COLLOQUIUM MATHEMATICUM

VOL. 75 1998 NO. 2

# THE BECKER–DÖRING MODEL WITH DIFFUSION. I. BASIC PROPERTIES OF SOLUTIONS

BY

PHILIPPE LAURENÇOT (NANCY)
AND DARIUSZ WRZOSEK (WARSZAWA)

**1. Introduction.** In this paper, we study the Becker–Döring model with diffusion. This model is a particular case of the discrete coagulation-fragmentation model with diffusion, which describes the evolution of a system of clusters when both coagulation and fragmentation of clusters are taken into account, together with spatial diffusion. In this model, each cluster consists of identical elementary units, and for $i \geq 1$, the concentration of $i$-clusters (i.e. clusters made of $i$ units) is denoted by $c_i$. The *Becker–Döring model with diffusion* then reads

$$\left.\begin{aligned} &\frac{\partial c_1}{\partial t} - d_1 \Delta c_1 = -W_1(c) - \sum_{j=1}^{\infty} W_j(c), \\ &\frac{\partial c_i}{\partial t} - d_i \Delta c_i = W_{i-1}(c) - W_i(c), \quad i \geq 2, \end{aligned}\right\} \text{ in } \Omega \times (0,\infty), \tag{1.1}$$

$$\frac{\partial c_i}{\partial \nu} = 0, \quad i \geq 1, \text{ on } \partial\Omega \times (0,\infty), \tag{1.2}$$

$$c_i(0) = c_i^0, \quad i \geq 1, \text{ in } \Omega, \tag{1.3}$$

where $c = (c_i)_{i \geq 1}$, and

$$W_i(c) = a_i c_1 c_i - b_{i+1} c_{i+1}, \quad i \geq 1. \tag{1.4}$$

Here $\Omega$ denotes a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary, $\nu$ the outward unit normal vector field to $\partial\Omega$ and

$$d_i > 0, \quad i \geq 1.$$

The *coagulation coefficient* $a_i$ and the *fragmentation coefficient* $b_i$ are nonnegative real numbers for each $i \geq 1$. The reaction part of (1.1) is

1991 *Mathematics Subject Classification*: 35K57, 92E20.

Both authors were supported by the French-Polish project 6058/1996. The second author was also partially supported by Polish KBN grant 2 PO3A 065 08.

a special case of the discrete coagulation-fragmentation equations (see [2], [16]). Indeed setting $a_{1,i} = a_i$, $b_{1,i} = b_{i+1}$ for $i \geq 2$, $a_{i,j} = b_{i,j} = 0$ for $i \geq 2$ and $j \geq 2$ and $a_{1,1} = 2a_1$, $b_{1,1} = 2b_2$ in the general coagulation-fragmentation model one obtains (1.1).

The Becker–Döring equations are thus viewed as describing situations in which the evolution is dominated by clusters gaining or losing just one particle. For a more precise description of the model we refer the reader to the fundamental work [4] where existence of solutions and their various properties are studied in the absence of diffusion. Here we mention that this model was used in [12] to describe the phase transition in a binary alloy. Similar models also appear in the theory of nucleation in chemical physics ([17]).

In recent years, several papers have been devoted to the analysis of the Becker–Döring model with or without diffusion. In the absence of diffusion ($d_i = 0$, $i \geq 1$), existence of solutions is proved in [15] and [4]. Results on the long time behaviour of solutions have subsequently been obtained in [4], [3] and [14].

Fewer results seem to be available for the Becker–Döring model with diffusion. When $d_i = D > 0$ for each $i \geq 1$, existence and uniqueness of strong solutions in $L^2$ is obtained in [7], while the case of different diffusion coefficients is considered in [16] where existence of weak solutions in $L^1$ is proved under a different set of assumptions on the kinetic coefficients than those in [7]. However, both papers [7] and [16] actually investigate existence of solutions to the general coagulation-fragmentation model with diffusion, which is more complicated, and thus require strong assumptions on the kinetic coefficients $(a_i)$ and $(b_i)$. It is our purpose in this work to prove existence of solutions to the Becker–Döring model with diffusion in the case of different diffusion coefficients under rather general assumptions on the kinetic coefficients, extending thereby the results of [7] and [16] for (1.1)–(1.3). Let us also mention at this point the related papers [6] and [9] where the pure coagulation and pure fragmentation models with diffusion are respectively studied.

We assume the same notion of solution as in the previous paper [9]. More precisely, let us define the Banach space

$$X = \Big\{ u = (u_i)_{i\geq 1} : u_i \in L^1(\Omega),\ \sum_{i=1}^{\infty} i|u_i|_{L^1} < \infty \Big\},$$

endowed with the norm

$$\|u\|_X = \sum_{i=1}^{\infty} i|u_i|_{L^1}, \qquad u \in X.$$

We also denote by $X^+$ the positive cone of $X$, i.e.

$$X^+ = \{u = (u_i)_{i\geq 1} \in X : u_i \geq 0 \text{ a.e. in } \Omega\}.$$

Notice that from the physical point of view a solution to (1.1)–(1.3) is expected to satisfy the mass conservation law which reads

$$\sum_{i=1}^{\infty} i|c_i(t)|_{L^1} = \sum_{i=1}^{\infty} i|c_i(0)|_{L^1}, \qquad t \geq 0. \tag{1.5}$$

Within our setting, the conservation of mass is just the conservation of the $X$-norm of a solution to (1.1)–(1.3).

Definition 1.1. A *solution* $c = (c_i)_{i\geq 1}$ to (1.1)–(1.3) is a mapping from $[0,\infty)$ to $X^+$ such that, for each $T > 0$,

1. $c_i \in \mathcal{C}([0,T]; L^1(\Omega))$ for each $i \geq 1$,
2. $\sum_{j=1}^{\infty} a_j c_1 c_j \in L^1(\Omega \times (0,T))$, $\sum_{j=1}^{\infty} b_{j+1} c_{j+1} \in L^1(\Omega \times (0,T))$,
3. $c_i$ is a mild solution to the $i$th equation of (1.1), i.e. for each $t \in [0,T]$,

$$c_1(t) = e^{d_1 L_1 t} c_1(0) - \int_0^t e^{d_1 L_1 (t-s)} \Big( W_1(c(s)) + \sum_{j=1}^{\infty} W_j(c(s)) \Big)\, ds,$$

$$c_i(t) = e^{d_i L_1 t} c_i(0) + \int_0^t e^{d_i L_1 (t-s)} (W_{i-1}(c(s)) - W_i(c(s)))\, ds, \qquad i \geq 2,$$

where $L_1$ is the closure in $L^1(\Omega)$ of the operator $L$ given by

$$D(L) = \{w \in H^2(\Omega) : \partial w/\partial\nu = 0 \text{ on } \partial\Omega\}, \qquad Lw = \Delta w,$$

and $e^{d_i L_1 t}$ denotes the linear $C_0$-semigroup in $L^1(\Omega)$ generated by $d_i L_1$.

Notice that $L$ is closable and accretive in $L^1(\Omega)$ and $L_1$ generates a compact positive and analytic semigroup in $L^1(\Omega)$ (see [1]). Throughout the paper, a solution to (1.1)–(1.3) in the sense of Definition 1.1 is called simply a solution.

We can now describe our main results. Similarly to the case without diffusion any solution to (1.1)–(1.3) conserves the initial mass, which is proved in Proposition 3.1. Existence of solutions is proved in Theorem 2.4 under the following hypotheses:

(H1) There exist $\kappa > 0$ and $\gamma > 0$ such that

(i) $0 < a_i \leq \kappa i$, $i \geq 1$,
(ii) $0 < b_i \leq \gamma a_i$, $i \geq 1$.

(H2) $c_1^0 \in L^\infty(\Omega)$, $\quad c^0 = (c_i^0)_{i\geq 1} \in X^+$.

As in the case of the general coagulation-fragmentation system a solution is constructed as a limit of solutions to suitably chosen truncated systems. It is worth mentioning that we cannot apply here any result from the quoted papers to prove existence of solutions under sufficiently general assumptions covering the case of linear growth of coagulation coefficients. We use the method of proof from [4], [2] and an idea similar to that used in [16] which enables us to show $L^\infty$-bounds for $c_1$.

In Section 3 we study various properties of solutions, starting with the mass conservation (1.5) (Proposition 3.1). Proposition 3.2 shows that any solution $c$ to (1.1)–(1.3) satisfies $c_i \in L^\infty(\Omega \times (0,T))$, $i \geq 1$, provided $c_i^0 \in L^\infty(\Omega)$ for each $i \geq 1$ and $c_1 \in L^\infty(\Omega \times (0,T))$. Under the latter assumption it is proved in Proposition 3.4 that any solution with non-zero initial mass has positive components: $c_i > 0$ on $\Omega \times [\delta, T)$ for any $\delta > 0$ and $i \geq 2$. The last result of this section (Proposition 3.5) identifies subsets of $X^+$ which are invariant through time evolution. Notice that both propositions may be applied to the solutions constructed in Theorem 2.4.

We then devote Section 4 to the question of uniqueness. We are not able to extend the uniqueness result from [4] to our case. Nevertheless, in Proposition 4.1 we provide *a posteriori* conditions under which solutions are uniquely determined. It is worth pointing out that under some additional assumptions (still physically relevant) these conditions are satisfied and a uniqueness result is provided in Theorem 4.2. We refer the reader to our paper [10] for a study of the long time behaviour of the solutions to (1.1)–(1.3) we construct in Theorem 2.4.

We use the following notations. The norm in the space $L^p(\Omega)$ is denoted by $|\cdot|_{L^p}$; otherwise the norm of a Banach space is $|\cdot|_B$. For $T > 0$, we will use the symbol $\Omega_T$ to denote the set $\Omega \times (0,T)$.

**2. Existence.** The solutions to (1.1)–(1.3) are constructed as a limit of solutions to suitable finite systems. We consider two different truncated systems called $(P^N)_\varrho$ for $\varrho = 0$ or $\varrho = 1$, $N = 2, 3, \ldots$, such that $(P^N)_1$ and $(P^N)_0$ consist of $N+1$ and $N$ equations respectively. Solutions to $(P^N)_1$ are mass-preserving for $N$ fixed in contrast to solutions to $(P^N)_0$. The particular form of $(P^N)_0$ is used in [10] in the derivation of a Lyapunov identity which is a crucial point in the study of the long time behaviour of solutions. The systems $(P^N)_\varrho$ for $\varrho = 1$ or $\varrho = 0$ read as follows:

$$\frac{\partial c_1^N}{\partial t} - d_1 \Delta c_1^N = -W_1(c^N) - \sum_{j=1}^{N-1} W_j(c^N) - a_N c_1^N c_N^N, \tag{2.1}$$

$$\frac{\partial c_i^N}{\partial t} - d_i \Delta c_i^N = W_{i-1}(c^N) - W_i(c^N), \quad 2 \leq i \leq N-1,$$

$$\frac{\partial c_N^N}{\partial t} - d_N \Delta c_N^N = W_{N-1}(c^N) - \varrho a_N c_1^N c_N^N,$$

$$\frac{\partial c_{N+1}^N}{\partial t} - d_{N+1} \Delta c_{N+1}^N = \varrho a_N c_1^N c_N^N,$$

with homogeneous Neumann boundary conditions and initial data

$$c_i^N(0) = \min(c_i^0\,, N) + \varepsilon_i/N, \qquad 1 \le i \le N+1, \tag{2.2}$$

where $(\varepsilon_i)_{i\ge1}$ is an arbitrary sequence of positive real numbers satisfying $\sum_{i\ge1} i\varepsilon_i = \Upsilon < \infty$.

Notice that both truncated systems for $\varrho = 0$ and $\varrho = 1$ are different from the one used in [4]. The latter may be obtained from $(P^N)_\varrho$ by removing the terms $a_N c_1^N c_N^N$ from the first and the last two equations. For such a system, however, we are not able to show uniform $L^\infty$-bounds for the sequence $(c_1^N)_{N\ge2}$.

PROPOSITION 2.1. *Assume that* (H1)(ii) *holds. The system* $(P^N)_\varrho$ ($\varrho = 0$ *or* $\varrho = 1$) *has a unique solution* $c^N$ *such that for each* $T > 0$ *and* $i \in \{1\,, \ldots, N+1\}$,

$$\begin{gathered} c_i^N \in \mathcal{C}([0,T]; L^2(\Omega)) \cap \mathcal{C}^1((0,T]; L^2(\Omega)) \cap L^\infty(\Omega_T) \cap L^\infty_{\mathrm{loc}}(0,T; D(L)), \\ c_i^N > 0 \quad \text{in } \Omega_T, \end{gathered} \tag{2.3}$$

*and each equation* (*in* $(P^N)_\varrho$) *is satisfied pointwise almost everywhere in* $\Omega_T$. *Moreover*,

$$\begin{aligned} \sum_{i=1}^{N} i|c_i^N(t)|_{L^1} &\le \sum_{i=1}^{N} i|c_i^N(0)|_{L^1}, \qquad t>0, \text{ for } \varrho = 0, \\ \sum_{i=1}^{N+1} i|c_i^N(t)|_{L^1} &= \sum_{i=1}^{N+1} i|c_i^N(0)|_{L^1}, \qquad t>0, \text{ for } \varrho = 1, \end{aligned} \tag{2.4}$$

*and*

$$c_1^N \le k_1 = \max(|c_1^0|_{L^\infty(\Omega)} + \varepsilon_1/N, 2\gamma) \qquad \text{in } \Omega\times(0,\infty). \tag{2.5}$$

P r o o f. Denote by $f_i^N$ the right-hand side of the $i$th equation of $(P^N)_\varrho$. The local-in-time solvability follows by classical arguments since $f_i^N$ are locally Lipschitz continuous functions. Since we have $f_i^N(\xi) \ge 0$ for each $\xi \in [0,\infty)^N$ such that $\xi_i = 0$, $(f_i^N)_{1\le i\le N+1}$ is quasi-positive and hence $u_i^N \ge 0$ on $\Omega\times(0\,, T_{\max}^N)$ (see e.g. [11]).

We shall prove that $T_{\max}^N = \infty$. To this end we multiply (2.1) by $(c_1^N - k_1)_+$ where $k_1$ is defined in (2.5). We then obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|(c_1^N-k_1)_+|_{L^2}^2 \le{}& \int_\Omega (2b_2c_2^N - a_2c_1^Nc_2^N)(c_1^N-k_1)_+ \\
&+ \int_\Omega \sum_{j=3}^N (b_jc_j^N - a_jc_1^Nc_j^N)(c_1^N-k_1)_+ \\
\le{}& \int_\Omega (2b_2-2\gamma a_2)c_2^N + \int_\Omega \sum_{j=3}^N (b_j-2\gamma a_j)c_j^N \le 0,
\end{aligned}
$$

which implies (2.5). Now, proceeding as in the proof of Proposition 3.2 below, we deduce that for arbitrary $T>0$,

$$\max_{1\le i\le N+1} |c_i^N|_{L^\infty(\Omega_T)} \le C(N,T).$$

Thus, the solutions to $(P^N)_\varrho$ are global in time.

Using the maximum principle one can show that

$$c_i^N(x,t) \ge \frac{\varepsilon_i}{N}e^{-t\alpha_i} \quad \text{in } \Omega_T$$

where $\alpha_i = a_ik_1 + b_i$ if $i\ge 2$ and $\alpha_1 = |\sum_{i=1}^N a_ic_i^N|_{L^\infty(\Omega_T)}$.

It remains to check (2.4). It will follow from the next lemma which contains a basic identity which will frequently be used in the sequel.

Lemma 2.2. *Let $N\ge 2$ and $c^N$ be a solution to $(P^N)_\varrho$. For any $(g_i)\in[0,\infty)^{N+1}$,*

$$
\begin{aligned}
\text{(2.6)}\quad \sum_{i=1}^{N+1} g_i\frac{\partial c_i^N}{\partial t} - \sum_{i=1}^{N+1} g_id_i\Delta c_i^N ={}& \sum_{i=1}^{N-1}(g_{i+1}-g_i-g_1)W_i(c^N) \\
&+ (\varrho g_{N+1} - \varrho g_N - g_1)a_Nc_1^Nc_N^N.
\end{aligned}
$$

Proof (of Lemma 2.2). We multiply the $i$th equation of $(P^N)_\varrho$ by $g_i$ and sum up the resulting equalities. This gives (2.6). ■

Inserting in (2.6) $g_i=i$ for $i\in\{1,\dots,N+1\}$ if $\varrho=1$, and for $i\in\{1,\dots,N\}$ if $\varrho=0$, and integrating over $\Omega\times(0,t)$ for $t>0$ we obtain (2.4), which completes the proof of Proposition 2.1. ■

Remark 2.3. In the sequel we will not use the positivity of solutions to $(P^N)_\varrho$. However, this property is necessary in order to derive the Lyapunov identity which we demonstrate in [10].

We now state and prove the main result of this section.

Theorem 2.4. *Assume that* (H1)–(H2) *hold. Then there exists a mass-preserving solution $c=(c_i)_{i\ge 1}$ to* (1.1)–(1.3) *with $c_1\in L^\infty(\Omega_T)$ for each $T>0$, i.e. a solution to* (1.1)–(1.3) *in the sense of Definition* 1.1 *with a bounded first component and satisfying* (1.5) *as well.*

Proof. We shall use the method introduced in [4] and [2] with suitable modifications. The proof concerns only the case of nonconservative approximation ($\varrho = 0$). The proof for $\varrho = 1$ is very similar and we omit it.

For each $N \geq 2$, we denote by $c^N$ the solution to $(P^N)_0$. We shall first find a bound on the tail of the series in the first equation of (1.1). Let $1 < m \leq (N-1)/2$ and put

$$X_m^N = \sum_{j=m}^{N} jc_j^N, \qquad Q_m^N = \sum_{j=m}^{2m} jc_j^N + 2m \sum_{j=2m+1}^{N} c_j^N.$$

We first show that

$$|X_m^N(t)|_{L^1} \leq |X_m^N(0)|_{L^1} + |Q_m^N(t)|_{L^1} + k_1 \kappa \int_0^t |X_m^N(s)|_{L^1}\, ds. \tag{2.7}$$

Indeed, taking in (2.6) $g_j = 0$ for $1 \leq j \leq m-1$, $g_j = j$ for $m \leq j \leq N$ and $g_{N+1} = 0$ we obtain

$$\sum_{i=m}^{N} i \left( \frac{\partial c_i^N}{\partial t} - d_i \Delta c_i^N \right) = m W_{m-1}(c^N) + \sum_{j=m}^{N-1} W_j(c^N).$$

Integrating over $\Omega \times (0,t)$ yields

$$|X_m^N(t)|_{L^1} = |X_m^N(0)|_{L^1} + \int_0^t \int_\Omega \Big( m W_{m-1}(c^N) + \sum_{j=m}^{N-1} W_j(c^N) \Big)\, dx\, ds. \tag{2.8}$$

Next setting in (2.6),

$$g_j = \begin{cases} 0 & \text{for } 1 \leq j \leq m-1 \text{ and } j = N+1, \\ j & \text{for } m \leq j \leq 2m, \\ 2m & \text{for } 2m+1 \leq j \leq N, \end{cases}$$

and integrating over $\Omega \times (0,t)$ we find

$$|Q_m^N(t)|_{L^1} = |Q_m^N(0)|_{L^1} + \int_0^t \int_\Omega \Big( m W_{m-1}(c^N) + \sum_{j=m}^{2m-1} W_j(c^N) \Big)\, dx\, ds. \tag{2.9}$$

Subtracting (2.9) from (2.8) we obtain

$$\begin{aligned} |X_m^N(t)|_{L^1} &= |X_m^N(0)|_{L^1} + |Q_m^N(t)|_{L^1} - |Q_m^N(0)|_{L^1} \\ &\qquad + \int_0^t \int_\Omega \sum_{j=2m}^{N-1} W_j(c^N)\, dx\, ds \\ &\leq |X_m^N(0)|_{L^1} + |Q_m^N(t)|_{L^1} + \int_0^t \int_\Omega \sum_{j=2m}^{N} a_j c_1^N c_j^N\, dx\, ds. \end{aligned}$$

Now using (H1)(i) and (2.5) we arrive at (2.7).

From (H1)(i), (2.2) and (2.4) we also obtain for $t > 0$,

$$\int_\Omega \sum_{j=1}^{N} a_j c_1^N(t) c_j^N(t)\,dx \le \kappa k_1 \sum_{j=1}^{N} \int_\Omega j c_j^N(t)\,dx \le \kappa k_1 \Big( \|c(0)\|_X + \frac{\Upsilon|\Omega|}{N} \Big),$$

$$\int_\Omega \sum_{j=1}^{N-1} b_{j+1} c_{j+1}^N(t)\,dx \le \gamma\kappa \sum_{j=1}^{N} \int_\Omega j c_j^N(t) dx \le \gamma\kappa \Big( \|c(0)\|_X + \frac{\Upsilon|\Omega|}{N} \Big).$$

Hence, the right-hand side of the $i$th equation of $(P^N)_0$ is uniformly bounded in the space $L^\infty(0,T;L^1(\Omega))$ for any $T > 0$. Using compactness results from [5] we can extract a subsequence $N_k$ such that for each $i \ge 1$ and $T > 0$,

$$\begin{aligned} c_i^{N_k} &\to c_i && \text{in } \mathcal{C}([0,T];L^1(\Omega)) \text{ and a.e. in } \Omega_T, \\ c_i(0) &= c_i^0 && \text{in } \Omega. \end{aligned} \tag{2.10}$$

For fixed $M \le N_k$, it follows from (2.2) and (2.4) that

$$\sum_{i=1}^{M} i |c_i^{N_k}|_{L^1} \le \|c^0\|_X + \frac{\Upsilon|\Omega|}{N_k}. \tag{2.11}$$

Hence, letting $N_k \to \infty$ and then $M \to \infty$ yields

$$\|c(t)\|_X \le \|c^0\|_X, \qquad t \ge 0. \tag{2.12}$$

Let $T > 0$ and $Q_m = \sum_{j=m}^{2m} j c_j + 2m \sum_{j=2m+1}^{\infty} c_j$. We shall show that for $m \ge 2$,

$$\lim_{N_k\to\infty} |Q_m^{N_k}(t) - Q_m(t)|_{L^1} = 0, \qquad 0 \le t \le T. \tag{2.13}$$

Indeed, by (2.11) and (2.12), for $0 \le t \le T$,

$$\Big| \sum_{j=L}^{\infty} c_j(t) - \sum_{j=L}^{N_k+1} c_j^{N_k}(t) \Big|_{L^1} \le \frac{2}{L} \|c^0\|_X, \qquad N_k \ge L. \tag{2.14}$$

Let $\varepsilon > 0$ and $L \ge 4(m + \|c^0\|_X)/\varepsilon$. Then by (2.14), for $N_k \ge L$,

$$\begin{aligned} |Q_m^{N_k}(t) - Q_m(t)|_{L^1} \le{}& \sum_{j=m}^{2m} j |c_j^{N_k}(t) - c_j(t)|_{L^1} \\ &+ 2m \sum_{j=2m+1}^{L-1} |c_j^{N_k}(t) - c_j(t)|_{L^1} + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ yields

$$\limsup_{N_k\to\infty} |Q_m^{N_k}(t) - Q_m(t)|_{L^1} \le \varepsilon$$

for any $\varepsilon > 0$, hence (2.13). Moreover,

$$|Q_m(t)|_{L^1} \le \sum_{j=m}^{\infty} j|c_j(t)|_{L^1} \to 0 \quad \text{for } t \in [0,T] \text{ as } m \to \infty. \tag{2.15}$$

From (2.7) and the Gronwall lemma we find

$$|X_m^N(t)|_{L^1} \le G_m^N(t) + k_1\kappa e^{k_1\kappa t}\int_0^t G_m^N(s)\,ds \tag{2.16}$$

with $G_m^N(s) = |X_m^N(0)|_{L^1} + |Q_m^N(s)|_{L^1}$. Using the Lebesgue dominated convergence theorem and (2.15), (2.12) we obtain

$$\lim_{m\to\infty}\int_0^T |Q_m(s)|_{L^1}\,ds = 0.$$

Now we shall show that for $t \in [0,T]$,

$$\sum_{j=1}^{N_k} jc_j^{N_k}(t) \to \sum_{j=1}^{\infty} jc_j(t) \quad \text{in } L^1(\Omega) \text{ as } N_k \to \infty. \tag{2.17}$$

Fix $t \in [0,T]$. It follows from (2.12), (H2) and (2.2) that for every $\varepsilon > 0$ there exists $M \ge 1$ such that

$$\sum_{j=M}^{\infty} j|c_j(t)|_{L^1} \le \varepsilon, \qquad \sum_{j=M}^{\infty} (j|c_j^0|_{L^1} + j\varepsilon_j) \le \varepsilon,$$
$$\int_0^T |Q_M(s)|_{L^1}\,ds \le \varepsilon, \qquad |Q_M(t)|_{L^1} \le \varepsilon.$$

It follows from (2.13) and the Lebesgue dominated convergence theorem that there exists $k_0$ such that for $k \ge k_0$ and $N_k \ge M$,

$$|Q_M^{N_k}(t)|_{L^1} \le 2\varepsilon, \qquad \int_0^T |Q_M^{N_k}(s)|_{L^1}\,ds \le 2\varepsilon.$$

Hence, for $k \ge k_0$,

$$G_M^{N_k}(t) \le 3\varepsilon, \qquad \int_0^T G_M^{N_k}(t)\,dt \le (T+2)\varepsilon,$$

which yields, thanks to (2.16),

$$|X_M^{N_k}(t)|_{L^1} \le (3 + k_1\kappa e^{k_1\kappa T}(T+2))\varepsilon. \tag{2.18}$$

Consequently, for $k \ge k_0$,

$$\begin{aligned}\Big|\sum_{j=1}^{\infty} jc_j(t) - \sum_{j=1}^{N_k} jc_j^{N_k}(t)\Big|_{L^1}&\\ &\le \sum_{j=1}^{M-1} j|c_j(t) - c_j^{N_k}(t)|_{L^1} + \sum_{j=M}^{\infty} j|c_j(t)|_{L^1} + |X_M^{N_k}(t)|_{L^1}\\ &\le \sum_{j=1}^{M-1} j|c_j(t) - c_j^{N_k}(t)|_{L^1} + (4 + k_1\kappa e^{k_1\kappa T}(2+T))\varepsilon\end{aligned}$$

This implies (2.17) after letting $N_k \to \infty$.

Setting in (2.6) $g_i = i$ for $1 \le i \le N_k$ and $g_{N_k+1} = 0$ and integrating over $\Omega \times (0,t)$, $t \in [0,T]$, we obtain

$$\int_\Omega \sum_{i=1}^{N_k} ic_i^{N_k}(t)\,dx + \int_0^t\int_\Omega a_{N_k} c_1^{N_k} c_{N_k}^{N_k}\,dx\,ds = \int_\Omega \sum_{i=1}^{N_k} ic_i^{N_k}(x,0)\,dx. \tag{2.19}$$

In view of (2.5) and (2.18),

$$\lim_{N_k\to\infty}\int_0^t |a_{N_k} c_1^{N_k} c_{N_k}^{N_k}|_{L^1}\,ds = 0.$$

Now passing to the limit in (2.19) and using (2.17) and (2.2) we arrive at

$$\|c(t)\|_X = \|c^0\|_X \quad \text{for } t \in [0,T]. \tag{2.20}$$

To complete the proof it is sufficient to show that the terms on the right-hand side of $(P^N)_0$ converge in $L^1(\Omega_T)$ to the appropriate limits. To this end notice that due to (2.5) and (2.10),

$$W_j(c^{N_k}) \to W_j(c) \quad \text{in } L^1(\Omega_T) \text{ as } N_k \to \infty.$$

We shall show that for each $t \in [0,T]$,

$$\sum_{j=1}^{N_k} a_j c_j^{N_k}(t) \to \sum_{j=1}^{\infty} a_j c_j(t) \quad \text{in } L^1(\Omega) \text{ as } N_k \to \infty. \tag{2.21}$$

Indeed, let $M \ge 2$. For $k$ large enough, we have $N_k > M$ and by (H1)(i),

$$\begin{aligned}\Big|\sum_{j=1}^{N_k} a_j c_j^{N_k}(t) - \sum_{j=1}^{\infty} a_j c_j(t)\Big|_{L^1} &\le \kappa |X_M^{N_k}(t)|_{L^1} + \kappa \sum_{j=M}^{\infty} j|c_j(t)|_{L^1}\\ &\quad + \kappa \sum_{j=1}^{M-1} j|c_j^{N_k}(t) - c_j(t)|_{L^1}.\end{aligned}$$

We then let $k \to \infty$ and use (2.10) and (2.17) to obtain

$$\limsup_{k\to\infty} \Big|\sum_{j=1}^{N_k} a_j c_j^{N_k}(t) - \sum_{j=1}^{\infty} a_j c_j(t)\Big|_{L^1} \le 2\kappa \sum_{j=M}^{\infty} j|c_j(t)|_{L^1}.$$

Letting $M \to \infty$ and using (2.20) then yield (2.21).

We now infer from (2.21), (2.20), (2.2), (2.4) and the Lebesgue dominated convergence theorem that

$$\sum_{i=1}^{N_k} a_j c_j^{N_k} \to \sum_{i=1}^{\infty} a_j c_j \quad \text{ in } L^1(\Omega_T).$$

Similarly one shows that

$$\sum_{j=1}^{N_k} b_j c_j^{N_k} \to \sum_{j=1}^{\infty} b_j c_j \quad \text{ in } L^1(\Omega_T).$$

We conclude from (2.5), (2.10) and (2.21) that

$$c_1^{N_k} \sum_{j=1}^{N_k} a_j c_j^{N_k} \to c_1 \sum_{j=1}^{\infty} a_j c_j \quad \text{ in } L^1(\Omega_T).$$

Notice that (2.3) and (2.20) imply that the solution constructed above belongs to $X^+$, which completes the proof. ∎

**3. Conservation of mass, $L^\infty$-bounds, positivity and higher moments.** In this section, we investigate various properties of solutions to (1.1)–(1.3). The following proposition shows that similarly to the case without diffusion, each solution to (1.1)–(1.3) satisfies the conservation of mass (1.5). It is worth pointing out that this is not true for the general coagulation-fragmentation system (see e.g. [2]).

PROPOSITION 3.1. *If $c = (c_i)_{i\ge1}$ is a solution to* (1.1)–(1.3) *then for each $t \in [0,\infty)$ and $M \ge 1$,*

$$\|c(t)\|_X = \|c^0\|_X, \tag{3.1}$$

$$\sum_{i=M+1}^{\infty} |c_i(t)|_{L^1} = \sum_{i=M+1}^{\infty} |c_i^0|_{L^1} + \int_0^t \int_\Omega W_M(c(s))\, dx\, ds. \tag{3.2}$$

Proof. Let $N > M \ge 1$ and $t \in (0,\infty)$. Since $c_i$ is a nonnegative mild solution to a linear heat equation with right-hand side in $L^1(\Omega_T)$, initial data in $L^1(\Omega)$ and homogeneous Neumann boundary conditions, we have

$$(3.3)\quad \sum_{i=M+1}^{N} g_i|c_i(t)|_{L^1} - \sum_{i=M+1}^{N} g_i|c_i^0|_{L^1} = \int_0^t\int_\Omega \sum_{i=M+1}^{N} (g_{i+1}-g_i)W_i(c)\,dx\,ds - \int_0^t\int_\Omega g_{N+1}W_N(c)\,dx\,ds + \int_0^t\int_\Omega g_{M+1}W_M(c)\,dx\,ds,$$

for any $g_i \ge 0$, $M+1 \le i \le N+1$. Now the proof runs very similarly to [4, Corollary 2.6]. By Definition 1.1 we have

$$\lim_{N\to\infty}\int_0^t\int_\Omega W_N(c)\,dx\,ds = 0.$$

Setting $g_i = 1$ in (3.3) and letting $N \to \infty$ we obtain (3.2) (recall that $c(s) \in X^+$ for each $s \in [0,\infty)$).

We next show that

$$(3.4)\quad \lim_{N\to\infty}(N+1)\int_0^t\int_\Omega W_N(c)\,dx\,ds = 0.$$

To this end notice that by Definition 1.1, $c(t) \in X$ and

$$(3.5)\quad \lim_{N\to\infty}(N+1)\sum_{i=N+1}^{\infty}|c_i(t)|_{L^1} \le \lim_{N\to\infty}\sum_{i=N+1}^{\infty} i|c_i(t)|_{L^1} = 0.$$

We then deduce from (3.2) that

$$\Big|(N+1)\int_0^t\int_\Omega W_N(c)\,dx\,ds\Big| \le (N+1)\sum_{i=N+1}^{\infty}|c_i(t)|_{L^1} + (N+1)\sum_{i=N+1}^{\infty}|c_i^0|_{L^1},$$

and (3.4) follows from (3.5) and the above inequality.

We now set $g_i = i$ for $M+1 \le i \le N+1$ in (3.3) and let $N \to \infty$ in the resulting identity. Using Definition 1.1 and (3.4), we obtain

$$(3.6)\quad \sum_{i=M+1}^{\infty} i|c_i(t)|_{L^1} - \sum_{i=M+1}^{\infty} i|c_i^0|_{L^1} = \int_0^t\int_\Omega \sum_{i=M+1}^{\infty} W_i(c)\,dx\,ds + (M+1)\int_0^t\int_\Omega W_M(c)\,dx\,ds.$$

Taking $M = 1$ in (3.6) and adding the first equation of (1.1) integrated over $\Omega \times (0,t)$ yield (3.1). ∎

Our next result states further regularity properties with respect to space and time variables of solutions to (1.1)–(1.3).

PROPOSITION 3.2. *Let $c = (c_i)_{i\geq 1}$ be a solution to* (1.1)–(1.3) *such that $c_i^0 \in L^\infty(\Omega)$ for each $i \geq 1$ and $c_1 \in L^\infty(\Omega_T)$ for some $T > 0$. Then also $c_i \in L^\infty(\Omega_T)$ for $i \geq 2$.*

The starting point for the proof of Proposition 3.2 is the following lemma.

LEMMA 3.3. *Let $T > 0$, $f \in L^1(\Omega_T)$ and consider a mild solution $u$ to*

$$\begin{aligned} u_t - D\Delta u + u &= f && \text{in } \Omega_T, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned}$$

*where $D > 0$ and $u_0 \in L^1(\Omega)$. Then $u \in L^p(\Omega_T)$ for any $p \in [1,(n+2)/n)$ and there is $C(n,D,p)$ depending only on $\Omega$, $n$, $D$ and $p$ such that*

$$|u|_{L^p(\Omega_T)} \leq C(n,D,p)(|u_0|_{L^1(\Omega)} + |f|_{L^1(\Omega_T)}), \qquad p \in [1,(n+2)/n).$$

Proof. We denote by $S(t)$ the $C_0$-semigroup in $L^1(\Omega)$ generated by the operator $-D\Delta + \mathrm{Id}$ with homogeneous Neumann boundary conditions. Smoothing effects are available for $S(t)$ and read (see e.g. [13, p. 25])

$$|S(t)z|_{L^p(\Omega)} \leq C(n,p)t^{-(n/2)(1-1/p)}|z|_{L^1(\Omega)}$$

for $z \in L^1(\Omega)$ and $p \in [1,\infty]$. Using the integral representation of $u$, we deduce that for $t \in (0,T)$ and $p \in [1,(n+2)/n)$,

$$\begin{aligned} |u(t)|_{L^p(\Omega)} &\leq |S(t)u_0|_{L^p(\Omega)} + \int_0^t |S(t-s)f(s)|_{L^p(\Omega)}\,ds \\ &\leq C(n,p)\Big(t^{-(n/2)(1-1/p)} + \int_0^t (t-s)^{-(n/2)(1-1/p)}|f(s)|_{L^1(\Omega)}\,ds\Big). \end{aligned}$$

Since $t \mapsto t^{-(n/2)(1-1/p)}$ belongs to $L^p(0,T)$ if $p \in [1,(n+2)/n)$ and $f \in L^1(\Omega_T)$, the above estimate and Young inequality for time convolution yield Lemma 3.3. ∎

*Proof of Proposition 3.2.* We consider each equation separately. Let $T > 0$ and $i \geq 2$. Since $c_i$ is a mild solution to

$$\frac{\partial c_i}{\partial t} - d_i\Delta c_i + c_i \in L^1(\Omega_T), \qquad c_i(0) \in L^\infty(\Omega),$$

with homogeneous Neumann boundary conditions, we infer from Lemma 3.3 that

$$c_i \in L^p(\Omega_T) \quad \text{for each } p \in [1,(n+2)/n). \tag{3.7}$$

Since (3.7) is valid for each $i \geq 2$ and $c_1 \in L^1(\Omega_T)$, $c_i$ is in fact a mild solution to

$$\frac{\partial c_i}{\partial t} - d_i \Delta c_i \in L^{\theta(n+2)/n}(\Omega_T), \qquad \frac{n}{n+2} \le \theta < 1,$$

for each $i \ge 2$. Classical $L^p$ regularity theory for linear parabolic equations ([8]) then yields

$$c_i \in W^{2,1}_{\theta(n+2)/n}(\Omega_T), \qquad \frac{n}{n+2} \le \theta < 1.$$

From the imbedding theorem (see [8, Lemma II.3.3]) it follows that

$$\begin{cases} c_i \in L^\infty(\Omega_T) & \text{for } n = 1, \\ c_i \in L^{\theta(n+2)/(n-2\theta)}(\Omega_T) & \text{for } n \ge 2. \end{cases} \tag{3.8}$$

The latter result also reads, for $n \ge 2$,

$$c_i \in L^p(\Omega_T) \quad \text{for } 1 \le p < \frac{n+2}{n-2} \text{ and } i \ge 2.$$

To complete the proof we shall show that for each $k \ge 1$ the following statement $(I_k)$ holds:

$$\begin{array}{lll} \text{if } n \ge 2k & \text{then} \quad c_i \in L^p(\Omega_T) & \text{for } 1 \le p < \dfrac{n+2}{n-2k} \text{ and } i \ge 2, \\ \text{if } 1 \le n < 2k & \text{then} \quad c_i \in L^\infty(\Omega_T) & \text{for } i \ge 2. \end{array}$$

Notice that by (3.8), $(I_1)$ holds true. Now assume $(I_k)$ for some $k \ge 1$ and consider $n \ge 2k$. Since $c_1 \in L^\infty(\Omega_T)$ we have

$$\frac{\partial c_i}{\partial t} - d_i \Delta c_i \in L^{\theta(n+2)/(n-2k)}(\Omega_T), \qquad \frac{n-2k}{n+2} \le \theta < 1.$$

Hence, $c_i \in W^{2,1}_{\theta(n+2)/(n-2k)}(\Omega_T)$ and using again [8, Lemma II.3.3] we find that

$$W^{2,1}_{\theta(n+2)/(n-2k)}(\Omega_T) \hookrightarrow \begin{cases} L^\infty(\Omega_T) & \text{if } 2k \le n < 2k+2, \\ L^{\theta(n+2)/(n-2(k+\theta))}(\Omega_T) & \text{if } n \ge 2k+2, \end{cases}$$

which yields $(I_{k+1})$. ■

We next turn to some positivity properties of solutions to (1.1)–(1.3).

PROPOSITION 3.4. *Assume that* $c = (c_i)_{i \ge 1}$ *is a solution to* (1.1)–(1.3) *such that* $\|c^0\|_X > 0$ *and* $c_1 \in L^\infty(\Omega_T)$ *for some* $T > 0$. *Then for* $i \ge 2$,

$$c_i(x,t) > 0 \quad \text{for } (x,t) \in \Omega_T$$

*after a possible modification on a set of measure zero.*

Proof. Fix $\tau > 0$ and set

$$E(\tau) = \{j \ge 2 : c_j(\cdot, t) = 0 \text{ a.e. on } \Omega \text{ for } t \in [0, \tau]\}.$$

We claim that $E(\tau)=\emptyset$ for $0<\tau\le T$. Assume that, on the contrary, $E(\tau)\ne\emptyset$ for some $\tau>0$, and put

$$i=\min E(\tau)\ge 2.$$

By Definition 1.1 we have

$$c_i(t)=e^{d_iL_1t}c_i^0+\int_0^t e^{d_iL_1(t-s)}(a_{i-1}c_1c_{i-1}-b_ic_i-a_ic_1c_i+b_{i+1}c_{i+1})\,ds.$$

Since $i\in E(\tau)$, for $t=\tau$ we obtain

$$\int_0^\tau e^{d_iL_1(t-s)}(a_{i-1}c_1c_{i-1}+b_{i+1}c_{i+1})\,ds=0.$$

However, $e^{d_iL_1t}$ is a positive semigroup and using the definition of a solution we conclude that for each $t\in[0,\tau]$,

$$a_{i-1}c_1(\cdot\,,t)c_{i-1}(\cdot\,,t)+b_{i+1}c_{i+1}(\cdot\,,t)=0\quad\text{ a.e. in } \Omega.$$

Thus, for each $t\in[0,\tau]$,

$$\begin{aligned} c_1(x,t)c_{i-1}(x,t)&=0\quad\text{ for a.e. } x\in\Omega,\\ c_{i+1}(x,t)&=0\quad\text{ for a.e. } x\in\Omega.\end{aligned}\tag{3.9}$$

Hence, proceeding by induction we may prove that in fact

$$E(\tau)=\{j\in\mathbb{N}:j\ge i\}.$$

If $i=2$ then (3.9) yields $c_1(\cdot\,,t)=0$ a.e. in $\Omega$ for $t\in[0\,,\tau]$, hence $c(\cdot\,,0)=0$ a.e., which contradicts the assumption. Thus, $i>2$.

Recall that $c_{i-1}$ is a solution to

$$\frac{\partial c_{i-1}}{\partial t}-d_{i-1}\Delta c_{i-1}+b_{i-1}c_{i-1}+a_{i-1}c_1c_{i-1}=a_{i-2}c_1c_{i-2}+b_ic_i.$$

Since $c_i=0$ on $[0,\tau]$, for $\varepsilon>0$ and $t\in[0,\tau]$ we have

$$\begin{aligned}|(c_{i-1}-\varepsilon)_+(t)|_{L^1}&\le|(c_{i-1}-\varepsilon)_+(0)|_{L^1}\\&\quad+\int_0^t\int_\Omega a_{i-2}c_1c_{i-2}\operatorname{sign}((c_{i-1}-\varepsilon)_+)\,dx\,ds,\end{aligned}\tag{3.10}$$

where $y_+=\max(y,0)$. It follows from (3.9) that for $(x,s)\in\Omega\times(0,\tau)$,

$$\begin{aligned}&\text{if}\quad c_{i-1}(x,s)\ge\varepsilon\quad\text{then}\quad c_1(x\,,s)=0,\\&\text{if}\quad c_{i-1}(x,s)<\varepsilon\quad\text{then}\quad (c_{i-1}-\varepsilon)_+(x,s)=0.\end{aligned}$$

In both cases $c_1\operatorname{sign}((c_{i-1}-\varepsilon)_+)=0$ a.e. and therefore from (3.10),

$$|(c_{i-1}-\varepsilon)_+(t)|_{L^1}\le|(c_{i-1}-\varepsilon)_+(0)|_{L^1}\quad\text{ for } t\in[0,\tau].\tag{3.11}$$

Now we show that $c_{i-1}(0)=0$ a.e. in $\Omega$. Assume that this is false; then

$$c_{i-1}\ge f\quad\text{ a.e. on } \Omega\times(0,\tau),$$

where $f(x,t) = e^{-(a_{i-1}k_1+b_{i-1})t}F(x,t)$, $k_1 = |c_1|_{L^\infty(\Omega_T)}$ and $F$ is given by

$$(3.12)\qquad \begin{aligned} \frac{\partial F}{\partial t} - d_{i-1}\Delta = 0 &\qquad \text{in } \Omega \times (0,\tau),\\ \frac{\partial F}{\partial \nu} = 0 &\qquad \text{on } \partial\Omega \times (0,\tau),\\ F(\cdot\,,0) = c_{i-1}^0(\cdot) &\qquad \text{in } \Omega. \end{aligned}$$

Notice that (3.12) has a classical solution for $t > 0$ and the strong maximum principle yields $F(x,t) > 0$ for $t > 0$, $x \in \Omega$. Hence after a possible modification on a set of measure zero, $c_{i-1}(x,t) > 0$ on $\Omega \times (0,\tau]$. Therefore it follows from (3.9) that $c_1(\cdot\,,t) = 0$ a.e. on $\Omega$ for $t \in [0,\tau]$. Then also $\sum_{j=1}^\infty a_j c_1 c_j = 0$ a.e. on $\Omega$ for $t \in [0,\tau]$. Hence,

$$c_1(t) = e^{d_1L_1t}c_1^0 + \int_0^t e^{d_1L_1(t-s)} \sum_{j=1}^\infty b_{j+1}c_{j+1}(s)\,ds$$

and for $t = \tau$,

$$\sum_{j=1}^\infty b_{j+1}c_{j+1} = 0 \qquad \text{a.e. on } \Omega \times (0,\tau).$$

Consequently, $c_{i-1} = 0$ a.e. in $\Omega_\tau$ and since $c_{i-1} \in \mathcal{C}([0,T];L^1(\Omega))$ we conclude that $c_{i-1}(t) = 0$ a.e. in $\Omega$ for $t \in [0,\tau]$, contrary to the definition of $i$.

Thus necessarily $c_{i-1}(0) = 0$ a.e. in $\Omega$ and (3.11) yields

$$c_{i-1}(\cdot\,,t) = 0 \qquad \text{a.e. in } \Omega \qquad \text{for } t \in [0,\tau].$$

This again contradicts the definition of $i$ and the claim is proved.

To complete the proof fix $j \geq 2$. Since $E(\tau) = \emptyset$ there is a sequence $t_k \to 0$ such that $|c_j(t_k)|_{L^1} > 0$ for $k \geq 1$. For fixed $k \geq 1$ and $t > t_k$ we have

$$c_j(x\,,t) \geq f(x,t-t_k)$$

where $f(x,t) = e^{-(b_j+a_jk_1)t}F(x,t)$ and

$$\begin{aligned} \frac{\partial F}{\partial t} - d_j\Delta F = 0 &\qquad \text{in } \Omega \times (0,T-t_k),\\ \frac{\partial F}{\partial \nu} = 0 &\qquad \text{on } \partial\Omega \times (0,T-t_k),\\ F(\cdot\,,0) = c_i(\cdot\,,t_k) &\qquad \text{in } \Omega. \end{aligned}$$

The strong maximum principle implies $F > 0$ and $c_j(x,t) > 0$ for $\Omega \times (t_k\,,T]$ after a modification on a set of measure zero. Since $t_k \to 0$ the result follows. ∎

Finally, we give examples of subspaces of $X^+$ which are invariant through the time evolution of solutions to (1.1)–(1.3).

PROPOSITION 3.5. *Assume* (H1)(i) *and consider a solution* $c = (c_i)_{i\ge1}$ *to* (1.1)–(1.3) *such that* $c_1 \in L^\infty(\Omega_T)$ *for some* $T > 0$ *and*

$$\sum_{i=1}^{\infty} g_i |c_i^0|_{L^1} < \infty \tag{3.13}$$

*for some nondecreasing sequence* $(g_i)_{i\ge1}$ *of nonnegative real numbers satisfying*

$$(g_{i+1} - g_i) a_i \le K_0 g_i, \qquad i \ge 2. \tag{3.14}$$

*Then*

$$\sup_{t\in[0,T]} \sum_{i=1}^{\infty} g_i |c_i(t)|_{L^1} < \infty.$$

P r o o f. In the following we denote by $K$ any positive constant depending only on $\Omega$, $T$, $|c_1|_{L^\infty(\Omega_T)}$, $\sum g_i |c_i^0|_{L^1}$, $g_1$ and $K_0$. For $N \ge 2$, we have

$$\begin{aligned}\sum_{i=2}^{N} g_i |c_i(t)|_{L^1} \le \sum_{i=2}^{N} g_i |c_i^0|_{L^1} &+ \int_0^t \int_\Omega \sum_{i=1}^{N} (g_{i+1} - g_i) W_i(c)\, dx\, ds \\ &- \int_0^t \int_\Omega g_{N+1} W_N(c)\, dx\, ds + \int_0^t \int_\Omega g_1 W_1(c)\, dx\, ds.\end{aligned}$$

Since $(g_i)_{i\ge1}$ is nondecreasing, it follows from (3.14) and the nonnegativity of $(c_i)_{i\ge1}$ that

$$\begin{aligned}\sum_{i=2}^{N} g_i |c_i(t)|_{L^1} \le K &+ K_0 \int_0^t \int_\Omega \sum_{i=1}^{N} g_i c_1 c_i\, dx\, ds \\ &- \int_0^t \int_\Omega g_{N+1} W_N(c)\, dx\, ds + \int_0^t \int_\Omega g_1 a_1 c_1^2\, dx\, ds\end{aligned}$$

and so

$$\begin{aligned}\sum_{i=2}^{N} g_i |c_i(t)|_{L^1} &\le K\Big(1 + \int_0^t \sum_{i=2}^{N} g_i |c_i(s)|_{L^1}\, ds\Big) \\ &\quad - \int_0^t \int_\Omega g_{N+1} W_N(c)\, dx\, ds.\end{aligned} \tag{3.15}$$

From (3.2) and the monotonicity of $(g_i)_{i\ge1}$ we infer that

$$-\int_0^t \int_\Omega g_{N+1} W_N(c)\, dx\, ds \le g_{N+1} \sum_{i=N+1}^{\infty} |c_i^0|_{L^1} \le \sum_{i=N+1}^{\infty} g_i |c_i^0|_{L^1}. \tag{3.16}$$

We now combine (3.15) and (3.16) to find

$$\sum_{i=2}^{N} g_i |c_i(t)|_{L^1} \le K\Big(1 + \int_0^t \sum_{i=2}^{N} g_i |c_i(s)|_{L^1}\, ds\Big).$$

Applying the Gronwall lemma gives

$$\sum_{i=2}^{N} g_i |c_i(t)|_{L^1} \le K(1 + e^{Kt}).$$

We let $N \to \infty$ to complete the proof. ∎

Remark 3.6. It is easy to see that if (H1)(i) holds, then $g_i = i^\alpha$, $i \ge 1$, for some $\alpha > 1$ satisfies (3.14), and so does $g_i = i \ln i$, $i \ge 1$.

Remark 3.7. Notice that the solutions constructed in Theorem 2.4 satisfy the assumption $c_1 \in L^\infty(\Omega_T)$. Therefore the conclusions of both propositions above are valid for those solutions.

**4. Uniqueness.** In this section, we investigate uniqueness of solutions to (1.1)–(1.3). We notice that, so far, we have not been able to prove uniqueness results comparable to that for the case without diffusion (see [4, Theorem 3.6]). We first state a uniqueness result under some *a posteriori* conditions on the solution to (1.1)–(1.3). The remainder of the section is then devoted to verifying that in some cases, solutions to (1.1)–(1.3) satisfying those conditions do exist.

Proposition 4.1. *Under the hypothesis* (H1)(i) *there exists at most one solution* $c = (c_i)_{i\ge 1}$ *to* (1.1)–(1.3) *satisfying for some* $T > 0$,

$$\sum_{i=1}^{\infty} i a_i |c_i|_{L^\infty(\Omega)} \in L^\infty(0,T). \tag{4.1.}$$

Proof. Let $c$ and $v$ be two solutions to (1.1)–(1.3) with

$$c_1 \in L^\infty(\Omega_T), \qquad v_1 \in L^\infty(\Omega_T), \tag{4.2}$$

and put $y_i = c_i - v_i$, $i \ge 1$. In the following, we denote by $K$ any positive constant depending only on $\Omega$, $T$, $\kappa$, $|c_1|_{L^\infty(\Omega_T)}$ and $|v_1|_{L^\infty(\Omega_T)}$. Let $N \ge 2$. Since $-d_i\Delta$ is an accretive operator in $L^1(\Omega)$ for each $i \ge 1$, we have

$$\sum_{i=1}^{N} i|y_i(t)|_{L^1} \le \int_0^t \int_\Omega \sum_{i=N}^{\infty} (|W_i(c)| + |W_i(v)|)\, dx\, ds \tag{4.3}$$

$$+\int_0^t\int_\Omega\sum_{i=1}^{N-1}(W_i(c)-W_i(v))[(i+1)\operatorname{sign}(y_{i+1})-i\operatorname{sign}(y_i)-\operatorname{sign}(y_1)]\,dx\,ds$$

$$-\int_0^t\int_\Omega N(W_N(c)-W_N(v))\operatorname{sign}(y_N)\,dx\,ds.$$

Computations similar to those of [4, p. 675] give

$$W_i(c)-W_i(v)=a_i(v_1y_i+y_1c_i)-b_{i+1}y_{i+1} \tag{4.4}$$

and

$$\begin{aligned}[W_i(c)-W_i(v)]&[(i+1)\operatorname{sign}(y_{i+1})-i\operatorname{sign}(y_i)-\operatorname{sign}(y_1)]\\ &=v_1a_i|y_i|[(i+1)\operatorname{sign}(y_iy_{i+1})-i-\operatorname{sign}(y_iy_1)]\\ &\quad+|y_1|a_ic_i[(i+1)\operatorname{sign}(y_{i+1}y_1)-i\operatorname{sign}(y_iy_1)-1]\\ &\quad-b_{i+1}|y_{i+1}|[(i+1)-i\operatorname{sign}(y_iy_{i+1})-\operatorname{sign}(y_{i+1}y_1)]\\ &\le 2(v_1a_i|y_i|+ia_ic_i|y_1|).\end{aligned} \tag{4.5}$$

Combining (H1)(i), (4.2), (4.3) and (4.5) yields

$$\begin{aligned}\sum_{i=1}^{N}i|y_i(t)|_{L^1}\le&\int_0^t\int_\Omega\sum_{i=N}^{\infty}(|W_i(c)|+|W_i(v)|)\,dx\,ds\\ &+K\int_0^t\int_\Omega\Big(\sum_{i=1}^{N-1}i|y_i|+|y_1|\sum_{i=1}^{N-1}ia_ic_i\Big)\,dx\,ds\\ &-\int_0^t\int_\Omega N(W_N(c)-W_N(v))\operatorname{sign}(y_N)\,dx\,ds.\end{aligned} \tag{4.6}$$

Next we have for $M>N$,

$$\begin{aligned}\sum_{i=N+1}^{M}|y_i(t)|_{L^1}\le&\int_0^t\int_\Omega\sum_{i=N}^{M-1}(\operatorname{sign}(y_{i+1})-\operatorname{sign}(y_i))(W_i(c)-W_i(v))\,dx\,ds\\ &+\int_0^t\int_\Omega\operatorname{sign}(y_N)(W_N(c)-W_N(v))\,dx\,ds\\ &-\int_0^t\int_\Omega\operatorname{sign}(y_M)(W_M(c)-W_M(v))\,dx\,ds,\end{aligned}$$

hence by (4.4),

$$\sum_{i=N+1}^{M} |y_i(t)|_{L^1} \le \int_0^t\int_\Omega \sum_{i=N}^{M-1} ((a_i v_1 |y_i| + b_{i+1}|y_{i+1}|)(\mathrm{sign}(y_{i+1}y_i) - 1))\,dx\,ds$$
$$+ 2\int_0^t\int_\Omega |y_1| \sum_{i=N}^{M-1} a_i c_i\,dx\,ds$$
$$+ \int_0^t\int_\Omega \mathrm{sign}(y_N)(W_N(c) - W_N(v))\,dx\,ds$$
$$- \int_0^t\int_\Omega \mathrm{sign}(y_M)(W_M(c) - W_M(v))\,dx\,ds.$$

Since $|\mathrm{sign}(y_M)| \le 1$, the first term of the right-hand side of the above estimate is nonnegative and we may let $M \to \infty$ and use Definition 1.1 and (H1)(i) to obtain

$$\sum_{i=N+1}^{\infty} |y_i(t)|_{L^1} \le K\int_0^t\int_\Omega \sum_{i=N}^{\infty} a_i c_i\,dx\,ds$$
$$+ \int_0^t\int_\Omega \mathrm{sign}(y_N)(W_N(c) - W_N(v))\,dx\,ds,$$

hence

$$-N\int_0^t\int_\Omega \mathrm{sign}(y_N)(W_N(c) - W_N(v))\,dx\,ds \le K\int_0^t\int_\Omega N\sum_{i=N}^{\infty} a_i c_i\,dx\,ds. \tag{4.7}$$

Combining (4.6) and (4.7) finally gives

$$\sum_{i=1}^{N} i|y_i(t)|_{L^1} \le \int_0^t\int_\Omega \sum_{i=N}^{\infty} (|W_i(c)| + |W_i(v)|)\,dx\,ds \tag{4.8}$$
$$+ K\int_0^t\int_\Omega \Big(\sum_{i=1}^{N-1} i|y_i| + |y_1| \sum_{i=1}^{N-1} i a_i c_i\Big)\,dx\,ds$$
$$+ K\int_0^t\int_\Omega N\sum_{i=N}^{\infty} a_i c_i\,dx\,ds.$$

The proof is now almost complete. Using (4.1), we obtain

$$\sum_{i=1}^{N} i|y_i(t)|_{L^1} \le \int_0^t\int_\Omega \sum_{i=N}^{\infty} (|W_i(c)| + |W_i(v)|)\,dx\,ds + K\int_0^t \sum_{i=N}^{\infty} i a_i |c_i|_{L^\infty(\Omega)}\,ds$$
$$+ K\int_0^t\int_\Omega \Big(\sum_{i=1}^{N-1} i|y_i| + \Big(\sum_{i=1}^{\infty} i a_i |c_i|_{L^\infty(\Omega)}\Big)|y_1|\Big)\,dx\,ds.$$

Owing to Definition 1.1, (3.1) and (4.1), we may let $N \to \infty$ in the above estimate and obtain

$$\sum_{i=1}^{\infty} i|y_i(t)|_{L^1} \le K' \int_0^t \sum_{i=1}^{\infty} i|y_i(s)|_{L^1}\, ds$$

for some constant $K'$ depending on $\Omega$, $T$, $\kappa$ and the $L^\infty(0,T)$-norms of $\sum ia_i|c_i|_{L^\infty(\Omega)}$ and $\sum ia_i|v_i|_{L^\infty(\Omega)}$. Proposition 4.1 then follows from the above inequality by the Gronwall lemma. ∎

Though Proposition 4.1 is stated in a rather general way, it turns out that we are, unfortunately, not able to construct a solution to (1.1)–(1.3) satisfying the requirements of Proposition 4.1 in the general case (compare (4.1) or (4.3) to (3.1)). Still, assuming the initial data $c^0 = (c_i^0)_{i\ge1}$ to be sufficiently decreasing with respect to $i$, we have the following result.

THEOREM 4.2. *Assume that* (H1)–(H2) *hold and there are* $\alpha \in [0,1]$, $\beta \ge 0$ *and an integer* $k > n/2$ *such that*

$$a_i \le K_0 i^\alpha, \quad i \ge 1, \tag{4.9}$$

$$K_1 i^{-\beta} \le d_i \le K_2, \quad i \ge 1, \tag{4.10}$$

*for some positive real numbers* $K_j$, $j = 0,1,2$, *and*

$$\sum_{i=1}^{\infty} i^{1+\alpha+(\alpha+\beta)k} |c_i^0|_{L^\infty(\Omega)} < \infty. \tag{4.11}$$

*Then* (1.1)–(1.3) *has a unique solution* $c = (c_i)_{i\ge1}$ *with* $c_1 \in L^\infty(\Omega_T)$ *for any* $T > 0$.

The key point of the proof of Theorem 4.2 is the following lemma.

LEMMA 4.3. *Assume that* (H1)–(H2) *and* (4.9)–(4.10) *hold and consider a solution* $c = (c_i)_{i\ge1}$ *to* (1.1)–(1.3) *satisfying*

$$\sum_{i=1}^{\infty} i^\lambda |c_i^0|_{L^\infty(\Omega)} < \infty, \tag{4.12}$$

$$\sum_{i=1}^{\infty} i^{\alpha+\beta+\lambda} |c_i|_{L^p(\Omega)} \in L^\infty(0,T) \quad \text{and} \quad c_1 \in L^\infty(\Omega_T), \tag{4.13}$$

*for some positive real numbers* $\lambda$, $T$ *and* $p \in [1,\infty]$. *Then*

$$\sum_{i=1}^{\infty} i^\lambda |c_i|_{L^q(\Omega)} \in L^\infty(0,T), \tag{4.14}$$

*where*

$$q \in \begin{cases} [1, np/(n-2p)) & \text{if } n > 2p, \\ [1,\infty) & \text{if } n = 2p, \\ [1,\infty] & \text{if } n < 2p. \end{cases} \tag{4.15}$$

Proof. We denote by $K$ any constant depending only on $\Omega$, $T$, $(\kappa,\gamma)$ in (H1), $|c_1|_{L^\infty(\Omega_T)}$, $\alpha$, $\beta$, $K_0$, $K_1$ and $K_2$. We first consider the case $n > 2p$.

Let $q \in [1, np/(n-2p))$. For $i \geq 2$, we infer from (1.1), (H1), (4.13) and [13, p. 25] that, for $t \in (0,T)$,

$$\begin{aligned}
&|c_i(t)|_{L^q(\Omega)} \\
&\quad \leq |c_i^0|_{L^q(\Omega)} \\
&\qquad + K \int_0^t \min(1, d_i(t-s))^{-(n/2)(1/p-1/q)} |W_{i-1}(c(s)) - W_i(c(s))|_{L^p(\Omega)}\, ds \\
&\quad \leq K(q)\Big( |c_i^0|_{L^\infty(\Omega)} \\
&\qquad + i^\beta \int_0^t \Big( \sum_{j=i-1}^{i+1} j^\alpha |c_j|_{L^p(\Omega)} \Big) \min(1, K_1(t-s))^{-(n/2)(1/p-1/q)}\, ds \Big).
\end{aligned}$$

Summing up the above inequalities with respect to $i$, we obtain for $M \geq 2$ and $t \in (0,T)$,

$$\begin{aligned}
&\sum_{i=2}^M i^\lambda |c_i(t)|_{L^q(\Omega)} \\
&\qquad \leq K(q) \Big( \sum_{i=1}^\infty i^\lambda |c_i^0|_{L^\infty(\Omega)} \\
&\qquad\quad + \int_0^t \min\,(1, K_1(t-s))^{-(n/2)(1/p-1/q)} \sum_{i=1}^\infty i^{\alpha+\beta+\lambda} |c_i|_{L^p(\Omega)}\, ds \Big).
\end{aligned}$$

Since $q \in [1, np/(n-2p))$, it follows from (4.12) and (4.13) that the right-hand side of the above estimate is bounded uniformly with respect to $M \geq 2$ and $t \in (0,T)$, hence (4.14) holds by the monotone convergence theorem.

The cases $n = 2p$ and $n < 2p$ are then handled by a similar argument. ■

*Proof of Theorem 4.2.* Let $c = (c_i)_{i\geq 1}$ be a solution to (1.1)–(1.3) with $c_1 \in L^\infty(\Omega_T)$ for any $T > 0$. We fix $T > 0$ and claim that

$$\sum_{i=1}^\infty i^{1+\alpha} |c_i|_{L^\infty(\Omega)} \in L^\infty(0,T). \tag{4.16}$$

We first consider the case $n = 1$. It follows from (4.11) and Proposition 3.5 that

$$\sum_{i=1}^{\infty} i^{1+\alpha+(\alpha+\beta)k} |c_i|_{L^1(\Omega)} \in L^\infty(0,T). \tag{4.17}$$

Hence by Lemma 4.3 (with $p = 1$ and $\lambda = 1 + \alpha + (\alpha + \beta)(k - 1)$)

$$\sum_{i=1}^{\infty} i^{1+\alpha+(\alpha+\beta)(k-1)} |c_i|_{L^\infty(\Omega)} \in L^\infty(0,T),$$

which yields (4.16) since $k \geq 1$.

Assume next that $n \geq 2$. Again, we first infer from (4.11) and Proposition 3.5 that (4.17) holds. We put $\ell_0 = n/2$ if $n$ is even and $\ell_0 = (n-1)/2$ if $n$ is odd, and $p_l = n/(n-2l)$ for $l \in \{1, \ldots, \ell_0\}$.

STEP 1. We prove that for each $l \in \{1, \ldots, \ell_0\}$ and $q \in [1, p_l)$,

$$\sum_{i=1}^{\infty} i^{1+\alpha+(\alpha+\beta)(k-l)} |c_i|_{L^q(\Omega)} \in L^\infty(0,T). \tag{4.18}$$

We proceed by induction. It first follows from (4.11), (4.17) and Lemma 4.3 (with $p = 1$ and $\lambda = 1 + \alpha + (\alpha + \beta)(k - 1)$) that (4.18) holds for $l = 1$. Assume that it holds for some $l \in \{1, \ldots, \ell_0 - 1\}$, i.e.

$$\sum_{i=1}^{\infty} i^{1+\alpha+(\alpha+\beta)(k-l)} |c_i|_{L^q(\Omega)} \in L^\infty(0,T) \tag{4.19}$$

for each $q \in [1, p_l)$. Owing to (4.11) and (4.19), we may apply Lemma 4.3 (with $p = q$ and $\lambda = 1 + \alpha + (\alpha + \beta)(k - l - 1)$) and obtain

$$\sum_{i=1}^{\infty} i^{1+\alpha+(\alpha+\beta)(k-l-1)} |c_i|_{L^r(\Omega)} \in L^\infty(0,T)$$

for each $r \in [1, nq/(n-2q))$ and $q \in [1, p_l)$ (observe that $q < p_l$ and $l \leq \ell_0 - 1$ imply that $2q < n$). Since this is valid for each $q \in [1, p_l)$ and $r \in [1, nq/(n-2q))$, and $p_{l+1}(n - 2p_l) = np_l$, we conclude that (4.18) holds for $l + 1$.

STEP 2. We now infer from (4.18) with $l = \ell_0$ that

$$\sum_{i=1}^{\infty} i^{1+\alpha+(\alpha+\beta)(k-\ell_0)} |c_i|_{L^q(\Omega)} \in L^\infty(0,T) \tag{4.20}$$

for each $q \in [1, p_{\ell_0})$. Since $p_{\ell_0} = \infty$ if $n$ is even and $p_{\ell_0} = n$ if $n$ is odd, (4.20) is true for $q = (2n+1)/4 \in (n/2, n)$, which yields, together with (4.11) and

Lemma 4.3 (with $p = (2n+1)/4$ and $\lambda = 1 + \alpha + (\alpha+\beta)(k - \ell_0 - 1)$),

$$\sum_{i=1}^{\infty} i^{1+\alpha+(\alpha+\beta)(k-\ell_0-1)} |c_i|_{L^\infty(\Omega)} \in L^\infty(0,T). \tag{4.21}$$

Since $k > n/2$ is an integer, we have $k \geq \ell_0 + 1$ and (4.16) follows at once from (4.9) and (4.21).

Consequently, under the assumptions of Theorem 4.2, any solution $c = (c_i)_{i\geq 1}$ to (1.1)–(1.3) with $c_1 \in L^\infty(\Omega_T)$ for any $T > 0$ satisfies (4.16) for each $T > 0$. We then infer from Proposition 4.1 that under the assumptions of Theorem 4.2, (1.1)–(1.3) has at most one solution $c = (c_i)_{i\geq 1}$ with $c_1 \in L^\infty(\Omega_T)$ for any $T > 0$. Since Theorem 2.4 provides the existence of such a solution, the proof of Theorem 4.2 is complete. ■

REMARK 4.4. Notice that Theorem 4.2 implies uniqueness of solutions in the physically important situation in which the coagulation-fragmentation process starts from a single concentration of monoclusters.

**Acknowledgements.** This work was done during visits of the first author to the Interdisciplinary Center for Mathematical and Computational Modelling (ICM) and the Institute of Applied Mathematics and Mechanics of Warsaw University and of the second author to the Institut Elie Cartan of the University of Nancy I. We thank these institutions for their hospitality. The first author also thanks D. Schmitt for useful discussions.

*REFERENCES*

[1] H. Amann, *Dual semigroups and second order linear elliptic boundary value problems*, Israel J. Math. 45 (1983), 225–254.

[2] J. M. Ball and J. Carr, *The discrete coagulation-fragmentation equations: existence, uniqueness, and density conservation*, J. Statist. Phys. 61 (1990), 203–234.

[3] —, —, *Asymptotic behaviour of solutions to the Becker–Döring equations for arbitrary initial data*, Proc. Roy. Soc. Edinburgh Sect. A 108 (1988), 109–116.

[4] J. M. Ball, J. Carr and O. Penrose, *The Becker–Döring cluster equations: basic properties and asymptotic behaviour of solutions*, Comm. Math. Phys. 104 (1986), 657–692.

[5] P. Baras, J. C. Hassan et L. Véron, *Compacité de l'opérateur définissant la solution d'une équation d'évolution non homogène*, C. R. Acad. Sci. Paris Sér. I 284 (1977), 799–802.

[6] P. Bénilan and D. Wrzosek, *On an infinite system of reaction-diffusion equations*, Adv. Math. Sci. Appl. 7 (1997), 349–364.

[7] J. F. Collet and F. Poupaud, *Existence of solutions to coagulation-fragmentation systems with diffusion*, Transport. Theory Statist. Phys. 25 (1996), 503–513.

[8] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monographs 23, Amer. Math. Soc., Providence, 1968.

[9] Ph. Laurençot and D. Wrzosek, *Fragmentation-diffusion model. Existence of solutions and asymptotic behaviour*, Proc. Roy. Soc. Edinburgh Sect. A, to appear.
[10] —, —, *The Becker–Döring model with diffusion. II. Long time behaviour*, submitted.
[11] R. H. Martin and M. Pierre, *Nonlinear reaction-diffusion systems*, in: Nonlinear Equations in Applied Science, W. F. Ames and C. Rogers (eds.), Academic Press, Boston, 1992.
[12] O. Penrose and A. Buhagiar, *Kinetics of nucleation in a lattice gas model: microscopic theory and simulation compared*, J. Statist. Phys. 30 (1983), 219–241.
[13] F. Rothe, *Global Solutions of Reaction-Diffusion Systems*, Lecture Notes in Math. 1072, Springer, Berlin, 1984.
[14] M. Slemrod, *Trend to equilibrium in the Becker–Döring cluster equations*, Nonlinearity 2 (1989), 429–443.
[15] J. L. Spouge, *An existence theorem for the discrete coagulation-fragmentation equations*, Math. Proc. Cambridge Philos. Soc. 96 (1984), 351–357.
[16] D. Wrzosek, *Existence of solutions for the discrete coagulation-fragmentation model with diffusion*, Topol. Methods Nonlinear Anal. 9 (1997), 279–296.
[17] A. Ziabicki, *Generalized theory of nucleation kinetics. IV. Nucleation as diffusion in the space of cluster dimensions, positions, orientations and internal structure*, J. Chem. Phys. 85 (1986), 3042–3056.

Institut Elie Cartan–Nancy
Université de Nancy I
BP 239
F-54506 Vandœuvre-lès-Nancy Cedex, France
E-mail: laurenco@iecn.u-nancy.fr

Institute of Applied Mathematics and Mechanics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: darekw@appli.mimuw.edu.pl

*Received 15 May 1997*