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SOME STABILITY RESULTS FOR ASYMPTOTIC NORMING PROPERTIES OF BANACH SPACES

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1. Introduction. In this paper, we study certain stability results for w^*-Asymptotic Norming Properties (w^*-ANP). The w^*-ANP's are stronger properties than X being an Asplund space. These were first introduced by Z. Hu and B. L. Lin in [9] (see the end of this section for the relevant definitions). They showed that w^*-ANP-II and w^*-ANP-III are respectively equivalent to the property $(**)$ studied earlier by Namioka and Phelps [16] and Hahn–Banach smoothness considered by Sullivan [21]. This latter property in turn grew out of the concept of U-subspaces introduced by Phelps [18].

In Section 2, using the equivalence of Hahn–Banach smoothness with w^*-ANP-III, we show that if X is such that all of its separable subspaces are Hahn–Banach smooth, then X itself is Hahn–Banach smooth. This result has recently been proved by E. Oja and M. Põldvere [17] by different arguments. We next show that Hahn–Banach smoothness is preserved under c_0-sums. We also give a necessary condition for the ℓ_∞-sum of copies of the span of a unit vector to be a U-subspace of the ℓ_∞-sum of copies of the space. Using this we give an example showing that being a U-subspace is not preserved under arbitrary ℓ_∞-sums. We prove that if Y is a proper U-subspace of X, then for any nontrivial space Z, the ℓ_1-direct sum $Y \oplus_1 Z$ is not a U-subspace of $X \oplus_1 Z$, and use this to conclude that Hahn–Banach smoothness is not preserved under taking ℓ_1-direct sums in any nontrivial way. These techniques also enable us to show that if each renorming of a Banach space is Hahn–Banach smooth, then the space is reflexive.

Section 3 is devoted to the study of the Namioka–Phelps property and a weaker version of it, called property (II), introduced by Chen and Lin [3]. It is shown that property (II) is preserved under arbitrary ℓ_p-sums ($1 < p < \infty$). However, it is not preserved even under finite ℓ_1-sums. We also show that under an assumption of compact approximation of identity on X,

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if $\mathcal{L}(X)$ has property (II) then X must be finite-dimensional. We conclude the section by showing that for a compact set K, the space of operators $\mathcal{L}(X, C(K))$ has (II) if and only if X is reflexive, X^* has (II), and K is finite.

All the Banach spaces considered here are over the real scalar field. Most of our notations and terminology is standard and can be found in [5].

2. Throughout this paper B_X and S_X denote respectively the closed unit ball and sphere of the Banach space X. We recall some relevant definitions.

DEFINITION 2.1 [9], [1]. (a) Let X be a Banach space and X^* its dual. A sequence $\{x_n^*\} \subseteq S_{X^*}$ is said to be *asymptotically normed* by B_X if for any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $x \in B_X$ such that $x_n^*(x) > 1 - \varepsilon$ for all $n \geq N$.

(b) A sequence $\{x_n\}$ in X is said to have *property* κ ($\kappa =$ I, II, II′ or III) if

I. $\{x_n\}$ is convergent,
II. $\{x_n\}$ has a convergent subsequence,
II′. $\{x_n\}$ is weakly convergent,
III. $\bigcap_{n=1}^{\infty} \overline{\mathrm{co}}\{x_k : k \geq n\} \neq \emptyset$.

(c) X is said to have w^*-ANP-κ ($\kappa =$ I, II, II′ or III) if every asymptotically normed sequence in S_{X^*} has property κ ($\kappa =$ I, II, II′ or III).

In this paper we will only be dealing with w^*-asymptotic norming properties.

DEFINITION 2.2 [17]. Let X be a Banach space. A subspace Y of X is said to be a *U-subspace* if for any $y^* \in Y^*$ there exists a unique norm preserving extension of y^* in X^*.

In particular, X is said to be *Hahn–Banach smooth* if X is a U-subspace of X^{**} under the canonical embedding of X in X^{**}.

It is well known that Hahn–Banach smoothness, w^*-ANP-III and the coincidence of weak and w^*-topologies on S_{X^*} are equivalent. The proof of the equivalence of the first two can be found in [9] while that of the first and the third can be found in [21].

DEFINITION 2.3 [16], [2], [3]. (a) X is said to have the *Namioka–Phelps property* if the weak* and the norm topologies coincide on S_{X^*}.

(b) X is said to have the *Mazur Intersection Property* (*MIP*) if the w^*-denting points of B_{X^*} are norm dense in S_{X^*}.

(c) A Banach space X is said to have *property* (II) if the w^*-PC's of B_{X^*} are norm dense in S_{X^*} (this should not be confused with w^*-ANP-II that we have defined earlier).

There are equivalent formulations of MIP and property (II). We choose these as in this form property (II) is the natural weakening of both the Namioka–Phelps property and MIP.

Our first result gives a simpler proof of the following theorem by E. Oja and M. Põldvere [17].

THEOREM 2.1. *X is Hahn–Banach smooth if and only if every separable subspace of X is Hahn–Banach smooth.*

Proof. It is easy to see that Hahn–Banach smoothness is hereditary.

Conversely, let X be such that all its separable subspaces are Hahn–Banach smooth. We will show that X is Hahn–Banach smooth, i.e., X has w^*-ANP-III. Let $\{x_n^*\}$ be a sequence in S_{X^*} which is asymptotically normed by B_X. In view of [9, Theorem 2.3], it is enough to show that $\{x_n^*\}$ has property III. For $m, n \in \mathbb{N}$, select $x_{nm} \in B_X$ such that $x_n^*(x_{nm}) \geq 1 - 1/m$. Also, for each $k \in \mathbb{N}$, there exist $n_k \in \mathbb{N}$ and $x_k \in B_X$ such that $x_n^*(x_k) > 1 - 1/k$ for all $n \geq n_k$. Let $Y = \overline{\operatorname{span}}\,[\{x_{nm}\} \cup \{x_k\}]$. Clearly, $\{x_n^*\}$ is asymptotically normed by B_Y. By Proposition 2 of [20] there exists a separable $Y' \supset Y$ and a linear mapping $T : Y'^* \to X^*$ such that for each $f \in Y'^*$, Tf is a norm preserving extension of f and $TY'^* \supset \overline{\operatorname{span}}\{x_n^*\}$. Since Y' is separable, it has w^*-ANP-III. Hence $\{x_n^*\}$ has property III. ■

We next consider the stability of being a U-subspace under ℓ_1-sums.

THEOREM 2.2. *Let $Y \subset X$ be a proper subspace of X and let Z be any nonzero Banach space. Then the ℓ_1-direct sum $Y \oplus_1 Z$ is not a U-subspace of $X \oplus_1 Z$.*

Proof. Let $y^* \in Y^*$, $0 < \|y^*\| < 1$, and let $z^* \in S_{Z^*}$. Let $x^* \in X^*$ be a norm preserving extension of y^*. Since $\|x^*\| < 1$ and Y is a proper subspace of X, choose $\tau \in Y^\perp$ such that $\tau \neq 0$ and $\|x^* \pm \tau\| \leq \|x^*\| + \|\tau\| \leq 1$. Now $\|(x^* \pm \tau, z^*)\| = \max(\|x^* \pm \tau\|, \|z^*\|) = 1$. Thus $(x^* \pm \tau, z^*)$ are two distinct norm preserving extensions of (y^*, z^*). ■

Before our next result, let us recall the definition of an L-projection.

DEFINITION 2.4 [8]. Let X be a Banach space. A linear projection P is called an *L-projection* if

$$\|x\| = \|Px\| + \|x - Px\| \quad \text{for all } x \in X.$$

COROLLARY 2.3. *If X is nonreflexive and Hahn–Banach smooth, then X has no nontrivial L-projections.*

Proof. Suppose $X = Y \oplus_1 Z$ is a nontrivial L-decomposition. Since X is not reflexive, assume without loss of generality, Y is nonreflexive. Since $X = Y \oplus_1 Z$ is a U-subspace of $X^{**} = Y^{**} \oplus_1 Z^{**}$, it is a U-subspace of

$Y^{**} \oplus_1 Z$ as well. By Theorem 2.2, this is a contradiction. Hence there are no nontrivial L-projections on X. ■

The following corollary is easy to see from the above arguments.

COROLLARY 2.4. *Let $\{X_i\}_{i\in\Gamma}$ be a family of Banach spaces. Then the ℓ_1-direct sum $\bigoplus_{\ell_1(\Gamma)} X_i$ is Hahn–Banach smooth if and only if all but finitely many X_i's are trivial*, *i.e.*, *equal to* $\{0\}$, *and the remaining are reflexive.*

COROLLARY 2.5. *If for a Banach space X*, *every equivalent renorming is Hahn–Banach smooth*, *then X is reflexive.*

Proof. Let $X = Y \oplus Z$ be a nontrivial direct sum; then the norm defined by $\|x\|_1 = \|y\| + \|z\|$, where $x = y + z$, $y \in Y$, $z \in Z$, is an equivalent norm on X and this new norm has a nontrivial L-projection. Therefore every nonreflexive space can be renormed to fail Hahn–Banach smoothness. Hence the result. ■

REMARK 2.1. In [10] the authors showed that X is reflexive if and only if for any equivalent norm on X, X is Hahn–Banach smooth and has ANP-III. Corollary 2.5 above is a much stronger result with a simpler proof.

COROLLARY 2.6. *Hahn–Banach smoothness is not a three-space property.*

Proof. Let M be Hahn–Banach smooth and nonreflexive. Let $X = M \oplus_1 M$. Then X/M is isometrically isomorphic to M, hence Hahn–Banach smooth. Corollary 2.3 shows that X is not Hahn–Banach smooth. ■

THEOREM 2.7. *Let $\{X_i\}_{i\in\Gamma}$ be a family of Banach spaces. For each $i \in \Gamma$*, *let Y_i be a U-subspace of X_i. Then the c_0-direct sum $\bigoplus_{c_0(\Gamma)} Y_i$ is a U-subspace of $\bigoplus_{c_0(\Gamma)} X_i$.*

Proof. Let $X = \bigoplus_{c_0(\Gamma)} X_i$; then $X^* = \bigoplus_{\ell_1(\Gamma)} X_i^*$. Similarly, $Y = \bigoplus_{c_0(\Gamma)} Y_i$ and $Y^* = \bigoplus_{\ell_1(\Gamma)} Y_i^*$. Let $y^* \in Y^*$. Let $x^* = (x_i^*)_{i\in\Gamma}$ and $z^* = (z_i^*)_{i\in\Gamma}$ be norm preserving extensions of $y^* = (y_i^*)_{i\in\Gamma}$. Clearly, $x_i^* \neq 0$ if and only if $y_i^* \neq 0$ if and only if $z_i^* \neq 0$. Thus $x_i^* = y_i^* = z_i^*$ on Y_i^* for all i. Now $\|x^*\| = \|y^*\|$ implies $\sum(\|x_i^*\| - \|y_i^*\|) = 0$. Since $\|x_i^*\| \geq \|y_i^*\|$, we have $\|x_i^*\| = \|y_i^*\|$ for all i. Similarly for z_i^*. Thus $\|z_i^*\| = \|x_i^*\|$ for all i. Since each Y_i is a U-subspace of X_i, it follows that $x_i^* = z_i^*$ for all i. Hence $z^* = x^*$. ■

Recall from [8] that a closed subspace $M \subset X$ is said to be an *M-ideal* if there exists a closed subspace $N \subset X^*$ such that $X^* = M^\perp \oplus_1 N$. As remarked in [8] any M-ideal is a U-subspace. An easy way of generating M-ideals is to consider any family $\{X_i\}_{i\in\Gamma}$ of Banach spaces and observe that $\bigoplus_{c_0(\Gamma)} X_i$ is an M-ideal in the ℓ_∞-direct sum $\bigoplus_{\ell_\infty(\Gamma)} X_i$ (this can be easily proved using the "three-ball characterization" of M-ideals). We use this simple observation in our next result.

Corollary 2.8. *If $\{X_i\}_{i\in\Gamma}$ is a family of Hahn–Banach smooth spaces, then $\bigoplus_{c_0} X_i$ is Hahn–Banach smooth as well.*

Proof. Since X_i is Hahn–Banach smooth for all i, each X_i is a U-subspace of X_i^{**}. So by the above theorem, $\bigoplus_{c_0} X_i$ is a U-subspace of $\bigoplus_{c_0} X_i^{**}$. Now, $\bigoplus_{c_0} X_i^{**}$ is an M-ideal in $\bigoplus_{\ell_\infty} X_i^{**} = (\bigoplus_{c_0} X_i)^{**}$. Thus $\bigoplus_{c_0} X_i$ is a U-subspace of $(\bigoplus_{c_0} X_i)^{**}$. Hence $\bigoplus_{c_0} X_i$ is Hahn–Banach smooth. ■

Corollary 2.9. *Let K be a scattered compact space and suppose Y is a U-subspace of X. Then $C(K,Y)$ is a U-subspace of $C(K,X)$.*

Proof. We only need to observe that if K is a scattered compact space, then $C(K,X)^* = \bigoplus_{\ell_1(\Gamma)} X^*$ for some discrete set Γ. The conclusion then follows from arguments identical to the proof of Theorem 2.7. ■

Remark 2.2. Unlike the situation for ℓ_1-direct sums considered in Theorem 2.2, in the case of $C(K,X)$, the space $C(K,Y)$ may be a U-subspace of $C(K,X)$ for some U-subspace Y of X (without any extra topological assumptions on the compact set K).

Example 2.1. Let $Y \subset X$ be a proper M-ideal (for example, consider $X = \ell_\infty$ and $Y = c_0$). Then, for any compact Hausdorff space K, it is known [8, Proposition VI.3.1] that $C(K,Y)$ is an M-ideal in $C(K,X)$ and is thus a U-subspace. ■

Theorem 2.10. *Let X be a Banach space. Let $x_0 \in S_X$. Suppose the infinite sum $\bigoplus_\infty \operatorname{span}\{x_0\}$ is a U-subspace of $\bigoplus_\infty X$. Then x_0 is a smooth point. If x_0^* denotes the unique norming functional, then x_0^* is strongly exposed by x_0.*

Proof. Since any M-summand is a U-subspace, we may assume without loss of generality that the sum is countably infinite.

Suppose

$$\|x^*\| = \|y^*\| = x^*(x_0) = y^*(x_0) = 1.$$

Fix a Banach limit L on ℓ^∞. Define $L_1, L_2 : \bigoplus_\infty X \to \mathbb{R}$ by $L_1(\{x_n\}) = L(\{x^*(x_n)\})$ and $L_2(\{x_n\}) = L(\{y^*(x_n)\})$. Clearly, $\|L_1\| = \|L_2\| = 1$ and $L_1 = L_2$ on $\bigoplus_\infty \operatorname{span}\{x_0\}$ and they are of norm one here as well. Therefore by hypothesis $L_1 = L_2$. Treating an $x \in X$ as a constant sequence, we thus get $x^*(x) = y^*(x)$ for all $x \in X$.

We now show that x_0^* is strongly exposed by x_0. Let $\{x_n^*\} \subset B_{X^*}$ and $x_n^*(x_0) \to 1 = x_0^*(x_0)$.

Claim. *$x_n^* \to x_0^*$ in norm.*

Indeed, suppose $x_n^* \not\to x^*$ in norm. By passing to a subsequence if necessary, we may assume that there exists $\varepsilon > 0$ such that $\|x_n^* - x_0^*\| \ge \varepsilon$. Choose $y_n \in S_X$ such that $x_n^*(y_n) - x_0^*(y_n) \ge \varepsilon$.

Define now $L', L'' : \bigoplus_\infty X \to \mathbb{R}$ by $L'(\{x_n\}) = L(\{x_n^*(x_n)\})$ and $L''(\{x_n\}) = L(\{x_0^*(x_n)\})$. Since $x_n^*(x_0) \to 1$, it is clear that $\|L'\| = \|L''\| = 1$ and $L' = L''$ on $\bigoplus_\infty \operatorname{span}\{x_0\}$ and they are of norm one here as well. Thus by the hypothesis, $L' = L''$. However, $L(\{x_n^*(y_n) - x_0^*(y_n)\}) \geq \varepsilon$. But this contradicts the choice of the sequence $\{y_n\}$ and ε. Hence the claim. ∎

We are grateful to Dr. P. Bandyopadhyay for suggesting this form of Theorem 2.10.

Example 2.2. We now use the above theorem to show that being a U-subspace is not preserved under ℓ_∞-direct sums.

Indeed, suppose X is a reflexive Banach space that is strictly convex but fails the property H (i.e., there exists a sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ weakly, $\|x_n\| \to \|x\|$, but $x_n \not\to x$ in norm). Then in such a space X, there are $x_0 \in S_X$ and $\{x_n\} \subset S_X$ such that $x_n \to x_0$ weakly, but not in norm. Fix $x_0^* \in S_{X^*}$, $x_0^*(x_0) = 1$. Since X is strictly convex, $\operatorname{span}\{x_0^*\}$ is a U-subspace of X^*. However, x_0^* does not strongly expose x_0. Therefore $\bigoplus_\infty \operatorname{span}\{x_0^*\}$ is not a U-subspace of $\bigoplus_\infty X^*$.

One such example, due to M. A. Smith, given in [21], is the following renorming of ℓ_2: Let $\|x\|_0 = \max\{\|x\|_2/2, \|x\|_\infty\}$. Define $T : \ell_2 \to \ell_2$ by $T(\{\alpha_k\}) = \{\alpha_k/k\}$. Finally, $|||x||| = \|x\|_0 + \|Tx\|_2$ is an equivalent norm with the required property. ∎

Example 2.3. By considering $\mathbb{R}$ as a U-subspace of the Euclidean $\mathbb{R}^2$ and taking a nonatomic measure λ, we now show that $L^1(\lambda)$ is not a U-subspace of $L^1(\lambda, \mathbb{R}^2)$.

Indeed, let K denote the Stone space of $L^\infty[0,1]$ and denote by λ the image of the Lebesgue measure on K. With this identification, $L^1(\lambda, \mathbb{R}^2)^* = C(K, \mathbb{R}^2)$ and $L^1(\lambda)^* = C(K)$. $L^1(\lambda)$ is embedded in $L^1(\lambda, \mathbb{R}^2)$ as $f \to f \otimes e_1$, i.e., by identifying $f \in L^1(\lambda)$ with $(f, 0) \in L^1(\lambda, \mathbb{R}^2)$. Let A be any clopen subset of K such that $0 < \lambda(A) < 1$. Consider $f = \chi_A \in L^1(\lambda)^* = C(K)$. Let $f \otimes e_1 \in C(K, \mathbb{R}^2)$. For any $g \in L^1(\lambda)$,

$$\int_A g \, d\lambda = \int (f \otimes e_1)(g \otimes e_1) \, d\lambda.$$

Since $\|f \otimes e_1\| = 1$, $f \otimes e_1$ is a norm preserving extension of f. Let $h : K \to \mathbb{R}^2$ be given by $h(k) = (\chi_A(k), \chi_{A^c}(k))$. Clearly, $h \in C(K, \mathbb{R}^2)$ and $\|h(k)\| = 1$ for all k. Again for $g \in L^1(\lambda)$,

$$\int h(k) \cdot (g(k), 0) \, d\lambda = \int_A g \, d\lambda.$$

Thus h is a norm preserving extension of f different from $f \otimes e_1$. ∎

Definition 2.5. A Banach space X is said to have the *finite intersection property* (*FIP*) if every family of closed balls in X with empty intersection contains a finite subfamily with empty intersection.

It is well known that any dual space and its 1-complemented subspaces have FIP.

Theorem 2.11. *If X is Hahn–Banach smooth and has FIP then X is reflexive.*

Proof. It is known from [6] that X has FIP if and only if $X^{**} = X + C_X$ where $C_X = \{F \in X^{**} : \|F + \widehat{x}\| \geq \|x\| \text{ for all } x \in X\}$. Let $\Lambda \in C_X$ and $\|\Lambda\| = 1$. Then by [6], $\overline{\mathrm{co}}^{w^*} B_{\ker \Lambda} = B_{X^*}$. Let $\|x^*\| = 1$ and $x^*_\alpha \in B_{\ker \Lambda}$ such that $x^*_\alpha \xrightarrow{w^*} x^*$. Clearly, $\|x^*_\alpha\| \to 1$. Since X is Hahn–Banach smooth, the weak and weak* topologies coincide on S_{X^*}. So, $x^*_\alpha \to x^*$ weakly. In particular, $\Lambda(x^*_\alpha) \to \Lambda(x^*)$. Thus $\Lambda(x^*) = 0$ for all x^* such that $\|x^*\| = 1$, a contradiction. Hence $C_X = \{0\}$, and consequently, X is reflexive. ■

Remark 2.3. That Hahn–Banach smoothness for a dual space implies reflexivity was first remarked by Sullivan [21]. The same result for 1-complemented subspaces of a dual space was noted by Lima [14].

3. In this section we study w^*-ANP-II and related properties. The latter is actually equivalent to the Namioka–Phelps property [9]. It is also known that X has property (V) [21] if and only if X^* has w^*-ANP-II′ [1]. Proceeding similarly to Theorem 2.1, we obtain

Theorem 3.1. *w^*-ANP-κ $(\kappa = \mathrm{I}, \mathrm{II}, \mathrm{II}')$ is a separably determined property.*

We next consider the Namioka–Phelps property for c_0-direct sums.

Theorem 3.2. *Let $\{X_i\}_{i\in\Gamma}$ be a family of Banach spaces with the Namioka–Phelps property. Then $X = \bigoplus_{c_0} X_i$ also has this property.*

Proof. Let $x^* = (x^*(i))_{i\in\Gamma} \in S_{X^*}$ and $\{x^*_\alpha\}$ be a net in S_{X^*} such that $x^*_\alpha \xrightarrow{w^*} x^*$. Then $\lim_\alpha \|x^*_\alpha\| = \|x^*\|$. Since $x^*_\alpha \xrightarrow{w^*} x^*$, we have $x^*_\alpha(i) \xrightarrow{w^*} x^*(i)$ in X_i for all i. Again by w^*-lower semicontinuity of the norm in X_i, we have $\lim_\alpha \|x^*_\alpha(i)\| = \|x^*(i)\|$. Thus, $x^*_\alpha(i) \to x^*(i)$ in norm for all i.

For $\varepsilon > 0$, there exists a finite set $A \subset \Gamma$ with N elements such that $\sum_{n\notin A} \|x^*(n)\| \leq \varepsilon/4$. Also, since $x^*_\alpha(i) \to x^*(i)$ in norm for all i, there exists β such that $\|x^*_\alpha(n) - x^*(n)\| < \varepsilon/(4N)$ for all $n \in A$ and all $\alpha \geq \beta$. It is now easily seen that

$$|\|x^*_\alpha - x^*\| + \|x^*_\alpha\| - \|x^*\|| \leq \sum_{n\in A} 2\|(x^*_\alpha - x^*)(n)\| + \sum_{n\notin A} 2\|x^*(n)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $\alpha \geq \beta$. Hence X has the Namioka–Phelps property. ■

REMARK 3.1. (a) The last part of the proof of the above theorem is adapted from Yost's arguments in [22, Lemma 9].

(b) It follows that w^*-ANP-II is stable under c_0-sums.

(c) Since ℓ_1 (resp. ℓ_∞) is not strictly convex, it clearly follows from [9] and [1] that w^*-ANP-I and w^*-ANP-II$'$ are not stable under c_0-sums (resp. ℓ_1-sums).

(d) As noted in [19], the ℓ_1-direct sum of spaces with the Namioka–Phelps property always fails the Namioka–Phelps property.

An argument similar to Corollary 2.6 shows that

COROLLARY 3.3. *Let X be a Banach space. Then the following are true.*

(a) *If X has w^*-ANP-κ ($\kappa =$ I, II, II$'$) and is not reflexive, then X has no nontrivial L-projections.*

(b) *If every equivalent renorming of X has w^*-ANP-κ ($\kappa =$ I, II, II$'$) then X is reflexive.*

(c) w^*-ANP-κ ($\kappa =$ I, II, II$'$) *is not a three-space property.*

We next look at the stability results for property (II). Unlike those considered before, this is not a hereditary property [3]. Also, this property does not imply that the underlying space is Asplund [3]. Analogously to what we have shown in Theorem 2.11, we have the following result.

THEOREM 3.4. *If X has (II) and has FIP, then X is reflexive. In particular, any dual space with property (II) is reflexive.*

Proof. As before, we will show that $C_X = \{0\}$. Let $\Lambda \in C_X$. Since X has (II), the w^*-PC's of B_{X^*} are dense in S_{X^*}. Hence it suffices to show $\Lambda(x^*) = 0$ for any w^*-PC $x^* \in S_{X^*}$. But this follows from arguments similar to the proof of Theorem 2.11. ∎

We now consider property (II) for ℓ_p-direct sums $(1 < p < \infty)$.

PROPOSITION 3.5. *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces. Then $X = \bigoplus_{\ell_p} X_i$ $(1 < p < \infty)$ has (II) if and only if for each $i \in I$, X_i has (II).*

Proof. Since $X^* = \bigoplus_{\ell_q} X_i^*$, where $1/p + 1/q = 1$, and $x^* \in S_{X^*}$ is a w^*-PC of B_{X^*} if and only if for each $i \in I$, either $x_i^* = 0$ or $x_i^*/\|x_i^*\|$ is a w^*-PC of $B_{X_i^*}$ (cf. [11]), the proof is similar to that of [2, Theorem 3]. ∎

It is known that if (Ω, Σ, μ) is a nonatomic measure space, then $f \in S_{L_p(\mu,X)^*}$ is a w^*-PC if and only if it is a w^*-denting point of $B_{L_p(\mu,X)^*}$ (see [11]). Thus it follows from Theorem 8 of [2] that

COROLLARY 3.6. *Let X be a Banach space, λ denote the Lebesgue measure on $[0,1]$ and $1 < p < \infty$. The space $L_p(\lambda, X)$ has (II) if and only if it has MIP if and only if X has MIP and is Asplund.*

Remark 3.2. It follows that there exists a space X with (II) such that $L_p(\lambda, X)$ does not have (II). Clearly, any finite-dimensional space which does not have MIP (e.g., $\mathbb{R}^n$ with ℓ_1 or sup norm) serves as an example.

Proposition 3.7. *Let X, Y, Z be Banach spaces such that $X = Y \oplus_1 Z$. Then $(y^*, z^*) \in S_{X^*}$ is a w^*-PC if and only if $\|y^*\| = 1$, $\|z^*\| = 1$ and y^*, z^* are w^*-PC's of B_{Y^*} and B_{Z^*} respectively.*

Proof. First, let $\|y^*\| = \|z^*\| = 1$, and y^*, z^* be w^*-PC's. Then obviously (y^*, z^*) is a w^*-PC.

Conversely, suppose (y^*, z^*) is a w^*-PC of B_{X^*}. Let $\{y^*_\alpha\}$ be a net in B_{Y^*} such that $y^*_\alpha \to y^*$ in the w^*-topology. Thus $\|(y^*_\alpha, z^*)\| = 1$ and $(y^*_\alpha, z^*) \to (y^*, z^*)$ in the w^*-topology. This implies $(y^*_\alpha, z^*) \to (y^*, z^*)$ in norm. This in turn implies that $y^*_\alpha \to y^*$ in norm. This also implies $\|y^*\| = 1$. Similarly for z^*. ∎

Now it readily follows that

Corollary 3.8. *Let X be a Banach space. Then the following are true.*

(a) *If X has property (II) and is not finite-dimensional, then X has no nontrivial L-projections.*

(b) *If every equivalent renorming of X has property (II), then X is finite-dimensional.*

(c) *Property (II) is not a three-space property.*

Theorem 3.9. *Let X, Y, Z be Banach spaces such that $X = Y \oplus_\infty Z$. Then the following are true.*

(1) *If $x^* = (y^*, z^*) \in S_{X^*}$ is a w^*-PC and*

(a) *one of the coordinates of x^* is zero, then the other is a w^*-PC of the corresponding component.*

(b) *$0 < \|y^*\| < 1$ and $0 < \|z^*\| < 1$, then $y^*/\|y^*\|$ and $z^*/\|z^*\|$ are w^*-PC's of B_{Y^*} and B_{Z^*} respectively.*

(2) *Conversely, if y^* and z^* are w^*-PC's of B_{Y^*} and B_{Z^*} respectively, then $(\lambda y^*, (1-\lambda)z^*)$ is a w^*-PC of B_{X^*} for all $0 \le \lambda \le 1$.*

(3) *X has (II) if and only if Y and Z have (II).*

Proof. (1) (a) Obviously when one of the coordinates of x^* is zero, the other one is a w^*-PC of the unit ball of the corresponding space.

(b) Suppose $0 < \|y^*\|, \|z^*\| < 1$. Let $y^*_\alpha \xrightarrow{w^*} y^*/\|y^*\|$. Then $(\|y^*\| y^*_\alpha, z^*) \to (y^*, z^*)$ in the w^*-topology, and hence, in norm. Thus $y^*_\alpha \to y^*/\|y^*\|$ in norm. Hence, $y^*/\|y^*\|$ is a w^*-PC. Similarly for $z^*/\|z^*\|$.

(2) Let $x^* = (\lambda y^*, (1-\lambda)z^*)$, $0 < \lambda < 1$. Let $(y^*_\alpha, z^*_\alpha) \xrightarrow{w^*} (\lambda y^*, (1-\lambda)z^*)$. This implies $\|y^*_\alpha\| + \|z^*_\alpha\| \to \lambda\|y^*\| + (1-\lambda)\|z^*\| = 1$, i.e., $\|y^*_\alpha\| \to \lambda$ and $\|z^*_\alpha\| \to (1-\lambda)$. Thus $y^*_\alpha \to \lambda y^*$ or $y^*_\alpha/\|y^*_\alpha\| \xrightarrow{w^*} y^*$, i.e., $y^*_\alpha/\|y^*_\alpha\| \to y^*$ in

norm, which implies $y^*_\alpha \to \lambda y^*$ in norm. Similarly, $z^*_\alpha \to (1-\lambda)z^*$. Thus $(y^*_\alpha, z^*_\alpha) \to (\lambda y^*, (1-\lambda)z^*)$ in norm. Hence x^* is a w^*-PC. Obviously, if $\lambda = 0$, then $x^* = (y^*, 0)$ is a w^*-PC of B_{X^*}.

(3) Suppose Y, Z have (II), and let $(y^*, z^*) \in S_{X^*}$.

Case 1. If $z^* = 0$, then $y^* \in S_{X^*}$ and there exists a sequence $\{y^*_n\}$ of w^*-PC's of B_{Y^*} such that $y^*_n \to y^*$, and hence $(y^*_n, 0) \to (y^*, 0)$, and by the above, $(y^*_n, 0)$ is a w^*-PC of B_{X^*} for each n.

Case 2. If $0 < \|y^*\|, \|z^*\| < 1$ then there exist sequences $\{y^*_n\}$ and $\{z^*_n\}$ of w^*-PC's of B_{Y^*}, B_{Z^*} respectively such that $y^*_n \to y^*/\|y^*\|$ and $z^*_n \to z^*/\|z^*\|$. This implies that $(y^*_n\|y^*\|, z^*_n\|z^*\|)$ is a w^*-PC of B_{X^*} and $(y^*_n\|y^*\|, z^*_n\|z^*\|) \to (y^*, z^*)$ in norm. This proves X has (II).

Conversely, let X have (II). Let $y^* \in S_{Y^*}$. Then there exists a sequence $x^*_n = (y^*_n, z^*_n)$ of w^*-PC's of S_{X^*} such that $(y^*_n, z^*_n) \to (y^*, 0)$. This implies $z^*_n \to 0$ and $y^*_n \to y^*$ in norm. Clearly, $y^*_n/\|y^*_n\|$ is a w^*-PC of B_{Y^*} for each n and $y^*_n/\|y^*_n\| \to y^*$ in norm. Hence Y has (II). Similarly for Z. ■

Following arguments similar to those in Theorem 3.2, we have

Corollary 3.10. *Let $\{X_i\}_{i\in\Gamma}$ be a family of Banach spaces with property (II). Then $X = \bigoplus_{c_0} X_i$ also has (II).*

Remark 3.3. However, property (II) is not stable under ℓ_∞-sums. In fact, ℓ_∞ does not have (II) since it is a nonreflexive dual space.

Theorem 3.11. *If X is an L^1-predual and has (II) then X^* is isometric to $\ell_1(\Gamma)$ for some discrete set Γ.*

Proof. Let $A \subset \partial_e B_{X^*}$ (i.e., the set of extreme points of B_{X^*}) be such that $A \cap -A = \emptyset$ and $A \cup -A = \partial_e B_{X^*}$. Now $B_{X^*} = \overline{\mathrm{co}}^{w^*}(A \cup -A)$. For each $f \in A$, $\mathrm{span}\{f\}$ is an L-summand. Also, for any $f_1, \ldots, f_n \in A$, we have $B(\mathrm{span}\{f_1, \ldots, f_n\}) = \mathrm{co}\{\pm f_i : i = 1, \ldots, n\}$ (cf. [8]). Thus $\Phi : \ell_1(A) \to X^*$ defined by $\Phi(\alpha) = \sum \alpha(f) \cdot f$ is a linear isometry.

We shall show that Φ is onto. Let $\|f\| = 1$. Let $\lim_\alpha \sum \lambda^\alpha_i f_i = f$ (i.e., the w^*-limit) where $\{f_i\}^n_{i=1} \subset A$ and $\sum_{i=1}^{n_\alpha} |\lambda^\alpha_i| = 1$ for all α. If f is now a w^*-PC, then $f = \lim_\alpha \sum \lambda^\alpha_i f_i$ (in norm). Then any such $f \in \Phi(\ell_1(A))$. Since X has (II), we conclude that Φ is an onto isometry. ■

Remark 3.4. (i) In the above argument we actually use the following fact: If X has (II) and A is such that $B_{X^*} = \overline{\mathrm{co}}^{w^*}(A)$ then $B_{X^*} = \overline{\mathrm{co}}(A)$. Thus if X is separable and has (II), then since $B_{X^*} = \overline{\mathrm{co}}^{w^*}(\partial_e B_{X^*})$ and since the extreme points of B_{X^*} form a w^*-separable metric space, we conclude that X^* is a separable space.

(ii) The same argument gives an easier proof of the fact that if X^* has (II), then X is reflexive. We simply observe that if X^* has (II), then $B_{X^{**}} = \overline{\mathrm{co}}(B_X) = B_X$.

Corollary 3.12. *Let K be a compact Hausdorff space. Then $C(K)$ has property (II) if and only if K is finite.*

Proof. Suppose $C(K)$ has (II). Then by the above, $C(K)^*$ is isometric to $\ell_1(\Gamma)$ for some discrete set Γ. Hence K does not support a nonatomic measure.

Let K' denote the set of isolated points of K. Since K' is dense in K, we see that $C(K)^*_1 = \overline{\mathrm{co}}^{w^*}\{\pm\delta(k') : k' \in K'\}$. However, since $C(K)$ has (II), this w^*-closure is the same as the norm closure. Now if $k \in K$ is an accumulation point, it is clear that $\delta(k)$ cannot be approximated in norm by a sequence from $\mathrm{co}\{\pm\delta(k') : k' \in K'\}$. This shows that $K' = K$ and hence K is finite. ∎

We finally consider property (II) for the space $\mathcal{L}(X)$ of operators on a Banach space X. Since this is not a hereditary property, it is not clear whether if $\mathcal{L}(X)$ has property (II) then X and X^* should as well (which in turn will force X to be reflexive). Our first result shows that under a mild approximation condition, the finite-dimensional spaces are the only ones for which $\mathcal{L}(X)$ has property (II).

Theorem 3.13. *Let X be a Banach space such that there exists a bounded net $\{K_\alpha\} \subseteq \mathcal{K}(X)$ with $K_\alpha(x) \to x$ weakly for all $x \in X$. If $\mathcal{L}(X)$ has (II), then X is finite-dimensional.*

Proof. For any $x \in X$ and $x^* \in X^*$, if $x \otimes x^*$ denotes the functional defined on $\mathcal{L}(X)$ by $x \otimes x^*(T) = x^*(T(x))$, then $\|x \otimes x^*\| = \|x\| \cdot \|x^*\|$. Since $\|T\| = \sup_{\|x^*\|=1,\, \|x\|=1} x^*(T(x)) = \sup_{\|x^*\|=1, \|x\|=1} x \otimes x^*(T)$, it follows that $A = \{x \otimes x^* : \|x^*\| = 1,\ \|x\| = 1\}$ determines the norm on $\mathcal{L}(X)$. Therefore by an application of the separation theorem, $B(\mathcal{L}(X))^* = \overline{\mathrm{co}}^{w^*}(A)$. Since $\mathcal{L}(X)$ has (II), $B(\mathcal{L}(X))^* = \overline{\mathrm{co}}(A)$.

Claim. *$K_\alpha \to I$ weakly.*

Indeed, since $\{K_\alpha\}$ is bounded, it suffices to check that $x \otimes x^*(K_\alpha) \to x \otimes x^*(I)$ for all $\|x\| = 1$, $\|x^*\| = 1$, i.e., to check $x^*(K_\alpha(x)) \to x^*(x)$. But $K_\alpha(x) \to x$ weakly, hence the claim.

Now since $K_\alpha \to I$ weakly, I is a compact operator. Hence X is finite-dimensional. ∎

Without any assumptions about the compact approximation of the identity one can still say the following:

Theorem 3.14. *Let X^* be a dual Banach space such that $\mathcal{L}(X^*)$ has (II). Then X is finite-dimensional.*

Proof. It is known that $\mathcal{L}(X^*) = (X \otimes_\pi X^*)^*$ (with the projective tensor product of X and X^*). Since $\mathcal{L}(X^*)$ is now a dual space with (II), it is reflexive. But this implies X and $\mathcal{L}(X)$ are reflexive, which in turn implies X is finite-dimensional (see [13]). ∎

REMARK 3.5. Similar ideas can be used to show that if $\mathcal{L}(X)$ is Hahn–Banach smooth, then X is finite-dimensional.

We now use the ideas contained in the proof of Corollary 3.12 to give a simple proof of Theorem 6 of [12]. Before proving the theorem we need a lemma which is of independent interest.

LEMMA 3.1. *Let $M \subset X$ be an M-ideal in X. If $x_0^* \in S_{M^*}$ is a w^*-PC of B_{M^*} (when M^* is canonically embedded in X^*), then it is a w^*-PC of B_{X^*}.*

Proof. Recall from [8, p. 11] that $P : X^* \to X^*$ is an L-projection whose range is canonically identified with M^* and Px^* is the unique norm preserving extension of $x^*|_M$. Also, $\ker P = M^\perp$. Now if $\{x_\alpha^*\} \subset B_{X^*}$ and $x_\alpha^* \xrightarrow{w^*} x_0^*$, then since $\|x_0^*\| = 1$, we have $\lim \|x_a^*\| = 1$. Clearly, we have $x_\alpha^*|_M \xrightarrow{w^*} x_0^*$ in M^* and thus by hypothesis $\|x_\alpha^* - x_0^*\|_M \to 0$. By the nature of P, $\|Px_\alpha^* - x_0^*\| \to 0$. Again since $\|x_\alpha^*\|_M \to 1$ and $\|x_\alpha^*\| = \|Px_\alpha^* - x_\alpha^*\| + \|Px_\alpha^*\|$, we conclude that $\|Px_\alpha^* - x_\alpha^*\| \to 0$. Further, since

$$\|x_\alpha^* - x_0^*\| = \|Px_\alpha^* - x_0^*\| + \|x_\alpha^* - Px_\alpha^*\|$$

we get $\|x_\alpha^* - x_0^*\| \to 0$. ∎

THEOREM 3.15 [12]. *An element μ of $C(K,X)^*$ is a w^*-PC of the unit ball of $C(K,X)^*$ if and only if it has the form $\mu = \sum_{k\in I} \delta_k \otimes x_k^*$, where $I = \{k \in K : k$ is an isolated point of $K\}$ and for each $k \in I$, either $x_k^* = 0$ or $x_k^*/\|x_k^*\|$ is a w^*-PC of B_{X^*} and $\sum_{k\in I} \|x_k^*\| = 1$.*

Proof. Let us call a measure F in $B_{C(K,X)^*}$ a *simple* measure if F is a finite convex combination of measures of the form $\delta(k) \otimes x^*$, where $k \in K$ and $x^* \in \partial_{\mathrm{e}} B_{X^*}$. It is well known that $B_{C(K,X)^*}$ is the w^*-closure of the simple measures. Since μ is a w^*-PC, it is therefore in the norm closure of the set of simple measures. By arguments similar to those in the proof Theorem 3.17 below we assume that $|\mu|$ is supported on a countable subset of K. Hence

$$\mu = \sum \delta(k_i) \otimes \mu(\{k_i\})$$

If $\|\mu(\{k_{i_0}\})\| > 0$ for some i_0, we claim that k_{i_0} is an isolated point of K. Otherwise let $\{t_\alpha\}$ be a net in K with $t_\alpha \to k_{i_0}$ and with t_α's distinct. Now μ is a w^*-limit of a net of measures in $B_{C(K,X)^*}$ obtained by replacing the i_0th component in the expression of μ by $\delta(t_\alpha) \otimes \mu(\{k_{i_0}\})$. Again since μ is a w^*-PC, this net converges to μ in norm. This contradicts the fact that the t_α's are distinct. Hence k_0 is an isolated point. That the corresponding normalized vector is a w^*-PC of B_{X^*} is proved similarly.

Conversely, let K' denote the set of isolated points of K. Put

$$M = \{f \in C(K, X) : f(K \setminus K') = 0\}.$$

Since K' is an open set, clearly M is an M-ideal [8]. It now follows from arguments similar to the one given during the proof of Theorem 3.9 that μ is a w^*-PC of M^*, after identifying M^* with the ℓ_1-direct sum of $|K'|$ copies of X^*. Hence the conclusion follows by application of the above lemma. ∎

COROLLARY 3.16. *If K is a metric space and X is separable, then the points of w^*-sequential continuity are given by a similar description.*

Proof. We only need to observe that $C(K, X)$ is now a separable space and thus w^*-sequential continuity is equivalent to w^*-continuity. ∎

In the case of $\mathcal{L}(X, Y)$, we have some partial results.

THEOREM 3.17. *Let $Y = C(K)$. Then $\mathcal{L}(X, Y)$ has property (II) if and only if X is reflexive, X^* has (II) and K is finite.*

Proof. Suppose $\mathcal{L}(X, Y)$ has property (II). Since Y has the metric approximation property, by arguments similar to the one indicated before, we have

$$B_{\mathcal{L}(X,Y)^*} = \overline{\mathrm{co}}\{\delta(k) \otimes x : x \in B_X,\ k \in K\}$$

and hence

$$\mathcal{L}(X, Y) = \mathcal{K}(X, Y).$$

Now if μ is a probability measure on K, then, for any $\|x\| = 1$, since

$$\mathcal{K}(X, C(K))^* = C(K, X^*)^* = M(K, X^{**}),$$

$\mu \otimes x$ and thus μ is a discrete measure. Therefore K is scattered. Now arguments similar to those given during the proof of Corollary 3.12 show that K must be a finite set. Hence X^* has property (II) and thus X is reflexive. ∎

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