad a_0 > 0.$$ 

Further, let $|\alpha| = \max(|\sigma_1 \alpha|, \ldots, |\sigma_r \alpha|)$. Then

$$h(\alpha) = \frac{1}{r} \left( \log a_0 + \sum_{i=1}^{r} \log \max(1, |\sigma_i \alpha|) \right)$$

is called Weil’s height of $\alpha$.

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LEMMA 1 ([2]). Let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers, and let $A = b_1 \log \alpha_1 + \ldots + b_m \log \alpha_m$ for some $b_1, \ldots, b_m \in \mathbb{Z}$. If $A \neq 0$, then we have

$$|A| \geq \exp \left( -18(m + 1)!m^{m+1}(32d)^{m+2}(\log 2md) \left( \prod_{i=1}^{m} A_i \right) (\log B) \right),$$

where $d$ is the degree of $\mathbb{Q}(\alpha_1, \ldots, \alpha_m)$,

$$A_i = \max \left( h(\alpha_i), \frac{1}{d} \left| \log \alpha_i \right|, \frac{1}{d} \right), \quad i = 1, \ldots, m,$

and $B = \max(|b_1|, \ldots, |b_m|, e^{1/d})$.

LEMMA 2 ([4, Notes of Chapter 5]). Let $K$ be an algebraic number field of degree $d$, and $h_K$, $R_K$, $O_K$ be the class number, the regulator and the algebraic integer ring of $K$, respectively. Let $\mu \in O_K \setminus \{0\}$, and let $F(X, Y) = a_0X^n + a_1X^{n-1}Y + \ldots + a_nY^n \in O_K[X, Y]$ be a binary form of degree $n$. If $F(z, 1)$ has at least three distinct zeros, then all solutions $(x, y)$ of the equation

$$f(x, y) = \mu, \quad x, y \in O_K,$$

satisfy

$$\max(|x|, |y|) \leq \exp(5(d + 1)^{50(d+2)}n^6(h_K R_K)^7 \log \max(e^\psi, HM)),
$$

where $H = \max(|x|, |y|, \ldots, |x|)$ and $M = |\mu|$.

Proof of Theorem. Let $(k, q, n)$ be a solution of (1). By [3], we may assume that $q \geq 47$ and $n \geq 4$. From (1) we get

$$(2) \quad (2k - 1)^2 - 17 = (2k - 1 + \sqrt{17})(2k - 1 - \sqrt{17}) = 8q^n.$$

Let $K = \mathbb{Q}(\sqrt{17})$, and let $h_K$, $R_K$, $O_K$, $U_K$ be the class number, the regulator, the algebraic integer ring and the unit group of $K$, respectively. It is a well-known fact that $h_K = 1$, $R_K = \log(4 + \sqrt{17})$, $O_K = \{(a + b\sqrt{17})/2 \mid a, b \in \mathbb{Z}, a \equiv b \pmod{2}\}$ and $U_K = \{\pm(4 + \sqrt{17})^s \mid s \in \mathbb{Z}\}$. Since $5^2 - 17 = 8$ and $(4 + \sqrt{17})^2 = 33 + 8\sqrt{17}$, we see from (2) that

$$(3) \quad \frac{2k - 1 + \sqrt{17}}{2} = \left( \frac{5 + \delta_1 \sqrt{17}}{2} \right) \left( \frac{X_1 + \delta_2 Y_1 \sqrt{17}}{2} \right)^n (33 + 8\sqrt{17})^s,
$$

where $X_1, Y_1 \in \mathbb{N}$ satisfy

$$(4) \quad X_1^2 - 17Y_1^2 = 4q, \quad X_1 \equiv Y_1 \pmod{2}, \quad \gcd(X_1, Y_1) = \begin{cases} 1 & \text{if } 2 \nmid X_1, \\ 2 & \text{if } 2 \mid X_1. \end{cases}$$

For any $u, v \in \mathbb{Z}$ with $u^2 - 17v^2 = 1$, if $X + Y \sqrt{17} = (X_1 \pm Y_1 \sqrt{17}) \times (u + v\sqrt{17})$ and $X + Y \sqrt{17} \in \mathbb{Z}$, then $X, Y \in \mathbb{Z}$ satisfy
\[ X^2 - 17Y^2 = 4q, \quad X \equiv Y \pmod{2}, \quad \gcd(X, Y) = \begin{cases} 1 & \text{if } 2 \nmid X_1, \\ 2 & \text{if } 2 \mid X_1, \end{cases} \]

by (4). Therefore, we may assume that \( X_1 \) and \( Y_1 \) satisfy

\[ 1 < \frac{X_1 + Y_1 \sqrt{17}}{X_1 - Y_1 \sqrt{17}} < (33 + 8\sqrt{17})^2. \]

Notice that \( q \geq 47, n \geq 4, 2k - 1 \geq 6249, \)
\[ \left| \frac{2k - 1 + \sqrt{17}}{2k - 1 - \sqrt{17}} \right| < 1.02 \text{ and } 10.40 < \frac{5 + \sqrt{17}}{5 - \sqrt{17}} < 10.41. \]

Since

\[ \frac{2k - 1 - \sqrt{17}}{2} = \left( \frac{5 - \delta \sqrt{17}}{2} \right) \left( \frac{X_1 - \delta Y_1 \sqrt{17}}{2} \right)^n (33 - 8\sqrt{17}), \]

by (3), we find from (3), (5) and (6) that

\[ |s| \leq 2n. \]

Let \( \eta = (5 + \sqrt{17})/2, \eta = (5 - \sqrt{17})/2, \varepsilon = (X_1 + Y_1 \sqrt{17})/2, \varphi = (X_1 - Y_1 \sqrt{17})/2, \varphi = 33 + 8\sqrt{17}, \).

Further, let \( r = |s|, \alpha_1 = \eta/\eta, \alpha_2 = \varphi, \alpha_3 = \varepsilon/\varphi. \) Then we have

\[ h(\alpha_1) = \log(5 + \sqrt{17}), \quad h(\alpha_2) = \log(4 + \sqrt{17}). \]

Further, by (5), we get

\[ h(\alpha_3) = \log(X_1 + Y_1 \sqrt{17}) < \log 2q\sqrt{q}. \]

From (3) and (6), we have

\[ \log \frac{2k - 1 + \sqrt{17}}{2k - 1 - \sqrt{17}} = \lambda_1 \log \alpha_1 + 2\lambda_2 r \log \alpha_2 + \lambda_3 n \log \alpha_3. \]

\[ \lambda_1, \lambda_2, \lambda_3 \in \{-1, 1\}. \]

Let \( \Lambda = \lambda_1 \log \alpha_1 + 2\lambda_2 r \log \alpha_2 + \lambda_3 n \log \alpha_3. \) Since \( \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\sqrt{17}), \)

by Lemma 1, we get from (7), (8) and (9) that if \( \Lambda \neq 0, \)

\[ |A| \geq \exp(-18(4!)^3465(\log 12)(\log(5 + \sqrt{17})) \times (\log(4 + \sqrt{17}))(\log 2(33 + 8\sqrt{17}))(\log 4n)) \]

\[ > \exp(-5 \cdot 10^{14}(5 + \log \sqrt{q})(\log 4n)). \]

On the other hand, from (1) we get

\[ \log \frac{2k - 1 + \sqrt{17}}{2k - 1 - \sqrt{17}} = \frac{2\sqrt{17}}{2k - 1 - \sqrt{17}} \sum_{i=0}^{\infty} \frac{1}{2i + 1} \left( \frac{\sqrt{17}}{2k - 1} \right)^{2i} < \frac{3\sqrt{17}}{2k - 1} < \frac{4.4}{q^{\alpha/2}}. \]

Combination of (9), (11) and (12) yields

\[ \log 4.4 + 5 \cdot 10^{14}(5 + \log \sqrt{q})(\log 4n) > n \log \sqrt{q}. \]
whence we obtain

\[(13) \quad n < 2 \cdot 10^{17}.
\]

Let

\[F(X, Y) = \left(\frac{5 + \delta_1 \sqrt{17}}{2}\right) \varphi^n X^n - \left(\frac{5 - \delta_1 \sqrt{17}}{2}\right) \varphi^n Y^n \in O_K[X, Y].\]

Since \(n \geq 3\) and \((5 + \sqrt{17})/2\) is a prime in \(O_K\), \(F(z, 1)\) has at least three distinct zeros. We see from (3) and (6) that \((x, y) = ((X_1 + \delta_2 Y_1 \sqrt{17})/2, (X_1 - \delta_2 Y_1 \sqrt{17})/2)\) is a solution of the equation

\[(14) \quad F(x, y) = \sqrt{17}, \quad x, y \in O_K.\]

Therefore, by Lemma 2, from (4), (7) and (14) we get

\[(15) \quad \sqrt{q} < \frac{X_1 + Y_1 \sqrt{17}}{2} = \max \left(\frac{X_1 + \delta_2 Y_1 \sqrt{17}}{2}, \frac{X_1 - \delta_2 Y_1 \sqrt{17}}{2}\right) \leq \exp \left(5 \cdot 3^{200} n^6 (\log(4 + \sqrt{17}))^7 \log \left(\frac{5 + \sqrt{17}}{2}\right)(33 + 8\sqrt{17})n^2\right).\]

Substituting (13) into (15), we obtain \(q < 10^{10^{182}}\). Finally, from (2) we get \(k < 10^{10^{182}}\). The Theorem is proved.

REFERENCES


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