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ON SOME PROPERTIES OF THE CLASS OF STATIONARY SETS

BY

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Some new properties of the stationary sets (defined by G. Pisier in [12]) are studied. Some arithmetical conditions are given, leading to the non-stationarity of the prime numbers. It is shown that any stationary set is a set of continuity. Some examples of "large" stationary sets are given, which are not sets of uniform convergence.

1. Introduction, notations and definitions. Let G be an infinite metrizable compact abelian group, equipped with its normalized Haar measure dx, and Γ its dual (discrete and countable). G will be mostly the unit circle of the complex plane and then Γ will be identified with $\mathbb{Z}$ by $p \to e_p$, where $e_p(x) = e^{2i\pi px}$.

We shall denote by $\mathcal{P}(G)$ the set of trigonometric polynomials over G, i.e. finite sums $\sum_{\gamma\in\Gamma} a_\gamma\gamma$, where $a_\gamma \in \mathbb{C}$; this is also the vector space of functions over G spanned by Γ.

We shall denote by $C(G)$ the space of complex continuous functions over G, with the norm $\|f\|_\infty = \sup_{x\in G}|f(x)|$. This is also the completion of $\mathcal{P}(G)$ for $\|\cdot\|_\infty$.

$M(G)$ will denote the space of complex regular Borel measures over G, equipped with the total variation norm. If $\mu \in M(G)$, its *Fourier transform* at the point γ is defined by $\widehat{\mu}(\gamma) = \int_G \gamma(-x)\,d\mu(x)$.

$L^p(G)$ denotes the Lebesgue space $L^p(G, dx)$ with the norm

$$\|f\|_p = \begin{cases} (\int_G |f(x)|^p\,dx)^{1/p}, & 1 \le p < \infty,\\ \operatorname{ess\,sup}|f(x)|, & p = \infty.\end{cases}$$

The map $f \to f\,dx$ identifies $L^1(G)$ with a closed ideal of $M(G)$ equipped with the convolution.

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If B is a normed space of functions over G which is continuously injected in $M(G)$, and if Λ is a subset of Γ, we shall set

$$B_\Lambda = \{f \in B \mid \widehat{f}(\gamma) = 0\ \forall \gamma \notin \Lambda\}.$$

This is also the set of elements of B whose spectrum is contained in Λ.

$(\varepsilon_\gamma)_{\gamma\in\Gamma}$ will denote a Bernoulli sequence indexed by Γ, i.e. a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, taking values $+1$ and -1 with probability $1/2$. Also, $(g_\gamma)_{\gamma\in\Gamma}$ will denote a sequence of centred independent complex Gaussian random variables, normalized by $\mathbb{E}|g_\gamma|^2 = 1$.

$|E|$ will denote the cardinality of a finite set E.

Let us now recall some classical definitions of lacunary subsets of Γ.

Definition 1.1. Let Λ be a subset of Γ. Then Λ is a *Sidon set* if it satisfies one of the following equivalent conditions:

(i) There is $C > 0$ such that for all $P \in \mathcal{P}_\Lambda(G)$, $\sum_{\gamma\in\Lambda} |\widehat{P}(\gamma)| \le C\|P\|_\infty$.

(ii) There is $C > 0$ such that for all $f \in C_\Lambda(G)$, $\sum_{\gamma\in\Lambda} |\widehat{f}(\gamma)| \le C\|f\|_\infty$.

(iii) There is $C > 0$ such that for all $(b_\lambda)_{\lambda\in\Lambda} \in \ell^\infty(\Lambda)$ with $\|b\|_\infty = 1$ there exists $\mu \in M(G)$ with $\|\mu\| \le C$ such that $\widehat{\mu}(\lambda) = b_\lambda$ for all $\lambda \in \Lambda$.

(iv) There is $C > 0$ such that for all $(b_\lambda)_{\lambda\in\Lambda} \in c_0(\Lambda)$ with $\|b\|_\infty = 1$ there exists $f \in L^1(G)$ with $\|f\|_1 \le C$ such that $\widehat{f}(\lambda) = b_\lambda$ for all $\lambda \in \Lambda$.

For a deep study of Sidon sets, see [4], [9] or [13].

Definition 1.2. A subset A of Γ is *dissociated* (resp. *quasi-independent*) if for every $(n_\gamma)_{\gamma\in A} \in \{-2, \dots, 2\}^A$ (resp. for all $(n_\gamma)_{\gamma\in A} \in \{-1, 0, 1\}^A$) with almost all n_γ equal to zero,

$$\prod_{\gamma\in A} \gamma^{n_\gamma} = 1 \Rightarrow \forall \gamma \in A : \gamma^{n_\gamma} = 1.$$

We recall that if A is dissociated, then A is a Sidon set.

Definition 1.3. Let $(F_N)_{N\ge0}$ be an increasing sequence of finite subsets of Γ such that $\bigcup_{N=0}^\infty F_N = \Gamma$. Then a subset Λ of Γ is a *set of uniform convergence* relative to $(F_N)_{N\ge0}$ (for short *UC set*) if for every $f \in C_\Lambda(G)$, $(S_N f)_{N\ge0}$ converges to f in $C_\Lambda(G)$, where $S_N f = \sum_{\gamma\in F_N} \widehat{f}(\gamma)\gamma$.

We define, in this case, the *UC constant* (denoted by $U(\Lambda)$) as

$$\sup\{\|S_N(f)\|_\infty \mid f \in C_\Lambda(G),\ \|f\|_\infty = 1,\ N \ge 0\}.$$

We also recall that Λ (included in $\mathbb{Z}$) is a *CUC set* if it is a UC set such that $\sup_{p\in\mathbb{Z}} U(p + \Lambda)$ is finite.

Remark. This notion, closely linked with the choice of $(F_N)_{N\ge0}$, is particularly studied in two cases: $G = \mathbb{T}$ and G being the Cantor group. Here we shall be interested in the case $G = \mathbb{T}$, where the natural choice of

$(F_N)_{N\geq 0}$ is $F_N=\{-N,\dots,N\}$. For a (non-exhaustive) review on UC sets, one may read [8].

DEFINITION 1.4. Let Λ be included in $\mathbb{Z}$. Then Λ is a *set of continuity* if for each $\varepsilon>0$ there is $\delta>0$ such that for all $\mu\in M(\mathbb{T})$ with $\|\mu\|=1$,

$$\overline{\lim_{\mathbb{Z}\setminus\Lambda}}\,|\widehat{\mu}(n)|<\delta\Rightarrow\overline{\lim_{\Lambda}}\,|\widehat{\mu}(n)|<\varepsilon.$$

The links between the sets of continuity and some other thin sets (in particular UC; $\Lambda(1)$; p-Sidon) were studied in [6].

DEFINITION 1.5. Let $0<p<\infty$ and A be a subset of Γ. Then A is a *$\Lambda(p)$ set* if $L^p_A(G)=L^q_A(G)$ for some $0<q<p$.

Let us mention that, in this case, we have $L^p_A(G)=L^r_A(G)$ for all $r\in{]0,p[}$.

DEFINITION 1.6. Let $1\leq p<2$ and Λ be a subset of Γ. Then Λ is a *p-Sidon set* if

$$\exists C>0\ \forall f\in\mathcal{P}_\Lambda(G):\quad\Big(\sum_{\lambda\in\Lambda}|\widehat{f}(\lambda)|^p\Big)^{1/p}\leq C\|f\|_\infty.$$

The best constant C is called the *p-Sidonicity constant* of Λ and is denoted by $S_p(\Lambda)$ (see for example [1] or [3]). Obviously, Λ being a p-Sidon set implies Λ is a q-Sidon set for $q>p$. If Λ is a p-Sidon set and not a q-Sidon set for any $q<p$, then Λ is called a *true p-Sidon set*.

Let us also introduce a fairly exotic norm on $\mathcal{P}(G)$, the *$C^{\mathrm{a.s.}}$ norm* ("almost surely continuous"), defined by

$$[\![f]\!]=\int_\Omega\Big\|\sum_{\gamma\in\Gamma}\varepsilon_\gamma(\omega)\widehat{f}(\gamma)\gamma\Big\|_\infty d\mathbb{P}(\omega).\tag{1}$$

REMARK. Marcus and Pisier [10] showed that an equivalent norm is defined by taking a Gaussian sequence $(g_\gamma)_{\gamma\in\Gamma}$ instead of the Bernoulli sequence $(\varepsilon_\gamma)_{\gamma\in\Gamma}$ in (1).

$C^{\mathrm{a.s.}}(G)$ is, by definition, the completion of $\mathcal{P}(G)$ for the norm $[\![\cdot]\!]$. This is also the set of functions in $L^2(G)$ such that the integral in (1) is finite, or the set of functions in $L^2(G)$ such that, almost surely, $\varepsilon_\gamma(\omega)\widehat{f}(\gamma)=\widehat{f^\omega}(\gamma)$ with f^ω in $C(G)$ (for the equivalence of the quantitative and the qualitative definition, we refer to [7]); $C^{\mathrm{a.s.}}(G)$ is also called the space of almost surely continuous random Fourier series.

Following the spectacular result of Drury ("the union of two Sidon sets is a Sidon set"), a lot of improvements were achieved in the 70's about such sets Λ. Rider, in particular, showed that they may be characterized by the following "a priori" inequality:

$$\sum_{\gamma\in\Gamma} |\widehat{f}(\gamma)| \le C[\![f]\!], \quad \forall f \in \mathcal{P}_\Lambda(G),$$

and Pisier [12] then observed that they may also be characterized by the "a priori" inequality $\|f\|_\infty \le C[\![f]\!]$, for all $f \in \mathcal{P}_\Lambda(G)$, i.e. one has the continuous inclusion $C_\Lambda^{\text{a.s.}}(G) \subset C_\Lambda(G)$. This led him to consider the class $\mathcal{S}$ of subsets of Γ satisfying the reverse "a priori" inequality, $[\![f]\!] \le C\|f\|_\infty$, for all $f \in \mathcal{P}_\Lambda(G)$, which corresponds to the continuous inclusion $C_\Lambda(G) \subset C_\Lambda^{\text{a.s.}}(G)$. He called the elements of this class $\mathcal{S}$ stationary. We have the following precise:

Definition 1.7. A subset Λ of Γ is *stationary* (for short, $\Lambda \in \mathcal{S}$) if

$$\exists C > 0\ \forall f \in \mathcal{P}_\Lambda(G): \quad [\![f]\!] \le C\|f\|_\infty.$$

The best constant C is called the *stationarity constant* of Λ and is denoted by $K_{\mathrm{S}}(\Lambda)$.

Pisier showed that $\mathcal{S}$ contains Sidon sets and finite products of such sets. Thus $\mathcal{S}$ is strictly larger than the class of Sidon sets, because of the following: if $\Lambda_1, \dots, \Lambda_k$ are infinite Sidon subsets of the groups $G_1, \dots, G_k$, then $\Lambda_1 \times \dots \times \Lambda_k$ is a true $\frac{2k}{k+1}$-Sidon subset of the group $G_1 \times \dots \times G_k$. Bourgain [2] also proved that if A_1 and A_2 are infinite then

$$A_1 \times A_2 \in \mathcal{S} \Leftrightarrow A_1, A_2 \in \mathcal{S} \cap \Lambda(2).$$

In spite of these results, the class $\mathcal{S}$ does not seem to have been thoroughly investigated yet. In this work we compare it to some other class of lacunary sets of harmonic analysis, in particular UC sets and sets of continuity, which were previously defined.

We shall need some remarkable inequalities, related to the $[\![\cdot]\!]$ norm. The inequality of Salem–Zygmund [14] will be used in the following form:

$$\text{(2)} \quad \exists C > 0\ \forall (a_n)_{n\ge 0},\ |a_n| = 1,\ \forall N \ge 1: \quad \Big[\!\!\Big[\sum_{n=0}^{N-1} a_n e_n\Big]\!\!\Big] \ge C\sqrt{N\log N}.$$

The inequality of Marcus–Pisier [10] is as follows: there exists a (numerical) constant $D > 0$ such that, for every sequence $(a_\gamma)_{\gamma\in\Gamma}$, denoting by $(a_k^*)_{k\ge 0}$ the decreasing rearrangement of $(|a_\gamma|)_{\gamma\in\Gamma}$, one has

$$\text{(3)} \qquad \Big[\!\!\Big[\sum_{\gamma\in\Gamma} a_\gamma \gamma\Big]\!\!\Big]_{C(G)} \ge D\Big[\!\!\Big[\sum_{k\ge 0} a_k^* e_k\Big]\!\!\Big]_{C(\mathbb{T})}.$$

2. Preliminary results. In the sequel, we shall use the previous two inequalities in the following way (c denotes a numerical constant which can vary from line to line):

LEMMA 2.1. *Take* $P \in \mathcal{P}(G)$. *Set* $E_\delta = \{\gamma \in \Gamma \mid |\widehat{P}(\gamma)| \geq \delta\}$ *and* $N_\delta = |E_\delta|$ $(\delta > 0)$. *Then*

$$[\![P]\!] \geq c\delta\sqrt{N_\delta \log N_\delta}.$$

Proof. By the contraction principle [7] we have

$$2[\![P]\!] \geq \Big[\!\!\Big[\sum_{\gamma\in E_\delta} |\widehat{P}(\gamma)|\gamma \Big]\!\!\Big] \geq \delta \Big[\!\!\Big[\sum_{\gamma\in E_\delta} \gamma \Big]\!\!\Big].$$

Using (3), we obtain

$$[\![P]\!] \geq c\delta \Big[\!\!\Big[\sum_{k=0}^{N_\delta - 1} e_k \Big]\!\!\Big]_{C(\mathbb{T})}$$

and then using (2), we have

$$[\![P]\!] \geq c\delta\sqrt{N_\delta \log N_\delta}. \ \blacksquare$$

Similarly to Sidon sets, there are several equivalent functional definitions of stationary sets. Indeed, we have the following proposition:

PROPOSITION 2.2. *The following assertions are equivalent for a stationary subset* Λ *of* Γ:

(i) $C_\Lambda(G) \subset C_\Lambda^{\mathrm{a.s.}}(G)$.

(ii) *There is* $K > 0$ *such that* $[\![f]\!] \leq K\|f\|_\infty$ *for all* $f \in C_\Lambda(G)$.

(iii) *There is* $K > 0$ *such that for all* $(\mu_\alpha) \in L^\infty(\Omega, \mathcal{A}, \mathbb{P}, M(G))$ *with* $\|\mu_\alpha\| \leq 1$ $\mathbb{P}$-*a.s., there exists* $\mu \in M(G)$ *with* $\|\mu\| \leq K$ *such that* $\widehat{\mu}(\gamma) = \int_\Omega \widehat{\mu}_\alpha(\gamma)\varepsilon_\gamma(\alpha)\, d\mathbb{P}(\alpha)$ *for all* $\gamma \in \Lambda$.

Proof. (i)⇒(ii). Just use the closed graph theorem.

(ii)⇒(i). Trivial.

(ii)⇒(iii). Take (μ_α) in $L^\infty(\Omega, \mathcal{A}, \mathbb{P}, M(G))$ with $\|\mu_\alpha\| \leq 1$ $\mathbb{P}$-a.s. The map $T : \mathcal{P}_\Lambda(G) \to \mathbb{C}$ defined by

$$\forall f \in \mathcal{P}_\Lambda(G): \quad T(f) = \int_\Omega \mu_\alpha * f^\alpha(0)\, d\mathbb{P}(\alpha)$$

is a linear form on $\mathcal{P}_\Lambda(G)$, with norm bounded by K. Indeed, for all $f \in \mathcal{P}_\Lambda(G)$ we have

$$|T(f)| \leq \int_\Omega \|\mu_\alpha\| \cdot \|f^\alpha\|_\infty\, d\mathbb{P}(\alpha) \leq \int_\Omega \|f^\alpha\|_\infty\, d\mathbb{P}(\alpha) = [\![f]\!] \leq K\|f\|_\infty.$$

By the Hahn–Banach theorem, T extends to $\widetilde{T}$ belonging to $C(G)^*$ with $\|\widetilde{T}\| = \|T\| \leq K$. The Riesz representation theorem gives the existence of a measure μ in $M(G)$ with norm less than K such that

$$\forall f \in C(G): \quad \widetilde{T}(f) = \mu * f(0). \tag{4}$$

Testing (4) on γ belonging to Λ, we get

$$\forall\gamma \in \Lambda : \quad \widehat{\mu}(\gamma) = \widetilde{T}(\gamma) = \int\limits_{\Omega} \widehat{\mu}_\alpha(\gamma)\varepsilon_\gamma(\alpha)\, d\mathbb{P}(\alpha),$$

that is, we get (iii).

(iii)$\Rightarrow$(ii). Let $f \in \mathcal{P}_\Lambda(G)$. By [7], $C^{\mathrm{a.s.}}(G)$ embeds in $L^1(\Omega, \mathcal{A}, \mathbb{P}, C(G))$ and $(L^1(\Omega, \mathcal{A}, \mathbb{P}, C(G)))^* = L^\infty(\Omega, \mathcal{A}, \mathbb{P}, M(G))$, so we get

$$[\![f]\!] = \sup\Big\{\Big|\int\limits_{\Omega} \mu_\alpha * f^\alpha(0)\, d\mathbb{P}(\alpha)\Big| \;\Big|\; (\mu_\alpha) \in L^\infty(\Omega, \mathcal{A}, \mathbb{P}, M(G)) \text{ with } \|\mu_\alpha\| \le 1 \text{ a.s.}\Big\}$$

Therefore, for each (μ_α) in the unit ball of $L^\infty\big(\Omega, \mathcal{A}, \mathbb{P}, M(G)\big)$, the condition (iii) yields $\mu \in M(G)$ with $\|\mu\| \le K$ such that $\widehat{\mu}(\gamma) = \int_\Omega \widehat{\mu}_\alpha(\gamma)\varepsilon_\gamma(\alpha)\, d\mathbb{P}(\alpha)$ for $\gamma \in \Lambda$; then we have

$$\int\limits_{\Omega} \mu_\alpha * f^\alpha(0)\, d\mathbb{P}(\alpha) = \sum_{\gamma\in\Lambda} \widehat{f}(\gamma)\Big(\int\limits_{\Omega} \widehat{\mu}_\alpha(\gamma)\varepsilon_\gamma(\alpha)\, d\mathbb{P}(\alpha)\Big)\gamma = f * \mu(0),$$

and so

$$\Big|\int\limits_{\Omega} \mu_\alpha * f^\alpha(0)\, d\mathbb{P}(\alpha)\Big| \le \|\mu\| \cdot \|f\|_\infty \le K\|f\|_\infty.$$

Taking the upper bound of the left hand side over the unit ball of $L^\infty(\Omega, \mathcal{A}, \mathbb{P}, M(G))$, we get $[\![f]\!] \le K\|f\|_\infty$, that is, we get (ii). ∎

One may notice that the probabilistic point of view cannot be replaced by a topological one. More precisely, one cannot replace "almost sure convergence" by "quasi-sure convergence" in the foregoing. Indeed, for Λ being a subset of Γ, we consider the Cantor group $\{-1, 1\}^\Lambda$ with its usual topology and denote by $r_\gamma(\alpha)$ the γth coordinate of $\alpha \in \{-1, 1\}^\Lambda$, for γ belonging to Λ. Suppose that Λ has the following property:

(P) For each $f \in C_\Lambda(G)$ there is Ω_f, a dense G_δ in $\{-1, 1\}^\Lambda$, such that for every $\alpha \in \Omega_f$ there exists $f^\alpha \in C_\Lambda(G)$ with $\widehat{f^\alpha}(\gamma) = r_\gamma(\alpha)\widehat{f}(\gamma)$ for all $\gamma \in \Lambda$.

Then Λ is necessarily a Sidon set.

This follows from the following more general lemma:

LEMMA 2.3. *Let X be a Banach space. Assume that the sequence $(x_n)_{n\ge0}$ in X has the following property: there is Ω_1, a dense G_δ in $\{-1, 1\}^{\mathbb{N}}$, such that $\sum_{n\ge0} r_n(\alpha)x_n$ converges in X for all $\alpha \in \Omega_1$. Then $\sum_{n\ge0} x_n$ converges unconditionally in X.*

P r o o f. Fix $p \ge 1$. For every $q \ge 1$, set

$$F_q = \Big\{\omega \in \Omega_1 \ \Big|\ \forall m', m \ge q : \Big\| \sum_{n=m}^{m'} r_n(\omega) x_n \Big\| \le 1/p\Big\}.$$

The assumption gives $\bigcup_{q\in\mathbb{N}^*} F_q = \Omega_1$.

Let $\omega \in \overline{F}_q \cap \Omega_1$. Then for all $m' \ge m \ge q$ there is $\alpha \in F_q$ such that $r_n(\omega) = r_n(\alpha)$ for $n \le m'$. We then have

$$\Big\| \sum_{n=m}^{m'} r_n(\omega) x_n \Big\| = \Big\| \sum_{n=m}^{m'} r_n(\alpha) x_n \Big\| \le \frac{1}{p} \quad \text{for } \alpha \in F_q.$$

So $\omega \in F_q$ and F_q is closed in Ω_1.

Ω_1 is a Baire space (as an intersection of dense open subsets of the compact $\{-1,1\}^{\mathbb{N}}$). So we have

$$\exists q \ge 1 : \quad \mathring{F}_q^{(\Omega_1)} \ne \emptyset,$$

that is, there are $c \in \Omega_1$ and $N \ge 1$ with the property that for all $\omega' \in \Omega_1$ such that $r_n(\omega') = r_n(c)$ for each $n \le N$, one has, for every $m' \ge m \ge q$, $\| \sum_{n=m}^{m'} r_n(\omega') x_n \| \le 1/p$ (roughly speaking, $B(c,N) \subset F_q$). We set $\widetilde{q} = \max(N+1, q)$.

Take $m' \ge m \ge \widetilde{q}$, $\omega \in \{-1,1\}^{\mathbb{N}}$ and define ω_1 by

$$r_n(\omega_1) = \begin{cases} r_n(c) & \text{if } n \le N, \\ r_n(\omega) & \text{if } n \ge N+1. \end{cases}$$

Then the density of Ω_1 yields $\omega' \in \Omega_1$ such that $r_n(\omega') = r_n(\omega_1)$ for every $n \le m'$. We then obtain, for $m' \ge m \ge \widetilde{q} \ge N+1$,

$$\begin{aligned} \Big\| \sum_{n=m}^{m'} r_n(\omega) x_n \Big\| &= \Big\| \sum_{n=m}^{m'} r_n(\omega_1) x_n \Big\| = \Big\| \sum_{n=m}^{m'} r_n(\omega') x_n \Big\| \\ &\le \frac{1}{p} \quad \text{for } \omega' \in \Omega_1 \text{ and } \omega' \in B(c,N) \subset F_q. \end{aligned}$$

We conclude that $\sum_{n\ge1} r_n(\omega) x_n$ converges in X for each $\omega \in \{-1,1\}^{\mathbb{N}}$. ■

COROLLARY 2.4. *If a subset Λ of Γ has the property* (P), *then Λ is a Sidon set.*

P r o o f. Let $f \in C_\Lambda(G)$ and denote Λ by $(\lambda_n)_{n\ge0}$. We define $x_n = \widehat{f}(\lambda_n)\lambda_n$. The sequence $(x_n)_{n\ge0}$ satisfies the assumption of Lemma 2.3. Hence $\sum_{n\ge0} x_n$ converges unconditionally in $C_\Lambda(G)$. In particular, $\sum_{n\ge0} \widehat{f}(\lambda_n)\lambda_n$ is unconditionally convergent for each f in $C_\Lambda(G)$. So, $\{\lambda_n\}$ is an unconditional basis of $C_\Lambda(G)$ and Λ is a Sidon set. ■

In [12], G. Pisier showed, using the Rudin–Shapiro polynomials, that $\mathbb{Z}$ is not a stationary set and more generally that a stationary set cannot contain arbitrarily long arithmetic progressions. It is easy to see that no infinite discrete abelian group may be a stationary set. We shall show even more in the next proposition.

We recall that a *parallelepiped of size* $s \geq 1$ is a set of the form

$$P = \Big\{\beta \prod_{j=1}^{s} \lambda_j^{\varepsilon_j} \;\Big|\; \varepsilon_j \in \{0,1\} \text{ for } 1 \leq j \leq s\Big\} \tag{5}$$

with $\beta, \lambda_1, \ldots, \lambda_s$ in Γ and where the λ_j are distinct.

Proposition 2.5. *Let $\Lambda \subset \Gamma$ be a stationary set. Then Λ cannot contain parallelepipeds of arbitrarily large size.*

Proof. Assume that Λ contains some parallelepiped of size s, arbitrarily large; we may also assume that $\{\lambda_j\}$ is quasi-independent. Indeed, let P_N be a parallelepiped of size N included in Λ. With N fixed, P_N has the form (5). One can choose $\lambda_{j_1} \neq 1$ and we assume some elements $\lambda_{j_1}, \ldots, \lambda_{j_p}$ with $p \geq 1$ are such that $D_p = \{\lambda_{j_q}\}_{1 \leq q \leq p}$ is quasi-independent. We consider the set

$$A_p = \Big\{ \prod_{q=1}^{p} \lambda_{j_q}^{\varepsilon_q} \;\Big|\; \varepsilon_q \in \{-1,0,1\} \text{ for each } 1 \leq q \leq p\Big\},$$

which is of cardinality less than or equal to 3^p. So the set $\{z \in \{\lambda_j\}_{1 \leq j \leq N} \mid z \notin A_p\}$ has a cardinality greater than $N - 3|A_p|$, hence greater than $N - 3^p$. We can continue this construction as long as $N \geq 3^p + 1$, so we can extract $\psi(N)$ elements, forming a quasi-independent subset of Γ, with $\psi(N)$ growing as $\log N$, therefore diverging to ∞.

So, in the sequel, we suppose that the parallelepipeds of arbitrarily large size N have the form (5) with $\{\lambda_1, \ldots, \lambda_N\}$ quasi-independent.

Let us fix N and make the following construction, which generalizes that of Rudin and Shapiro: $R_0 = S_0 = \beta$; then we define by induction, for $0 \leq q \leq N-1$,

$$R_{q+1} = R_q + \lambda_{q+1} S_q, \qquad S_{q+1} = R_q - \lambda_{q+1} S_q.$$

From the parallelogram law, we get $|R_{q+1}|^2 + |S_{q+1}|^2 = 2(|R_q|^2 + |S_q|^2)$. So $|R_q|^2 + |S_q|^2 = 2^{q+1}$ and $\|R_q\|_\infty \leq 2^{(q+1)/2}$.

Now, the quasi-independence gives the following properties for the polynomial R_N:

$$R_N \in \mathcal{P}_\Lambda, \tag{6.1}$$

$$|\{\gamma \in \Gamma \mid \widehat{R}_N(\gamma) \neq 0\}| = 2^{N+1}, \tag{6.2}$$

$$\|R_N\|_\infty \leq 2^{(N+1)/2}, \tag{6.3}$$

$$\forall \gamma \in \Lambda \quad \widehat{R}_N(\gamma) \in \{-1,0,1\}. \tag{6.4}$$

Applying Lemma 2.1 to the polynomials R_N with $\delta = 1$, we get, using (6.2) and (6.4),

$$\exists c > 0 : \quad [\![R_N]\!] \geq c 2^{(N+1)/2}\sqrt{N+1}, \tag{7}$$

and the stationarity of Λ gives, by (6.1),

$$[\![R_N]\!] \leq K_{\mathrm{S}}(\Lambda)\|R_N\|_\infty, \quad \text{and by (6.3)}, \quad [\![R_N]\!] \leq K_{\mathrm{S}}(\Lambda)2^{(N+1)/2}. \tag{8}$$

Finally, the relations (7) and (8) lead to $N \leq (K_S(\Lambda)/c)^2$, which gives an upper bound for the size of the parallelepipeds that can be contained in Λ. This contradiction completes the proof.

Corollary 2.6. *Γ is not a stationary set.*

In the case $\Gamma = \mathbb{Z}$, we shall deduce more precise results from [11]. Let us recall that Miheev showed the following. If a set $\Lambda = \{n_j\}_{j\geq 0}$ of integers does not contain any parallelepiped of size $S \geq s$ (for some $s \geq 2$), then:

$$\begin{cases} \text{(i) there are } m > 1 \text{ and } c > 0 \text{ such that } n_j \geq cj^m,\ j = 1, 2, \ldots, \\ \text{(ii) } \sum_{j\geq 1} 1/n_j \text{ converges.} \end{cases} \tag{9}$$

Corollary 2.7. *Let $\Lambda = \{n_j\}_{j\geq 0}$ be a stationary set of integers. Then Λ enjoys property* (9).

From this, we easily deduce the following proposition:

Proposition 2.8. *The set of prime numbers $(p_j)_{j\geq 1}$ is not a stationary set.*

Proof. $\sum_{j\geq 1} 1/p_j = \infty$. ■

Corollary 2.9. *Let Λ be a stationary set in $\mathbb{Z}$. Then its upper density is zero*, *that is*,

$$\Delta^+(\Lambda) = \lim_N \sup_{a\in\mathbb{Z}} \frac{|\Lambda \cap \{a, \ldots, a+N\}|}{N+1} = 0.$$

3. Stationary sets and sets of continuity. In [6], the authors proved that if Λ is a UC set included in $\mathbb{N}$, then $\mathbb{Z}^- \cup \Lambda$ is a set of continuity. We shall prove a weaker result for stationary sets. The proof relies on the following proposition.

Proposition 3.1. *Let Λ be a stationary set in Γ and $\delta > 0$. Then*

$$\forall \mu \in M_\Lambda(G) : \quad |\{\gamma \in \Lambda \mid |\widehat{\mu}(\gamma)| \geq \delta\}| \leq \exp(c\|\mu\|^2/\delta^2) \tag{10}$$

where c is an absolute constant depending only on Λ. That is, *for each μ belonging to M_Λ*,

$$\{\widehat{\mu}(\gamma)\}_{\gamma\in\Lambda} \in \ell^{\psi,\infty} \tag{11}$$

where $\psi(t) = e^{t^2} - 1$ and $\ell^{\psi,\infty}$ denotes the space $\{(a_n) \mid \sup_{n\geq 1} \psi^{-1}(n)a_n^* < \infty\}$, (a_n^*) being the decreasing rearrangement of $\{|a_n|\}_{n\geq 1}$.

The proof of Proposition 3.1 uses the following lemma:

LEMMA 3.2. *Let Λ be a stationary set in Γ. Then*

$$\exists c>0\ \forall \mu\in M_\Lambda(G)\ \forall h\in L^2(G):\quad [\![\mu * h]\!]\le c\|\mu\|_M\|h\|_2. \tag{12}$$

Proof. Fix μ in $M_\Lambda(G)$. First observe that the operator $T_\mu : C(G)\to C^{\mathrm{a.s.}}(G)$ defined by $T_\mu(h)=\mu * h$ is bounded. Indeed, $\mu * f\in C_\Lambda(G)$ for $f\in C(G)$, hence

$$[\![T_\mu(f)]\!]=[\![f*\mu]\!]\le K_{\mathrm{S}}(\Lambda)\|f*\mu\|_\infty\le K_{\mathrm{S}}(\Lambda)\|\mu\|\cdot\|f\|_\infty.$$

Recall ([12]) that $C^{\mathrm{a.s.}}(G)^*$ can be identified with $M_{2,\psi}$, the space of multipliers from $L^2(G)$ to $L^\psi(G)$, hence for each m in $M_{2,\psi}$ and for each ω in Ω, one has $m^\omega\in M_{2,\psi}$ and $\|m^\omega\|_{M_{2,\psi}}=\|m\|_{M_{2,\psi}}$ ($M_{2,\psi}$ is a space admitting the characters as unconditional basis) where $m^\omega(n):=\varepsilon_n(\omega)m_n$.

So, by duality for each $\omega\in\Omega$, $m\to T_\mu^*(m^\omega)$ is bounded from $M_{2,\psi}$ to $M(G)$ and $\|T_\mu^*(m^\omega)\|_{M(G)}\le\|T_\mu\|\cdot\|m\|_{M_{2,\psi}}$.

Therefore $T_\mu^*(m^\omega)=(\mu*m)^\omega\in M(G)$ for all $\omega\in\Omega$ and so ([7]) $\mu*m\in L^2(G)$. Consequently, we have the diagram

$$\begin{tikzcd} M_{2,\psi} \arrow[rr, "T_\mu^*"] \arrow[dr, "U"'] & & M(G) \\ & L^2(G) \arrow[ur, "\text{injection}"'] & \end{tikzcd}$$

and by duality again, we have the following factorization:

$$\begin{tikzcd} C(G) \arrow[rr, "T_\mu"] \arrow[dr, "\text{injection}"'] & & C^{\mathrm{a.s.}}(G) \\ & L^2(G) \arrow[ur, "U^*"'] & \end{tikzcd}$$

that is,

$$\exists c>0\ \forall h\in C(G):\quad [\![T_\mu(h)]\!]=[\![U^*(h)]\!]\le c\|\mu\|\cdot\|h\|_2$$

and the density of $C(G)$ in $L^2(G)$ leads to

$$\exists c>0\ \forall h\in L^2(G):\quad [\![T_\mu(h)]\!]\le C\|\mu\|\cdot\|h\|_2. \ ■$$

REMARK. It may be noticed that it is easy to prove the same result using the Kahane–Katznelson–de Leeuw theorem:

$$\exists c>0\ \forall h\in L^2(G)\ \exists f\in C(G):\quad \|f\|_\infty\le c\|h\|_2$$

and

$$\forall\gamma\in\Gamma:\quad |\widehat{f}(\gamma)|\ge|\widehat{h}(\gamma)|.$$

Another proof, similar to the one given here, can be made through the Pietsch factorization theorem, noticing that T_μ is 2-summing.

Proof of Proposition 3.1. Let μ belong to $M_\Lambda(G)$ and $\delta > 0$. Let $\Lambda_\delta = \{\gamma \in \Lambda \mid |\widehat{\mu}(\gamma)| \geq \delta\}$; denote by Λ'_δ any finite subset of Λ_δ. Then

$$f := \frac{1}{|\Lambda'_\delta|^{1/2}} \sum_{\gamma \in \Lambda'_\delta} \gamma \in L^2(G) \quad \text{and} \quad \|f\|_2 = 1.$$

Upon using Lemma 3.2, (12) leads to

$$\exists c > 0 : \quad [\![f * \mu]\!] \leq c\|\mu\|. \tag{13}$$

By observing that

$$\forall \gamma \in \Lambda'_\delta : \quad \widehat{f * \mu}(\gamma) = \frac{1}{|\Lambda'_\delta|^{1/2}} \widehat{\mu}(\gamma),$$

Lemma 2.1 leads to the inequality

$$\exists c' > 0 : \quad [\![f * \mu]\!] \geq c' \frac{\delta}{|\Lambda'_\delta|^{1/2}} (|\Lambda'_\delta| \log |\Lambda'_\delta|)^{1/2} = c'\delta(\log |\Lambda'_\delta|)^{1/2}.$$

Consequently, via (13) we obtain

$$\exists c_1 > 0 : \quad c_1\|\mu\| \geq \delta(\log |\Lambda'_\delta|)^{1/2}.$$

Taking the upper bound over all finite subsets Λ'_δ of Λ_δ, we see that Λ_δ itself is finite and that $c_1\|\mu\| \geq \delta(\log |\Lambda_\delta|)^{1/2}$ for some $c_1 > 0$; equivalently,

$$\exists c_1 > 0 \ \forall \delta > 0 : \quad |\Lambda_\delta| \leq \exp(c_1^2\|\mu\|^2/\delta^2)$$

where c_1 does not depend on μ; this proves (11).

This can also be written

$$\exists D > 0 \ \forall \delta > 0 \ \forall \mu \in M_\Lambda : \quad |\Lambda_\delta| \leq \psi(D\|\mu\|/\delta).$$

Let $(b_j)_{j \geq 1}$ be the decreasing rearrangement of $\{|\widehat{\mu}(\gamma)|\}_{\gamma \in \Lambda}$. Given $n \in \mathbb{N}^*$ and $\ell \in \mathbb{N}^*$ such that $b_\ell \geq D\|\mu\|/\psi^{-1}(n)$, we apply the previous result with $\delta = (\psi^{-1}(n))^{-1}D\|\mu\|$ to get

$$n \geq |\{\gamma \in \Lambda \mid |\widehat{\mu}(\gamma)| \geq \delta\}| = |\{p \in \mathbb{N}^* \mid b_p \geq \delta\}| \geq \ell$$

so, in particular, $b_n \leq \delta$ and $\sup_n b_n\psi^{-1}(n) \leq D\|\mu\|$; this proves (11). ∎

An immediate corollary is:

COROLLARY 3.3. *Each stationary set Λ of Γ is a Rajchman set. That is,*

$$\forall \mu \in M_\Lambda(G) : \quad \lim_{\gamma \to \infty} \widehat{\mu}(\gamma) = 0.$$

We may also deduce the following stronger result.

THEOREM 3.4. *Every stationary subset of $\mathbb{Z}$ is a set of continuity.*

Proof. Let Λ be a stationary subset of $\mathbb{Z}$. Arguing by contradiction, assume that there is $\varepsilon > 0$ such that for each $\delta > 0$ there exists $\mu \in M(\mathbb{T})$

with $\|\mu\| = 1$ satisfying

$$\overline{\lim_{n\notin\Lambda}}\,|\widehat{\mu}(n)| \le \delta \quad \text{and} \quad \overline{\lim_{n\in\Lambda}}\,|\widehat{\mu}(n)| > \varepsilon;$$

we then have

$$\exists m = m(\delta)\ \forall n \notin \Lambda \text{ with } |n| \ge m(\delta): \quad |\widehat{\mu}(n)| \le \delta.$$

Let us choose a sequence $(h_j)_{j\ge 0}$ in Λ such that

$$\begin{cases} |\widehat{\mu}(h_j)| > \varepsilon \text{ for all } j \ge 0, \\ |h_p| \ge \displaystyle\sum_{j=0}^{p-1} |h_j| + m \text{ for } p \ge 1 \text{ and } |h_0| \ge m, \\ \{h_j\}_{j\ge 0} \text{ is a dissociated set.} \end{cases} \tag{14}$$

Let $N \ge 1$ and $\nu = \mu * R_N - \sum_{n\notin\Lambda} \widehat{\mu * R}_N(n) e_n$, R_N being the Riesz product $\prod_{j=1}^N [1 + \mathrm{Re}(e_{h_j})]$. Since ν belongs to M_Λ, applying Proposition 3.1 to ν, we find that there exists $C > 0$ such that for all $\varepsilon_1 > 0$,

$$\varepsilon_1^2 \log |\Lambda_{\varepsilon_1}| \le C\|\nu\|^2 \le C\Big[\|\mu * R_N\| + \Big\|\sum_{n\notin\Lambda} \widehat{\mu * R}_N(n) e_n\Big\|\Big]^2 \tag{15}$$

(where Λ_{ε_1} denotes the set $\{n \in \mathbb{Z} \mid |\widehat{\nu}(n)| \ge \varepsilon_1\}$). But

$$\|\mu * R_N\| \le \|\mu\| \cdot \|R_N\|_1 \le 1 \tag{16}$$

and

$$\Big\|\sum_{n\notin\Lambda} \widehat{\mu * R}_N(n) e_n\Big\| \le \Big\|\sum_{n\notin\Lambda} \widehat{\mu * R}_N(n) e_n\Big\|_2. \tag{17}$$

One notices that

$$\|R_N\|_2^2 = \sum_{\substack{s=\sum_{k=1}^N \varepsilon_k h_k \\ \varepsilon_k=-1,0,1}} |\widehat{R}_N(s)|^2 = \sum_{t=0}^N C_N^t \frac{1}{4^t} = \left(\frac{5}{4}\right)^N. \tag{18}$$

In fact, if $s = \sum_{k=1}^N \varepsilon_k h_k$ with $\varepsilon_k = -1, 0, 1$ and $\sum_{k=1}^N |\varepsilon_k| = t$, then $\widehat{R}_N(s) = 1/2^t$. On the other hand, $\widehat{R}_N(s) \ne 0$ only for $s = \sum_{k=1}^N \varepsilon_k h_k$ with $\varepsilon_k \in \{-1, 0, 1\}$ (and in that case, $|s| \ge m$). So, in this case, for $s \notin \Lambda$,

$$|\widehat{\mu}(s)| \le \delta. \tag{19}$$

Therefore, (17)–(19) lead to

$$\Big\|\sum_{n\notin\Lambda} \widehat{\mu * R}_N(n) e_n\Big\|_M \le \delta(5/4)^{N/2}. \tag{20}$$

For $1 \leq p \leq N$ we have $|\widehat{\mu}(h_p)| \cdot |\widehat{R}_N(h_p)| \geq \varepsilon/2$ hence $h_p \in \Lambda_{\varepsilon/2}$, so $\{h_1, \ldots, h_N\} \subset \Lambda_{\varepsilon/2}$ and $|\Lambda_{\varepsilon/2}| \geq N$; we have therefore we get from (15), (16) and (20) the inequality

$$(\varepsilon/2)^2 \log N \leq C[1 + \delta(5/4)^{N/2}]^2. \tag{21}$$

Now, take N such that $(\varepsilon/2)^2 \log N > 4C$ and δ such that $\delta < (5/4)^{-N/2}$. Then (21) leads to a contradiction ∎

4. Stationary sets and UC sets. Let us recall that G. Pisier proved the existence of some stationary sets that are not Sidon (conversely, any Sidon set is trivially stationary). We shall generalize this result by exhibiting a class of stationary sets that are not UC sets. Thus, it is possible to construct stationary subsets of $\mathbb{Z}$ rather large in the following sense: for each $k \geq 1$ there is a stationary Λ_k and $\delta_k > 0$ such that

$$\forall N \geq 1 : \quad |\Lambda_k \cap [-N, N]| \geq \delta_k (\log N)^k.$$

THEOREM 4.1. *Let E be a dissociated set in Γ, $E = \{\lambda_j\}_{j \geq 1}$. Let $k > 1$ be an integer. Then*

$$\Lambda_k := \Big\{ \prod_{p=1}^{k} \lambda_{j_p}^{\varepsilon_p} \;\Big|\; \varepsilon_p \in \{-1, 1\},\ (j_p)_{1 \leq p \leq k} \textit{ distinct} \Big\}$$

is a stationary subset of Γ.

Proof. We first follow the method of Blei [1]. In fact, we have

$$\Lambda_k = \Big\{ \prod_{p=1}^{k} \lambda_{j_p} \;\Big|\; j_p \text{ distinct} \Big\} \cup \bigcup_{l=0}^{k-1} \Big\{ \prod_{p=1}^{l} \lambda_{j_p} \prod_{p=l+1}^{k} \overline{\lambda}_{j_p} \;\Big|\; j_p \text{ distinct} \Big\}$$

so that every f in $\mathcal{P}_{\Lambda_k}(G)$ can be written as (in the following $\sum'_{(j_p)}$ will mean $j_1 < \ldots < j_l$ and $j_{l+1} < \ldots < j_k$ for $0 \leq l \leq k-1$, and $j_1 < \ldots < j_k$ for $l = k$)

$$\begin{aligned} f = {} & \sum_{l=0}^{k-1} \Big(\sum_{(j_p)}{}' \widehat{f}(\lambda_{j_1} \ldots \lambda_{j_l} \overline{\lambda}_{j_l+1} \ldots \overline{\lambda}_{j_k}) \lambda_{j_1} \ldots \lambda_{j_l} \overline{\lambda}_{j_{l+1}} \ldots \overline{\lambda}_{j_k} \Big) \\ & + \sum_{(j_p)}{}' \widehat{f}\Big(\prod_{p=1}^{k} \lambda_{j_p} \Big) \prod_{p=1}^{k} \lambda_{j_p}. \end{aligned}$$

Define F in $\mathcal{P}\,(\underbrace{G\times\ldots\times G}_{k\,\text{times}})$ by $F=\sum_{l=0}^{k}F_l$ where

$$(22)\qquad \begin{cases} F_k=\displaystyle\sum_{\substack{(j_p)\\ \text{distinct}}}\widehat{f}\Big(\prod_{p=1}^{k}\lambda_{j_p}\Big)\lambda_{j_1}\otimes\ldots\otimes\lambda_{j_k} \qquad (F_k\in\mathcal{P}_{E\times\ldots\times E}(G^k)),\\ F_l=\displaystyle\sum_{\substack{(j_p)\\ \text{distinct}}}\ \sum_{\substack{\varepsilon_i=\pm1\\ \varepsilon_1+\ldots+\varepsilon_k=2l-k}}\widehat{f}(\lambda_{j_1}\ldots\lambda_{j_l}\overline{\lambda}_{j_{l+1}}\ldots\overline{\lambda}_{j_k})\\ \qquad\qquad\qquad\qquad\qquad\times\lambda_{j_1}^{\varepsilon_1}\otimes\ldots\otimes\lambda_{j_k}^{\varepsilon_k}\qquad(0\le l\le k-1). \end{cases}$$

In the sequel, the cases $l=0$ and $l=k$ are treated in the same way. Fixing $0\le l\le k-1$, F_l is symmetrized by writing

$$(23)\qquad F_l=\sum_{m=1}^{k}(-1)^{m+k}\sum\widehat{F}_l(\lambda_{j_1},\ldots,\lambda_{j_l},\overline{\lambda}_{j_{l+1}},\ldots,\overline{\lambda}_{j_k})\\ \times\psi_S(\lambda_{j_1})\ldots\psi_S(\lambda_{j_l})\overline{\psi_S(\lambda_{j_{l+1}})}\ldots\overline{\psi_S(\lambda_{j_k})}$$

where the second sum runs over the subsets S of $\{1,\ldots,k\}$ with cardinality m and over the distinct indices (j_p) $(1\le p\le k)$ and where $\psi_S(\gamma)(g_1,\ldots,g_k)$ is equal to $\sum_{r\in S}\gamma(g_r)$ with $(g_1,\ldots,g_k)\in G^k$.

Fixing (again) m in $\{1,\ldots,k\}$ and S included in $\{1,\ldots,k\}$ with $|S|=m$, we write $\widetilde{F}$ for

$$\sum_{\substack{(j_p)\\ \text{distinct}}}\widehat{F}_l(\lambda_{j_1},\ldots,\lambda_{j_l},\overline{\lambda}_{j_{l+1}},\ldots,\overline{\lambda}_{j_k})\psi_S(\lambda_{j_1})\ldots\psi_S(\lambda_{j_l})\overline{\psi_S(\lambda_{j_{l+1}})}\ldots\overline{\psi_S(\lambda_{j_k})}$$

(noticing that $\psi_S(\overline{\gamma})=\overline{\psi_S(\gamma)}$). One has $\widetilde{F}\in\mathcal{P}_{E\times\ldots\times E\times\overline{E}\times\ldots\times\overline{E}}$. Fix $g_1,\ldots,g_k$ in G and set

$$(24)\qquad V:=\widetilde{F}(g_1,\ldots,g_k).$$

Introducing the measure ν defined by the Riesz product

$$\prod_{\gamma\in E}[1+\mathrm{Re}(e^i\gamma)],$$

we have

$$\widehat{\nu}(\lambda_{j_1}\ldots\lambda_{j_l}\overline{\lambda}_{j_{l+1}}\ldots\overline{\lambda}_{j_k})=\frac{e^{il}e^{-i(k-l)}}{2^k}=:a_l.$$

There is a polynomial P_l (depending only on k and l) such that

$$P_l(a_l)=1\quad\text{and}\quad P_l(a_t)=0\quad\text{whenever}\quad t\neq l.$$

We now set $\mu_l=P_l(\nu)$ (where the product on $M(G)$ is convolution) and observe that

(25) $\widehat{\mu}_l(\lambda_{j_1}\ldots\lambda_{j_t}\overline{\lambda}_{j_{t+1}}\ldots\overline{\lambda}_{j_k})=\delta_{t,l}$ (Kronecker's symbol) for any $(j_1,\ldots,j_k)$ distinct, and $\|\mu_l\|\le C_k$ with C_k depending only on k.

Finally, we consider the Riesz product

$$\mathcal{R}=\prod_{j\ge 0}[1+\mathrm{Re}(\eta_j\lambda_j)]\quad\text{where}\quad \eta_j=\frac{\psi_S(\lambda_j)}{2^m}(g_1,\ldots,g_k)$$

(notice that $|\eta_j|\le 1$). One easily checks that (remember (24)) $V=2^{mk}2^k\mu_l*\mathcal{R}*f(0)$ and concludes, using (25), that

$$|V|\le 2^{mk}2^k\|\mu_l\|_M\|\mathcal{R}\|_M\|f\|_\infty\le 2^{mk}2^kC_k\|f\|_\infty,$$

and then, taking the upper bound over G^k, we obtain

$$\|\widetilde{F}\|_\infty\le 2^{mk}2^kC_k\|f\|_\infty.$$

Now, considering (23) and the previous majorization, we have

$$\|F\|_\infty\le\sum_{l=0}^{k}\|F_l\|_\infty\le\sum_{l=0}^{k}\sum_{m=1}^{k}\sum_{|S|=m}2^{mk}2^kC_k\|f\|_\infty$$

and

(26) $$\|F\|_\infty\le A_k\|f\|_\infty\qquad(\text{with } A_k=k2^k(2^k+1)^kC_k).$$

E being a dissociated set is a Sidon set; so is $E\cup\overline{E}$, hence by the Pisier theorem ([12]), $\underbrace{E_1\times\ldots\times E_1}_{k\ \text{times}}$ is a stationary set in Γ^k, E_1 denoting $E\cup\overline{E}$. So, there is $B_k>0$ such that for all $F\in\mathcal{P}_{E_1\times\ldots\times E_1}(G^k)$,

(27) $$\int_\Omega\Big\|\sum_{\beta\in E_1^k}\mathcal{E}_\beta(\omega)\widehat{F}(\beta)\beta\Big\|_\infty d\mathbb{P}(\omega)\le B_k\|F\|_\infty.$$

On the other hand, fixing f in $\mathcal{P}_{\Lambda_k}(G)$, one observes that

$$\begin{aligned}\int_\Omega\Big\|\sum_{l=0}^{k}\sideset{}{'}\sum_{(j_p)}&\mathcal{E}_{l,j_1,\ldots,j_k}(\omega)\widehat{f}(\lambda_{j_1}\ldots\lambda_{j_l}\overline{\lambda}_{j_{l+1}}\ldots\overline{\lambda}_{j_k})\\ &\times\lambda_{j_1}\ldots\lambda_{j_l}\overline{\lambda}_{j_{l+1}}\ldots\overline{\lambda}_{j_k}\Big\|_{C(G)}d\mathbb{P}(\omega)\\ \le\int_\Omega\Big\|\sum_{l=0}^{k}&\sum_{\substack{(j_p)\\ \text{distinct}}}\mathcal{E}_{l,j_1,\ldots,j_k}(\omega)\widehat{f}(\lambda_{j_1}\ldots\lambda_{j_l}\overline{\lambda}_{j_{l+1}}\ldots\overline{\lambda}_{j_k})\\ &\times\lambda_{j_1}\otimes\ldots\otimes\lambda_{j_l}\otimes\overline{\lambda}_{j_{l+1}}\otimes\ldots\otimes\overline{\lambda}_{j_k}\Big\|_{C(G^k)}d\mathbb{P}(\omega).\end{aligned}$$

By the contraction principle, this gives

(28) $$[\![f]\!]\le[\![F]\!]\quad\text{where } F \text{ has the form given by (22).}$$

Combining (26)–(28), we obtain $[\![f]\!] \le A_k B_k \|f\|_\infty$ and Λ_k is a stationary subset of Γ. ∎

COROLLARY 4.2. *There are some stationary subsets of* $\mathbb{Z}$ *that are not UC sets.*

Proof. For example, if H is a Hadamard sequence, then $H - H + H$ is not a UC set ([5] or [8]) but it is stationary by Theorem 4.1. More generally, recall that if E, F are infinite sets included in $\mathbb{N}$, then $E - F$ is not a CUC set. ∎

Let us recall some known facts from [6].

DEFINITION 4.3. A pair (Q, R) of subsets of $\mathbb{Z}$ is an *alternating pair* of size N if $|Q| = |R| = N$ and, writing $Q = (q_n)_{1\le n\le N}$ and $R = (r_n)_{1\le n\le N}$, one has $q_2 - q_1 \le r_2 - r_1 < q_3 - q_1 \le r_3 - r_1 < \ldots$

In [6], the authors show that UC sets included in $\mathbb{N}$ cannot contain such differences of alternating pairs for too large sizes (the bound on the size depending on the UC constant).

On the other hand, let us construct a particular dissociated set E. We consider the sets

$$\Lambda_1 = \bigcup_{n\,\mathrm{odd}} (4^{n!} + E_n) \quad \text{and} \quad \Lambda_2 = \bigcup_{n\,\mathrm{even}} (4^{n!} - E_n)$$

where

$$E_n = \begin{cases} \{4^{n!-k}\}_{k\,\mathrm{odd},\,1\le k<n} & \text{for } n \text{ odd},\\ \{4^{(n-1)!-k}\}_{k\,\mathrm{even},\,2\le k<n} & \text{for } n \text{ even}.\end{cases}$$

The set E is $\Lambda_1 \cup \Lambda_2$.

COROLLARY 4.4. *There is a stationary set included in* $\mathbb{N}$ *containing translates of the difference* $Q - R$ *arising from an alternating pair* (Q, R) *of arbitrarily large size.*

Proof. With the previous notations, it suffices to consider the set $\Lambda = E + E$. By Theorem 4.1, Λ is stationary. On the other hand, Λ is not a UC set.

Indeed, assume otherwise. For odd n, Λ contains translates of the difference $E_n - E_{n+1}$. As Λ is included in $\mathbb{N}$, it would be a CUC set too (e.g. [8]) and then $\Lambda - 4^{n!}$ would be a UC set with bounded UC-constant. On the other hand, $\Lambda - 4^{n!}$ contains the differences arising from the alternating pair (E_n, E_{n+1}). This is a contradiction for n large enough ([6]). ∎

COROLLARY 4.5. *For all* $k \ge 1$, *there are some stationary sets* Λ_k *satifying*

$$\exists \delta_k > 0\ \forall N \ge 1 : \quad |\Lambda_k \cap [-N, N]| \ge \delta_k (\log N)^k.$$

Proof. It suffices to consider the set $\Lambda_k = \{3^{m_1}+\ldots+3^{m_k} \mid m_j \text{ distinct}\}$. One easily checks that for all $N \geq 2k$,

$$|\Lambda_k \cap \{0,\ldots,k3^N\}| \geq (N+2)\ldots(N+2-k) \geq \left(\frac{N+2}{2}\right)^k. \ \blacksquare$$

REMARK. Theorem 4.1 is optimal in the sense that there is a Sidon set E such that $\mathbb{N} \subset E+E$ and then $E+E$ is not stationary in $\mathbb{Z}$. Indeed, it suffices to consider $E = \{10^n + n\}_{n\in\mathbb{N}} \cup \{-10^n\}_{n\in\mathbb{N}}$.

General remark. Essentially we used the fact that $[\![\cdot]\!]$ is an unconditional norm for characters satisfying: there exists $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{x\to\infty} \psi(x) = \infty$ and for each finite $A \subset \Gamma$,

$$\Big[\!\!\Big[\sum_{\gamma\in A} \gamma \Big]\!\!\Big] \geq \psi(|A|) \Big\| \sum_{\gamma\in A} \gamma \Big\|_2.$$

So, it may be noticed that the previous results hold for some other lacunary sets of harmonic analysis, for example p-Sidon sets. Using the same methods, it is easy to rewrite all of them essentially replacing *stationary* by *p-Sidon*.

More precisely, Proposition 3.1, for example, can be strengthened when Λ is a p-Sidon set $(1 \leq p < 2)$: we recover, by other methods, an inequality due to Edwards (see e.g. [6]). Denoting $2p/(2-p)$ by r, there exists a constant $C > 0$ such that

$$\forall \mu \in M_\Lambda(G): \quad \|\widehat{\mu}\|_r \leq C\|\mu\|.$$

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