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EMBEDDINGS OF KRONECKER MODULES INTO THE CATEGORY OF PRINJECTIVE MODULES AND THE ENDOMORPHISM RING PROBLEM

BY

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1. Introduction. Throughout this paper K is a field, R is a commutative ring with an identity element and $(I, \preceq)$ is a finite poset (i.e. partially ordered set) with respect to the partial order $\preceq$. We shall write $i \prec j$ if $i \preceq j$ and $i \neq j$. For simplicity we write I instead of $(I, \preceq)$.

One of the aims of this paper is to show that the category $\mathrm{prin}(KI)$ of finite-dimensional prinjective right modules over the incidence K-algebra KI of I (defined below) is of infinite representation type if and only if the category $\mathrm{prin}(KI)$ contains a full exact subcategory which is equivalent to the category

$$\mathrm{mod}\begin{pmatrix} K & K^2 \\ 0 & K \end{pmatrix}$$

of Kronecker K-modules (see Theorem 1.7).

This together with Theorem 1.2 of [10] shows that if the category $\mathrm{prin}(KI)$ is of infinite representation type then the category $\mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(KI)$ of propartite modules (defined below) over the incidence K-algebra KI contains large objects with prescribed endomorphism K-algebras and contains large rigid direct systems (see Theorem 1.8).

This is a Corner type result (see [3]) studied by many authors for nice subcategories $\mathcal{A}$ of module categories $\mathrm{Mod}(\Lambda)$ over rings Λ with an identity element (see [1], [2], [5], [6], [8], [9], [17], [19]).

Throughout denote by $\max I$ the set of all maximal elements of I (called *peaks* of I). We suppose that

$$I = \{1, \ldots, n, p_1, \ldots, p_r\}, \quad \max I = \{p_1, \ldots, p_r\}$$

and that the order relation $\preceq$ in I is such that $i \prec j$ implies that $i < j$ in

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the natural order. We can always achieve this by a suitable renumbering of the elements in I.

Usually we view the poset I as a quiver with the commutativity relations induced by the ordering $\prec$, and we denote by RI the path algebra of I with coefficients in the ring R (see Chapter 14 of [22]). It follows from our assumption on the order $\preceq$ that RI has the following upper triangular $I \times I$-matrix form:

$$(1.1) \qquad RI = \begin{pmatrix} R & R_{12} & \dots & R_{1n} & R_{1p_1} & R_{1p_2} & \dots & R_{1p_r} \\ 0 & R & \dots & R_{2n} & R_{2p_1} & R_{2p_2} & \dots & R_{2p_r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & R & R_{np_1} & R_{np_2} & \dots & R_{np_r} \\ 0 & 0 & \dots & 0 & R & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & R & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & R \end{pmatrix}$$

where $R_{ij} = R$ if $i \prec j$ and $R_{ij} = 0$ otherwise. For $i \preceq j$ we denote by $e_{ij} \in RI$ the matrix having 1 at the i-j-th position and zeros elsewhere. Given j in I we denote by $e_j = e_{jj}$ the standard primitive idempotent corresponding to j.

Right RI-modules will be identified with the systems

$$X = (X_i;\ {}_jh_i)_{i,j \in I}$$

where $X_i = Xe_i$ is an R-module and ${}_jh_i : X_i \to X_j$, $i \prec j$, are R-homomorphisms such that ${}_th_j \cdot {}_jh_i = {}_th_i$ for all $t \prec j \prec i$ in I. The module X is finitely generated if X_j is a finitely generated R-module for any $j \in I$.

If K is a field then following [15] (see also [20], [28]) we call the right KI-module X *prinjective* if X is finitely generated and the right KI^--module Xe^- is projective, where

$$(1.2) \qquad I^- = I \setminus \max I$$

and $e^- = \sum_{j \in I^-} e_j$. It is easy to prove that a module X in $\mathrm{mod}(KI)$ is prinjective if and only if there exists a short exact sequence

$$0 \to P_1 \to P_0 \to X \to 0$$

in $\mathrm{mod}(KI)$, where P_0, P_1 are projective KI-modules and P_1 is semisimple of the form $P_1 = \bigoplus_{p \in \max I} (e_pKI)^{t_p}$, $t_p \geq 0$.

We denote by $\mathrm{mod}(KI)$ the category of finitely generated right KI-modules and by $\mathrm{prin}(KI)$ the full subcategory of $\mathrm{mod}(KI)$ consisting of the prinjective modules. It follows from [15] that the category $\mathrm{prin}(KI)$ is additive, has the finite unique decomposition property, is closed under extensions in $\operatorname{mod}(KI)$, has Auslander–Reiten sequences, source maps and sinks maps, and has enough relative projective and relative injective objects. An interpretation of $\operatorname{prin}(KI)$ as a bimodule matrix problem in the sense of Chapter 17 of [22] is given in [23].

Following [23] we define the poset I to be of *finite prinjective type* if the category $\operatorname{prin}(KI)$ is of finite representation type, that is, the number of isomorphism classes of indecomposable modules in $\operatorname{prin}(KI)$ is finite. It follows from [23] that the definition does not depend on the choice of K.

We recall that a right module X over a bipartite generalized triangular matrix ring

$$S = \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix}$$

can be identified with the system

$$X = (X'_A, X''_B, \varphi)$$

where X'_A is a right A-module, X''_B is a right B-module and $\varphi : X' \otimes {}_AM_B \to X''_B$ is a B-homomorphism (see [22]). We recall from [26] that the S-module X is said to be *propartite* if X'_A is a projective R-module and X''_B is a projective B-module. We denote by $\operatorname{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(S)^A_B$ the category of propartite right S-modules, and by $\operatorname{mod}^{\mathrm{pr}}_{\mathrm{pr}}(S)^A_B$ the full subcategory of $\operatorname{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(S)^A_B$ consisting of the finitely generated modules.

For any ring R with an identity element the generalized matrix R-algebra

$$\Gamma_2(R) = \begin{pmatrix} R & R^2 \\ 0 & R \end{pmatrix} \tag{1.3}$$

is called the *Kronecker R-algebra*, where the multiplication is defined by the formula

$$\begin{pmatrix} D & U \\ 0 & C \end{pmatrix} \begin{pmatrix} F & V \\ 0 & E \end{pmatrix} = \begin{pmatrix} DF & DV + UE \\ 0 & CE \end{pmatrix}.$$

The right $\Gamma_2(R)$-modules are called *Kronecker R-modules*. Following the remarks above, the category $\operatorname{Mod}(\Gamma_2(R))$ of Kronecker R-modules X can be identified with the category of R-representations of the Kronecker quiver (see [17] and [22])

$$\bullet \overset{\varphi'}{\underset{\varphi''}{\rightrightarrows}} \bullet$$

that is, the systems

$$X = (X', X'', \varphi', \varphi'') \tag{1.4}$$

where X' and X'' are R-modules and $\varphi', \varphi'' : X' \to X''$ are R-homomorphisms. A morphism from $X = (X', X'', \varphi', \varphi'')$ to $X_1 = (X'_1, X''_1, \varphi'_1, \varphi''_1)$ is a pair (f', f'') of R-module homomorphisms $f' : X' \to X'_1$, $f'' : X'' \to X''_1$ such that $\varphi'_1 f' = f''\varphi'$ and $\varphi''_1 f' = f''\varphi''$.

The category $\mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(\varGamma_2(R))$ will be called the category of *R-projective Kronecker modules*. It is easy to see that $\mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(\varGamma_2(R))$ can be identified with the category of R-projective representations of the Kronecker quiver, that is, the R-representations $P = (P', P'', \varphi', \varphi'')$, where P' and P'' are projective R-modules.

Throughout we shall view RI as a bipartite R-algebra as follows:

$$RI = \begin{pmatrix} RI^- & M \\ 0 & B \end{pmatrix} \tag{1.5}$$

where $B = R \times \ldots \times R$ ($|\max I|$ times), the free R-module

$$M = \bigoplus_{p\in\max I} \bigoplus_{\substack{i\prec p\\ i\in I^-}} e_{ip}R$$

is viewed as an RI^--B-bimodule in the obvious way and multiplication is matrix multiplication.

It is easy to see that the right RI-module X identified with the system $X = (X_i; {}_jh_i)_{i,j\in I}$ is propartite if and only if X_p is a projective R-module for any $p \in \max I$ and the restriction $X^- = \bigoplus_{j\in I^-} X_j$ of X to I^- is projective when viewed as a right RI^--module. It follows that if $R = K$ is a field then $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(KI) = \mathrm{prin}(KI)$.

It was shown in [12] that there exists a full and faithful exact embedding functor

$$T : \mathrm{mod} \begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix} \to \mathrm{prin}(KI)$$

if and only if there exists a nonzero vector $v \in \mathbb{N}^I$ such that $q_I(v) < 0$, where $q_I : \mathbb{Z}^I \to \mathbb{Z}$ is the Tits quadratic form

$$q_I(x) = \sum_{i\in I} x_i^2 + \sum_{\substack{i\prec j\\ j\in I^-}} x_i x_j - \sum_{p\in\max I} \Big(\sum_{i\prec p} x_i\Big) x_p \tag{1.6}$$

of the poset I. Following an idea of [12] we prove the following theorem which is one of the main theorems of this paper.

THEOREM 1.7. *Let I be a finite poset. The following statements are equivalent.*

(a) *The poset I is of infinite prinjective type.*

(b) *The Tits quadratic form $q_I : \mathbb{Z}^I \to \mathbb{Z}$ is not weakly positive*, *that is*, *there exists a nonzero vector $v \in \mathbb{N}^I$ with $q_I(v) \le 0$, where* $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

(c) *The poset I contains as a peak subposet one of the critical posets $\mathcal{P}_1, \ldots, \mathcal{P}_{110}$ listed in Section* 5 *of* [23].

(d) *For any commutative field K there exists a full and faithful exact functor T* : $\mathrm{mod}(\varGamma_2(K)) \to \mathrm{prin}(KI)$, *where $\varGamma_2(K)$ is the Kronecker K-algebra* (1.3).

(e) *For any commutative ring A which is an algebra over a field K there exist full*, *faithful and exact functors*

$$T:\mathrm{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(\mathit{\Gamma}_2(A))\to\mathrm{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(AI)\quad\text{and}\quad T':\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(\mathit{\Gamma}_2(A))\to\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(AI)$$

where $\mathit{\Gamma}_2(A)$ is the Kronecker A-algebra (1.3).

We recall from [23] that a subposet L of I is a *peak subposet* if $L\cap(\max I)=\max L$.

By applying Theorem 1.7 and the results of [8] and [9] we prove our second main theorem.

THEOREM 1.8. *Let I be a finite poset of infinite prinjective type*, *K a field and KI the K-incidence algebra of I.*

(a) *For any K-algebra A generated by at most λ elements*, *where λ is an infinite cardinal number*, *there exists a direct system*

$$\mathbb{F}=\{\mathbb{F}_\beta,u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$$

of K-linear additive functors $\mathbb{F}_\beta:\mathrm{Mod}(A)\to\mathrm{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(KI)$ connected by injective functorial morphisms $u_{\beta\gamma}:\mathbb{F}_\beta\to\mathbb{F}_\gamma$ satisfying the following conditions:

(i) *If M is a module in* $\mathrm{Mod}(A)$ *which is A-projective*, *then the KI-module $\mathbb{F}_\beta(M)$ is A-projective for all $\beta\subseteq\lambda$.*

(ii) *Suppose that M and N are modules in* $\mathrm{Mod}(A)$. *Then*

$$\mathrm{Hom}_{KI}(\mathbb{F}_\beta(M),\mathbb{F}_\gamma(N))=0\quad\text{if }\beta\not\subseteq\gamma,$$

and the natural R-homomorphism

$$\mathrm{Hom}_A(M,N)\to\mathrm{Hom}_{KI}(\mathbb{F}_\beta(M),\mathbb{F}_\gamma(N)),$$

$f\mapsto u_{\beta\gamma}(N)\circ\mathbb{F}_\beta(f)$, *is an isomorphism for all $\beta\subseteq\gamma\subseteq\lambda$.*

(b) *Any K-algebra A is isomorphic to an algebra of the form* $\mathrm{End}\,X$, *where X is a module in* $\mathrm{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(KI)$.

In view of Theorem 3.1 of [23] the proof of Theorem 1.7 reduces to the proof of the implications (a)⇒(d)⇒(e). The first implication will follow from Proposition 3.1 and Theorem 4.4, because Proposition 2.2 and Lemma 2.8 reduce the problem to the case where I is any of the peak-irreducible posets listed in Tables 4.7 of Section 4. For any such poset I we construct in Theorem 4.4 a pair of modules $S(a)$ and V in $\mathrm{prin}(KI)$ satisfying the conditions (i)–(iii) of Proposition 3.1, and inducing an embedding functor T required in Theorem 1.7(d). The implication (d)⇒(e) is proved in Theorem 5.8 of Section 5. The proofs of Theorems 1.7 and 1.8 are presented in Section 6.

The statement (b) of Theorem 1.8 follows from (a). In view of Theorem 1.7, the statement (a) of Theorem 1.8 reduces to an analogous statement proved in [10] with $\mathrm{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(RI)$ and $\mathrm{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(\mathit{\Gamma}_2(R))$ interchanged.

A result analogous to Theorem 1.8 for loop finite K-algebras is proved in [10, Section 4] by applying the characterization of minimal representation-infinite loop-finite artin algebras given by Skowroński in [27].

2. Preliminaries and notation. We recall from [15] and [23] that the *coordinate vector* $\mathbf{cdn}\, X \in \mathbb{N}^I$ of a module X in $\mathrm{prin}(KI)$ is defined by the formula

$$(2.1)\qquad (\mathbf{cdn}\, X)(j) = \begin{cases} \dim_K(X_j) & \text{for } j \in \max I,\\ \dim_K(\mathrm{top}\, X)e_j & \text{for } j \in I \setminus \max I.\end{cases}$$

We consider $\mathbf{cdn}\, X$ as a map $\mathbf{cdn}\, X : I \to \mathbb{N}$. Note that if X is an indecomposable module in $\mathrm{prin}(KJ)$ which is not isomorphic to any of the simple projective modules $e_{p_1}KI, \ldots, e_{p_r}KI$ then the projective cover $P(X)$ of X has the form

$$P(X) = \bigoplus_{j \in I^-} (e_j KI)^{(\mathbf{cdn} X)(j)}.$$

Throughout we shall frequently use the following connection between prinjective modules over KI and over KJ for a peak subposet J of I.

Proposition 2.2. *Let J be a peak subposet of I and consider the idempotents $e_J = \sum_{j\in J} e_j$ and $e_{J\cup I^-} = \sum_{j \in J\cup I^-} e_j$ of KI. Then there exists an algebra isomorphism $e_J(KI)e_J \cong KJ$, and the subposet induction functor*

$$(2.3)\qquad \widehat{T}_J = (-) \otimes_{KJ} e_J(KI)e_{J\cup I^-} : \mathrm{prin}(KJ) \to \mathrm{prin}(KI)$$

is full, faithful and carries exact sequences to exact ones. Moreover, given X in $\mathrm{prin}(KJ)$ we have

$$(2.4)\qquad (\mathbf{cdn}\, \widehat{T}_J X)(j) = \begin{cases} (\mathbf{cdn}\, X)(j) & \text{for } j \in I,\\ 0 & \text{for } j \notin I.\end{cases}$$

For the proof we refer to Proposition 2.4 of [12]. ∎

We extend Proposition 2.2 in Proposition 5.2 to propartite modules over RI, where R is a commutative ring.

Most of the problems studied in this paper for prinjective modules over KI reduce to the case when I is a peak-irreducible poset. For this purpose we recall from [11] and [12] that if I is a poset with $|\max I| \geq 2$ then $q \in \max I$ is said to be a *reducible peak* if there exists an element $c \in I$ such that

(i) The set $q^\nabla \cap (\bigcup_{q\neq p \in \max I} p^\nabla)$ consists of one element c.
(ii) There is no element $t \in I$ such that $c \prec t \prec q$.
(iii) The subposet $I_q := q^\nabla \setminus \{c\}$ of I is linearly ordered.

Here given $j \in I$ we set $j^\nabla = \{i \in I : i \preceq j\}$.

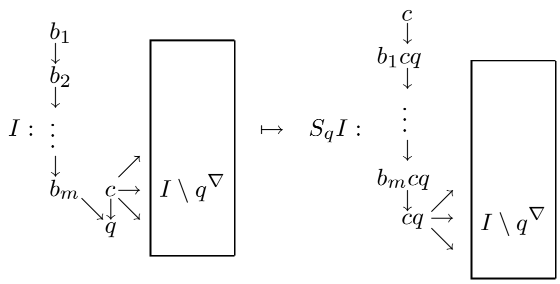
The poset I is said to be *peak-reducible* if there exists a reducible peak in I; otherwise I is said to be *peak-irreducible*. If $q \in I$ is a reducible peak, we associate with I a *q-reflection poset* S_qI defined as follows. We assume that $I_q = \{b_1 \prec \ldots \prec b_m\}$ and we set

$$S_qI = \{c, b_1cq, \ldots, b_mcq, cq\} \cup (I \setminus q^\nabla) \tag{2.5}$$

where $b_1cq, \ldots, b_mcq, cq$ are new points. The partial order in S_qI is generated by the partial order $\preceq$ in $I \setminus q^\nabla$ and the following relations:

(i) $c \prec b_1cq \prec \ldots \prec b_mcq \prec cq$;
(ii) $cq \prec i$ if $c \prec i$ in I and $i \in I \setminus q^\nabla$.

The construction $I \mapsto S_qI$ can be visualized by the following picture:



The poset I is said to be *peak-reducible to a poset* L if there exists a sequence of posets $J_0, J_1, \ldots, J_s$ such that $J_0 \cong I$, $J_s \cong L$ and for any $i = 1, \ldots, s$ there exists a reducible peak q_{i-1} in J_{i-1} such that $S_{q_{i-1}}J_{i-1} \cong I_i$.

Let I be a peak-reducible poset with a reducible peak q and let S_qI be the q-reflection form (2.5) of I. Following [21], [13] and [14] we define the functor

$$\widetilde{\mathbf{G}}_q : \mathrm{prin}(KS_qI) \to \mathrm{prin}(KI) \tag{2.6}$$

as follows. Given a prinjective module X in $\mathrm{prin}(KS_qI)$ we view it as a system $X = (X_i, {}_jh_l)_{i,j,l \in S_qI, i \prec j}$, where $X_i = Xe_i$ and ${}_jh_l : X_l \to X_j$ is the K-linear map defined by multiplication by $e_{lj} \in KS_qI$. Here $e_{lj} \in KS_qI$ is the matrix having 1 at the l-j-th entry and zeros elsewhere. We set $\widetilde{\mathbf{G}}_q(X) = (\overline{X}_i, {}_j\overline{h}_l)_{i,j,l \in I, l \prec j}$, where

$$\overline{X}_i = \begin{cases} X_i & \text{if } i \in I \setminus q^\nabla, \\ X_{cq} & \text{if } i = c, \\ X_{cq}/\mathrm{Im}\,{}_{cq}h_c & \text{if } i = q, \\ X_{bcq}/\mathrm{Im}\,{}_{cq}h_c & \text{if } i \in I_q = q^\nabla \setminus \{c\}. \end{cases} \tag{2.7}$$

Here we use the notation from (2.5). We set ${}_j\overline{h}_l = {}_jh_l$, ${}_j\overline{h}_c = {}_jh_{cq}$ for $j, l \in I \setminus q^\nabla$, ${}_q\overline{h}_c : \overline{X}_c \to \overline{X}_q$ is the natural projection, and we take for ${}_{b'}\overline{h}_b$

and ${}_q\overline{h}_b$ the K-linear maps induced by ${}_{b'cq}h_{bcq}$ and ${}_{cq}h_{bcq}$, respectively, if $b, b' \in S_q I$. The functor $\widetilde{\mathbf{G}}_q$ is defined on morphisms in a natural way. It is easy to check that $\widetilde{\mathbf{G}}_q$ carries prinjective modules to prinjective ones.

The following lemma follows from Theorem 2.15 of [14].

Lemma 2.8. *If I is a peak-reducible poset with a reducible peak q then the functor $\widetilde{\mathbf{G}}_q : \mathrm{prin}(KS_qI) \to \mathrm{prin}(KI)$ is full, faithful and exact.*

We shall extend Lemma 2.8 in Lemma 5.4 to propartite modules over RI, where R is a commutative ring.

3. The main embedding functor. Our construction of embeddings of Kronecker modules into categories of prinjective modules and of propartite modules announced in Theorem 1.7 will essentially depend on the following result.

Proposition 3.1. *Let R be a commutative ring and let Λ be an R-algebra which is a finitely generated R-module. Assume that there exists a pair (U, V) of modules U and V in $\mathrm{mod}(\Lambda)$ satisfying the following three conditions*:

(i) $\mathrm{End}_\Lambda(U) \cong \mathrm{End}_\Lambda(V) \cong R$,
(ii) $\mathrm{Hom}_\Lambda(U, V) = \mathrm{Hom}_\Lambda(V, U) = 0$,
(iii) *The* $\mathrm{End}_\Lambda(U)$-$\mathrm{End}_\Lambda(V)$*-bimodule* $\mathrm{Ext}^1_\Lambda(V, U)$ *contains a subbimodule N, which is a free R-module of rank two when viewed as an R-module.*

Then there exists a full faithful exact functor

$$T_{V,U} : \mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}} \begin{pmatrix} R & R^2 \\ 0 & R \end{pmatrix} \to \mathrm{Mod}(\Lambda)$$

satisfying the following conditions.

(a) *The image category* $\mathrm{Im}\, T_{V,U}$ *of the functor* $T_{V,U}$ *is equivalent to the full subcategory* $\widehat{\mathcal{EXT}}_N(V, U)$ *of* $\mathrm{Mod}(\Lambda)$ *formed by all Λ-modules Z which are middle terms of exact sequences*

$$\mathbf{e}_Z : \quad 0 \to Z_2 \to Z \to Z_1 \to 0 \tag{3.2}$$

in $\mathrm{Mod}(\Lambda)$, *where Z_1 is a direct summand of a direct sum $\bigoplus_{j\in\Omega} V_j$, $V_j \cong V$, Z_2 is a direct summand of a direct sum $\bigoplus_{i\in\Sigma} U_i$, $U_i \cong U$, and the equivalence class $[\mathbf{e}_Z]$ of $\mathbf{e}_Z$ in the extension group $\mathrm{Ext}^1_\Lambda(Z_1, Z_2)$ belongs to the subgroup*

$$\widehat{\mathbf{N}}(Z_1, Z_2) = \prod_{j\in\Omega} \bigoplus_{i\in\Sigma} N(V_j, U_i)$$

of $\mathrm{Ext}^1_\Lambda(Z_1, Z_2)$.

(b) *For any module X in* $\operatorname{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} R & R^2 \\ 0 & R \end{pmatrix}$ *there exists a short exact sequence*

$$0 \to U_0 \to T_{V,U}(X) \to V_0 \to 0 \tag{3.3}$$

in $\operatorname{mod}(\Lambda)$, *where* U_0 *and* V_0 *are direct summands of* U^r *and* V^t *with* $r, t \geq 0$.

Proof. If R is a field, the proposition generalizes Lemmata 1.5 and 8.6 of [16] (see also Gabriel [7]).

Suppose that R is an arbitrary commutative ring. We shall apply the bimodule matrix problem technique developed by the second author in [26], and the categories of extensions developed in [24] and [25]. We follow the proof of Theorem 3.12 of [25].

For this purpose we consider two full additive subcategories $\widehat{\mathbb{K}} = \operatorname{Add}(V)$ and $\widehat{\mathbb{L}} = \operatorname{Add}(U)$ of $\operatorname{Mod}(\Lambda)$ consisting of the modules isomorphic to direct summands of direct sums of copies of the module V and U, respectively. Denote by $\mathbb{K} = \operatorname{add}(V)$ and $\mathbb{L} = \operatorname{add}(U)$ the full additive subcategories of $\widehat{\mathbb{K}}$ and $\widehat{\mathbb{L}}$ consisting of the finitely generated modules. We associate with the $\operatorname{End}(U)$-$\operatorname{End}(V)$-subbimodule N of $\operatorname{Ext}^1_\Lambda(V, U)$ two R-linear bifunctors

$$\widehat{\mathbf{N}}(-,-) : \widehat{\mathbb{K}}^{\mathrm{op}} \times \widehat{\mathbb{L}} \to \operatorname{Mod}(R), \qquad \mathbf{N}(-,-) : \mathbb{K}^{\mathrm{op}} \times \mathbb{L} \to \operatorname{mod}(R)$$

defined by the formulae

$$\widehat{\mathbf{N}}\Big(\bigoplus_{j\in\Omega} V_j, \bigoplus_{i\in\Sigma} U_i\Big) = \prod_{j\in\Omega}\bigoplus_{i\in\Sigma} N(V_j, U_i),$$
$$\mathbf{N}\Big(\bigoplus_{j=1}^{s} V_j, \bigoplus_{i=1}^{r} U_i\Big) = \bigoplus_{j=1}^{s}\bigoplus_{i=1}^{r} N(V_j, U_i).$$

We shall prove the proposition by constructing a commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{EXT}}_N(V,U) \arrow[r, "\widehat{\mathbb{S}}'", "\simeq"'] & \mathbf{Mat}({}_{\widehat{\mathbb{L}}}\widehat{\mathbf{N}}_{\widehat{\mathbb{K}}}) \arrow[r, "\widehat{\mathbb{S}}''", "\simeq"'] & \operatorname{Mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} R & R^2 \\ 0 & R \end{pmatrix} \\
\mathcal{EXT}_N(V,U) \arrow[r, "\mathbb{S}'", "\simeq"'] \arrow[u, hook] & \mathbf{Mat}({}_{\mathbb{L}}\mathbf{N}_{\mathbb{K}}) \arrow[r, "\mathbb{S}''", "\simeq"'] \arrow[u, hook] & \operatorname{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} R & R^2 \\ 0 & R \end{pmatrix} \arrow[u, hook]
\end{tikzcd}$$

where $\mathcal{EXT}_N(V,U)$ is the full subcategory of $\widehat{\mathcal{EXT}}_N(V,U)$ formed by the finitely generated modules and the vertical arrows denote the natural embedding functors.

Here $\mathbf{Mat}({}_{\mathbb{L}}\mathbf{N}_{\mathbb{K}})$ is the category of matrices in the sense of Drozd [4], that is, the objects of $\mathbf{Mat}({}_{\mathbb{L}}\mathbf{N}_{\mathbb{K}})$ are triples (x, y, m), where x is an object of $\mathbb{K}$, y is an object of $\mathbb{L}$ and $m \in \mathbf{N}(x, y)$. A morphism from (x, y, m) to (x', y', m') in $\mathbf{Mat}({}_{\mathbb{L}}\mathbf{N}_{\mathbb{K}})$ is a pair (φ, ψ), where $\varphi \in \mathbb{K}(x, x')$ and $\psi \in \mathbb{L}(y, y')$ are such that $\mathbf{N}(x, \psi)m = \mathbf{N}(\varphi, y')m'$ (see also [22, Chapter 17, p. 466]).

It is easy to check that $\mathbf{Mat}({}_{\mathbb{L}}\mathbf{N}_{\mathbb{K}})$ is an additive category. The direct sum of two objects (x, y, m) and (x', y', m') of $\mathbf{Mat}({}_{\mathbb{L}}\mathbf{N}_{\mathbb{K}})$ is the object $(x \oplus x', y \oplus y', m \oplus m')$, where

$$m \oplus m' = \begin{pmatrix} m & 0 \\ 0 & m' \end{pmatrix} \in \begin{pmatrix} \mathbf{N}(x, y) & \mathbf{N}(x, y') \\ \mathbf{N}(x', y) & \mathbf{N}(x', y') \end{pmatrix} = \mathbf{N}(x \oplus x', y \oplus y').$$

under the obvious identifications. The category $\mathbf{Mat}({}_{\widehat{\mathbb{L}}}\widehat{\mathbf{N}}_{\widehat{\mathbb{K}}})$ is defined in a similar way.

In order to define the functors $\widehat{\mathbb{S}}'$ and $\mathbb{S}'$ we observe that the module Z in the exact sequence (3.2) determines the modules Z_1 and Z_2 uniquely up to isomorphism, because of the assumption $\mathrm{Hom}_{\Lambda}(U, V) = 0$. Given such a module Z we set

$$\widehat{\mathbb{S}}'(Z) = (Z_1, Z_2, m)$$

where $m = [\mathbf{e}_Z]$ is the element of $\widehat{\mathbf{N}}(Z_1, Z_2) \subseteq \mathrm{Ext}^1_{\Lambda}(Z_1, Z_2)$ determined by the short exact sequence $\mathbf{e}_Z$.

If $f : Z \to W$ is a Λ-homomorphism and the modules Z and W are in $\widehat{\mathcal{EXT}}_N(V, U)$, then in view of the equality $\mathrm{Hom}_{\Lambda}(Z_2, W_1) = 0$ there exists a commutative diagram

$$\begin{array}{lccccccccc} \mathbf{e}_Z : & 0 & \to & Z_2 & \to & Z & \to & Z_1 & \to & 0 \\ & & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \\ \mathbf{e}_W : & 0 & \to & W_2 & \to & W & \to & W_1 & \to & 0 \end{array}$$

and the Λ-homomorphisms f_1 and f_2 are uniquely determined by f. It is easy to check by a standard pull-back and push-out arguments that $(f_1, f_2) : (Z_1, Z_2, [\mathbf{e}_Z]) \to (W_1, W_2, [\mathbf{e}_W])$ is a morphism in the category of matrices (see [25, Section 3]). We set $\widehat{\mathbb{S}}'(f) = (f_1, f_2)$. The functor $\mathbb{S}'$ is defined in a similar way.

It follows from the definition and our assumptions that the functors $\widehat{\mathbb{S}}'$ and $\mathbb{S}'$ are equivalences of categories.

The functors $\mathbb{S}''$ and $\widehat{\mathbb{S}''}$ will be constructed by applying [26, Theorem 2.8] to our situation. We note that in view of the R-algebra isomorphisms $\mathrm{End}(V) \cong R$ and $\mathrm{End}(U) \cong R$ the Yoneda correspondences $\boldsymbol{\omega}(-) = \mathbb{K}(V, -)$ and $\boldsymbol{\omega}'(-) = \mathbb{L}(U, -)$ induce the equivalences of categories

$$\mathbb{K} \xrightarrow[\simeq]{\omega} \mathrm{pr}(\mathrm{End}(V)) \cong \mathrm{pr}(R) \quad \text{and} \quad \mathbb{L} \xrightarrow[\simeq]{\omega'} \mathrm{pr}(\mathrm{End}(U)) \cong \mathrm{pr}(R)$$

where $\mathrm{pr}(A)$ is the category of finitely generated projective modules over a ring A. Since by our assumption the $\mathrm{End}(U)$-$\mathrm{End}(V)$-bimodule $N(V, U)$ is isomorphic to R^2, Theorem 2.8 and the formula (2.9) of [26] apply to yield a functorial isomorphism

$$\mu_{x,y} : \mathbf{N}(x, y) \xrightarrow[\simeq]{} \mathrm{Hom}_R(\boldsymbol{\omega}(x) \otimes_R (R^2)_R, \boldsymbol{\omega}'(y))$$

for all objects x in $\mathbb{K}$ and y in $\mathbb{L}$. We define the functor $\mathbb{S}''$ by the formula

$$\mathbb{S}''(x, y, m) = (\boldsymbol{\omega}(x), \boldsymbol{\omega}'(y), \mu_{x,y}(m));$$

on morphisms it is defined in a natural way. It is easy to check that $\mathbb{S}''$ is an equivalence of categories.

The equivalence $\mathbb{S}''$ extends to an equivalence $\widehat{\mathbb{S}''}$ in a natural way. For this purpose we note that given $Z_1 = \bigoplus_{j\in\Omega} V_j$ in $\widehat{\mathbb{K}}$ with $V_j \cong V$, and $Z_2 = \bigoplus_{i\in\Sigma} U_i$ in $\widehat{\mathbb{L}}$ with $U_i \cong U$, we get the composed functorial isomorphism $\widehat{\mu}$

$$\begin{aligned}\widehat{\mathbf{N}}(Z_1, Z_2) &\underset{\simeq}{\longrightarrow} \prod_{j\in\Omega}\bigoplus_{i\in\Sigma}\mathbf{N}(V_j, U_i)\\ &\underset{\simeq}{\xrightarrow{\mu}} \prod_{j\in\Omega}\bigoplus_{i\in\Sigma}\mathrm{Hom}_R(\boldsymbol{\omega}(V_j)\otimes_R (R^2)_R, \boldsymbol{\omega}'(V_i))\\ &\underset{\simeq}{\longrightarrow} \mathrm{Hom}_R(\widehat{\boldsymbol{\omega}}(Z_1)\otimes_R (R^2)_R, \widehat{\boldsymbol{\omega}'}(Z_2))\end{aligned}$$

where we set $\widehat{\boldsymbol{\omega}}(Z_1) = \bigoplus_{j\in\Omega}\boldsymbol{\omega}(V_j)$ and $\widehat{\boldsymbol{\omega}'}(Z_2) = \bigoplus_{i\in\Sigma}\boldsymbol{\omega}'(V_i)$. Hence we derive equivalences of categories

$$\widehat{\boldsymbol{\omega}} : \widehat{\mathbb{K}} \xrightarrow{\simeq} \mathrm{Pr}(R) \quad \text{and} \quad \widehat{\boldsymbol{\omega}'} : \widehat{\mathbb{L}} \xrightarrow{\simeq} \mathrm{Pr}(R)$$

and a functorial isomorphism

$$\widehat{\mu}_{\widehat{x},\widehat{y}} : \widehat{\mathbf{N}}(\widehat{x}, \widehat{y}) \underset{\simeq}{\longrightarrow} \mathrm{Hom}_R(\widehat{\boldsymbol{\omega}}(\widehat{x})\otimes_R (R^2)_R, \widehat{\boldsymbol{\omega}'}(\widehat{y}))$$

for all objects $\widehat{x}$ in $\widehat{\mathbb{K}}$ and $\widehat{y}$ in $\widehat{\mathbb{L}}$, where $\mathrm{Pr}(R)$ is the category of projective R-modules. We define the functor $\widehat{\mathbb{S}''}$ by the formula

$$\widehat{\mathbb{S}''}(\widehat{x}, \widehat{y}, m) = (\widehat{\boldsymbol{\omega}}(\widehat{x}), \widehat{\boldsymbol{\omega}'}(\widehat{y}), \widehat{\mu}_{\widehat{x},\widehat{y}}(m));$$

on morphisms it is defined in a natural way. It is easy to check that $\widehat{\mathbb{S}''}$ is an equivalence of categories. The details are left to the reader (see [22, Theorem 17.89] and [26, Theorem 2.8]).

We take for the functor

$$T_{V,U} : \mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} R & R^2 \\ 0 & R\end{pmatrix} \to \mathrm{Mod}(\Lambda)$$

the inverse of the functor $\widehat{\mathbb{S}''}\circ\widehat{\mathbb{S}}'$. It is easy to check that $T_{V,U}$ has the required properties. This finishes the proof of the proposition. ■

4. Embeddings of Kronecker modules over a field. Throughout we fix a maximal element $a \in I^- = I \setminus \max I$ and we set

$$I'_a = I \setminus \{a\}.$$

We view I'_a as a peak subposet of I.

Let $S(a) = e_aKI^-/\mathrm{rad}(e_aKI^-)$ be the simple KI-module corresponding to the point a. We shall denote by

(4.1) $$\widehat{T}_a : \mathrm{prin}(KI'_a) \to \mathrm{prin}(KI)$$

the full, faithful and exact functor $\widehat{T}_{I'_a}$ induced by the subposet $J = I'_a$ of I according to Proposition 2.2.

The proof of the implication (a)⇒(d) of Theorem 1.7 essentially depends on the following two results, which are of type similar to the main lemmata of [12].

LEMMA 4.2. *Assume that* $a \in I^- = I \setminus \max I$ *is a maximal element and let* $V = (V_j, {}_jh_i : V_i \to V_j)_{i \preceq j}$ *be an indecomposable module in* $\mathrm{prin}(KI)$.

(a) *Suppose that there exists* $p \in \max I$ *such that* $p \succ a$ *and* $V_p \neq 0$, *and the element* a *is such that the subposet* $a^\nabla = \{j \in I : j \preceq a\}$ *of* $I^- = I \setminus \max I$ *is linearly ordered and each of the elements of* a^∇ *is incomparable with all elements of* $I^- \setminus a^\nabla$. *Then the induced map*

$$h_a = ({}_ph_a)_{p \succ a} : V_a \to \bigoplus_{a \prec p \in \max I} V_p$$

is injective and $\mathrm{Hom}_{KI}(S(a), V) = 0$.

(b) *If* V *is such that the map* $h_a = ({}_ph_a)_{p \succ a}$ *is injective and* $(\mathbf{cdn}\, V)(a) = 0$ *then* $\mathrm{Hom}_{KI}(S(a), V) = 0$ *and* $\dim_K \mathrm{Ext}^1_{KI}(S(a), V) = \ell_a(\mathbf{cdn}\, V)$, *where we have set*

$$\ell_a(v) = \sum_{a \prec p \in \max I} v(p) - \sum_{i \prec a} v(i) \tag{4.3}$$

for any $v \in \mathbb{N}^I$ (see (3.7) in [12]).

Proof. By our assumption the subposet $C := a^\nabla = \{j \in I : j \preceq a\}$ of I'_a is linearly ordered. Assume that $C = \{a_1 \to \ldots \to a_s = a\}$. Since V is prinjective, by our assumption the restriction

$$V_C = (V_{a_1} \to \ldots \to V_{a_{s-1}} \to V_a)$$

of V to C is a projective KC-module and without loss of generality we can suppose that the arrows in V_C are represented by K-space injections $V_{a_{t-1}} \subseteq V_{a_t}$ for $t = 2, \ldots, s$.

Now suppose to the contrary that the map $h_a = ({}_ph_a)_{p \succ a} : V_a \to \bigoplus_{a \prec p \in \max I} V_p$ is not injective, and fix a nonzero element y in

$$\mathrm{Ker}\, h_a = \bigcap_{a \prec p \in \max I} \mathrm{Ker}_p\, h_a.$$

Define the KC-projective submodule $U = (U_{a_1} \subseteq \ldots \subseteq U_{a_{s-1}} \subseteq U_a)$ of V_C by setting $U_a = Ky$ and $U_{a_j} = Ky \cap V_{a_j}$ for $j = 1, \ldots, s-1$. Note that the induced maps $V_{a_j}/U_{a_j} \to V_{a_{j+1}}/U_{a_{j+1}}$ are injective and therefore the factor KC-module V_C/U is projective (see Corollary 5.7 of [22]). Consequently, the projective KC-module U is a KC-module direct summand of V_C. By the choice of y and our assumption on C the module U is a projective KI^--module and it is a KI-module direct summand of V. Since V is indecomposable, it follows that $V = U$ and $V_p = U_p = 0$ for any $p \in \max I$, $p \succ a$. This contradicts our assumption.

In order to prove the second statement of (a) and the statement (b) we note that there exists an exact sequence

$$0 \to \bigoplus_{a \prec p \in \max I} e_p KI \xrightarrow{(u_p)} e_a KI \xrightarrow{\varepsilon} S(a) \to 0$$

as a is maximal in I^-. There is a natural isomorphism $\mathrm{Hom}_{KI}(e_j KI, V) \cong Ve_j = V_j$ for any $j \in I$, and so the sequence above induces the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{KI}(e_a KI, V) & \xrightarrow{(u_p^*)} & \mathrm{Hom}_{KI}(\bigoplus_{a \prec p \in \max I} e_p KI, V) & \to \mathrm{Ext}^1_{KI}(S(a), V) \to 0 \\ \downarrow \cong & & \downarrow \cong & \downarrow \mathrm{id} \\ V_a & \xrightarrow{h_a} & \bigoplus_{a \prec p \in \max I} V_p & \to \mathrm{Ext}^1_{KI}(S(a), V) \to 0 \end{array}$$

with exact rows and $\mathrm{Ker}(u_p^*) \cong \mathrm{Im}\,\mathrm{Hom}_{KI}(\varepsilon, \mathrm{id}_V) \cong \mathrm{Hom}_{KI}(S(a), V)$. We conclude that $\mathrm{Hom}_{KI}(S(a), V) = 0$ if and only if the map h_a is injective. In particular, this finishes the proof of (a).

Further, the exactness of the lower row of the diagram and the injectivity of h_a yield

$$\dim_K \mathrm{Ext}^1_{KI}(S(a), V) = \sum_{a \prec p \in \max I} v(p) - \dim_K V_a$$

and the required equality will hold if we prove that $\dim_K V_a = \sum_{i \prec a} v(i)$. In order to prove this we set $v = \mathbf{cdn}\, V$ and we note that the restriction $V^- = Ve^-$ of V to I^- is a projective module. By applying the formula (2.1) to $X = V$ we get $V^- \cong \bigoplus_{a \neq j \in I^-} (e_j KI^-)^{v(j)}$. Since $V_a = Ve_a = V^- e_a$, $e_j(KI^-)e_a = 0$ if $j \not\preceq a$, and $e_j(KI^-)e_a = K$ if $j \preceq a$, we get

$$\begin{aligned} \dim_K V_a &= \dim_K V^- e_a = \dim_K \Big[\bigoplus_{a \neq j \in I^-} (e_j KI^- e_a)^{v(j)} \Big] \\ &= \sum_{j \prec a} v(j) \dim_K (e_j KI^- e_a) = \sum_{j \prec a} v(j). \end{aligned}$$

This finishes the proof of (b) and the lemma follows. ■

Theorem 4.4. *Let I be a peak-irreducible minimal poset of infinite prinjective type among the posets $\mathcal{P}_1, \ldots, \mathcal{P}_{110}$ listed in Section* 5 *of* [23]. *Then*:

(a) *I is one of the* 28 *posets listed in Tables* 4.7 *at the end of this section.*

(b) *There exist a maximal element $a \in I^- = I \setminus \max I$ and an indecomposable module V' in* $\mathrm{prin}(KI'_a)$, $I'_a = I \setminus \{a\}$, *satisfying the following conditions*:

(i) *The subposet* $a^{\nabla} = \{j \in I : j \preceq a\}$ *of* I^- *is linearly ordered and each element of* a^{∇} *is incomparable with any elements of* $I^- \setminus a^{\nabla}$.

(ii) *The difference*

$$\widehat{\ell}_a(v') = \sum_{a \prec p \in \max I} v'(p) - \sum_{i \prec a} v'(i) \tag{4.5}$$

is equal to two, *where* $v' = \mathbf{cdn}\, V'$ (see (3.10) in [12]).

(c) *If* a *and* V' *are as in* (b) *and we set* $V = \widehat{T}_a(V')$, *where* $\widehat{T}_a : \operatorname{prin}(KI'_a) \to \operatorname{prin}(KI)$ *is the functor* (2.3) *with* $J = I'_a$, *then*

(i) $\operatorname{End}_{KI}(S(a)) \cong \operatorname{End}_{KI}(V) \cong K$,

(ii) $\operatorname{Hom}_{KI}(S(a), V) = \operatorname{Hom}_{KI}(V, S(a)) = 0$,

(iii) $\dim_K \operatorname{Ext}^1_{KI}(S(a), V) = 2$.

P r o o f. (a) It is easy to see that a minimal poset $\mathcal{P}_j \in \{\mathcal{P}_1, \ldots, \mathcal{P}_{110}\}$ of infinite prinjective type is peak-irreducible if and only if $\mathcal{P}_j$ is one of the 28 posets listed in Tables 4.7 at the end of this section (see Remark 5.0′ in [11, p. 275]).

(b) Let $I = \mathcal{P}_j$ be any of the 28 peak-irreducible posets listed in Tables 4.7. In each such $\mathcal{P}_j$ we have marked in Tables 4.7 a point $\mathbf{a} \in \mathcal{P}_j$ such that $\mathbf{a} \in \max(\mathcal{P}_j \setminus \max \mathcal{P}_j)$ and the condition (b)(i) is satisfied.

Now we shall find for any such $\mathcal{P}_j$ an indecomposable prinjective module $V'_{(j)}$ in $\operatorname{prin}(K\mathcal{P}'_j)$, where $\mathcal{P}'_j = \mathcal{P}_j \setminus \{\mathbf{a}\}$, such that the coordinate vector

$$v'_{(j)} = \mathbf{cdn}\, V'_{(j)}$$

satisfies the equation

$$\widehat{\ell}_a(v') = 2.$$

We shall prove this by a case-by-case inspection of the 28 peak-irreducible critical posets listed in Tables 4.7.

If $\mathcal{P}_j$ is any of the posets $\mathcal{P}_{1,n+1}, \mathcal{P}'_{2,n+1}, \mathcal{P}'_{3,n+1}, \mathcal{P}''_{3,n+1}$ in Part 1 of Tables 4.7 we take for $V'_{(j)}$ the following prinjective module in $\operatorname{prin}(K\mathcal{P}'_j)$:

$$V'_{1,n+1}: \quad \begin{tikzcd}[column sep=small] K \arrow[d] \arrow[dr] & K \arrow[d] \arrow[dr] & K \arrow[d] \arrow[dr] & \cdots \arrow[dr] & K \arrow[d] \arrow[dr] & \\ K & K & K & & K & K \end{tikzcd}$$

$$V'_{2,n+1}: \quad \begin{tikzcd}[column sep=small] K \arrow[dr] & K \arrow[d] & K^2 \arrow[dl] \arrow[dr] & & \cdots & & K^2 \arrow[dl] \arrow[dr] & K \arrow[d] \\ & K & & K^2 & & K^2 & & K^2 \end{tikzcd}$$

$$V'_{3,n+1}: \quad \begin{tikzcd}[column sep=small, row sep=small] & K \arrow[d] & & & & & & \\ & K^2 \arrow[dl] \arrow[d] & K^2 \arrow[dl] \arrow[d] & K^2 \arrow[dl] \arrow[d] & \cdots \arrow[dl] & K^2 \arrow[dl] \arrow[dr] & K \arrow[d] \arrow[dr] & \\ K & K^2 & K^2 & K^2 & \, & & K^2 & K \end{tikzcd}$$

$$V''_{3,n+1}: \quad \begin{tikzcd}[column sep=small, row sep=small] K \arrow[d] & & & & & & \\ K^2 \arrow[d] \arrow[dr] & & K^2 \arrow[dl] \arrow[d] & \cdots \arrow[dl] & K^2 \arrow[dl] \arrow[d] & K^2 \arrow[dl] \arrow[dr] & K \arrow[d] \\ K^2 & K^2 & K^2 & \, & K^2 & & K^2 \end{tikzcd}$$

respectively. It follows that the vectors

$$\begin{aligned} v'_{(1,n)} &= \mathbf{cdn}\, V'_{(1,n)} = \left[\begin{smallmatrix} 1 & \cdots & 1 & \\ 1 & \cdots & 1 & 1 \end{smallmatrix}\right], \\ v'_{(2,n+1)} &= \mathbf{cdn}\, V'_{(2,n+1)} = \left[\begin{smallmatrix} 1 & 1 & 2 & & \cdots & 2 & 1 \\ & 1 & & 2 & \cdots & & 2 \end{smallmatrix}\right] \\ v'_{(3,n+1)} &= \mathbf{cdn}\, V'_{(3,n+1)} = \left[\begin{smallmatrix} & 1 & & & & & & \\ & 1 & 2 & \cdots & 2 & 2 & 1 & \\ 1 & 2 & 2 & \cdots & 2 & & 2 & 1 \end{smallmatrix}\right], \\ v''_{(3,n+1)} &= \mathbf{cdn}\, V''_{(3,n+1)} = \left[\begin{smallmatrix} 1 & & & & & & & \\ 1 & & 2 & 2 & \cdots & & 2 & 1 \\ 1 & 2 & 2 & 2 & \cdots & 2 & & 2 \end{smallmatrix}\right] \end{aligned}$$

satisfy the required condition.

Next we consider the posets $\mathcal{P}_4 = \mathcal{K}_1^*, \ldots, \mathcal{P}_8 = \mathcal{K}_5^*$ of Part 2 of Tables 4.7, which are one-peak enlargements of the critical Kleiner posets $\mathcal{K}_1, \ldots, \mathcal{K}_5$. In this case $\mathcal{P}'_j = \mathcal{P}_j \setminus \{\mathbf{a}\}$ is the sincere poset $\mathcal{F}(j-2)^*$ of Kleiner (see Chapter 10 of [22]) and $V'_{(j)}$ can be chosen from the list of the sincere representations of the poset $\mathcal{F}(j-2)^*$ presented in Tables 10.20 of [22], for $j = 4, \ldots, 8$. It is easy to check by consulting Tables 10.20 of [22] that the modules

$$V'_{(4)} = \mathcal{S}_2^2, \quad V'_{(5)} = \mathcal{S}_2^3, \quad V'_{(6)} = \mathcal{S}_5^4, \quad V'_{(7)} = \mathcal{S}_6^5, \quad V'_{(8)} = \mathcal{S}_{10}^6$$

have the required property, and $v'_{(4)} = \mathbf{cdn}\, \mathcal{S}_2^2$, $v'_{(5)} = \mathbf{cdn}\, \mathcal{S}_2^3$, $v'_{(6)} = \mathbf{cdn}\, \mathcal{S}_5^4$, $v'_{(7)} = \mathbf{cdn}\, \mathcal{S}_6^5$, $v'_{(8)} = \mathbf{cdn}\, \mathcal{S}_{10}^6$ are the following vectors:

$v'_{(4)}$	$v'_{(5)}$	$v'_{(6)}$	$v'_{(7)}$	$v'_{(8)}$
$\begin{smallmatrix} 1 & 1 & 1 \\ & 2 & \end{smallmatrix}$	$\begin{smallmatrix} 1 & 1 & \\ 1 & 1 & 1 \\ & 3 & \end{smallmatrix}$	$\begin{smallmatrix} & & 1 \\ & 1 & 1 \\ 2 & 1 & 1 \\ & 4 & \end{smallmatrix}$	$\begin{smallmatrix} & & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ & 5 & \end{smallmatrix}$	$\begin{smallmatrix} & & 1 \\ & & 1 \\ & 2 & 1 \\ 3 & 2 & 1 \\ & 6 & \end{smallmatrix}$

Now we consider the remaining 19 peak-irreducible posets $\mathcal{P}_j$ presented in Part 3 of Tables 4.7. It follows from Theorem 3.1 and Corollary 3.2 of [23] that for each such $\mathcal{P}_j$ the poset $\mathcal{P}'_j = \mathcal{P}_j \setminus \{\mathbf{a}\}$ is of finite prinjective type, the Auslander–Reiten quiver $\Gamma(\mathrm{prin}\, K\mathcal{P}'_j)$ of the category $\mathrm{prin}(K\mathcal{P}'_j)$ is finite and coincides with its preprojective component, $\mathrm{End}(X) \cong K$ for any indecomposable module X in $\mathrm{prin}(K\mathcal{P}'_j)$, and X is determined by the vector $\mathbf{cdn}\, X$ uniquely up to isomorphism.

By the modification given in [14, Section 4] of the algorithm described in Theorems 11.52 and 11.80 (see also 11.87) of [22] we shall construct the finite Auslander–Reiten quiver $\Gamma(\operatorname{prin} K\mathcal{P}'_j)$ for each of the posets $\mathcal{P}_j$ presented in Part 3 of Tables 4.7 and we shall find an indecomposable sincere module $V'_{(j)}$ in $\operatorname{prin}(K\mathcal{P}'_j)$ such that $\operatorname{End}(V'_{(j)}) \cong K$ and $v'_{(j)} = \mathbf{cdn}\, V'_{(j)}$ is the corresponding vector in the following tables:

$v'_{(10)}$	$v'_{(11)}$	$v'_{(12)}$	$v_{(15)}$	$v'_{(19)}$	$v'_{(20)}$	$v'_{(21)}$	$v'_{(23)}$	$v'_{(25)}$	$v'_{(27)}$
			1					1	1
1			1	1		1 1	1 1	1	1 2
1	1 1	1 2 1	1 1	1 2	1 1 1	1 2	1 1	1 3	1 2
1 1	1 2 1	2 1	1	3 1 2	3 1 2	2 1	1 2	3 2	2
2 2	2 3	3 3	3 2	4 4	3 4	4 3	4 2	5 4	5 3

$v'_{(29)}$	$v'_{(31)}$	$v'_{(47)}$	$v'_{(48)}$	$v'_{(49)}$	$v'_{(100)}$	$v'_{(106)}$	$v'_{(108)}$	$v'_{(110)}$
	1							
	1							
1	1						1	
1	1			1	1	2	1	
1 1	1	1	2	1	1 3	1 1	1	1 1
2 3	2	1 1	1 1 1	1 1 1	2 1	2 1	1 2	1 2 1
5 2	3 4	1 2 1	2 2 2	2 2 1	4 3 2	3 3 2	2 3 2	1 2 3

If $j = 10$ or $j = 47$, we can easily construct the representation $V'_{(j)}$ without looking at the Auslander–Reiten quiver $\Gamma(\operatorname{prin} K\mathcal{P}'_j)$. We note that the representations

$$V'_{(10)}: \begin{tikzcd} & K \arrow[d] & \\ K \arrow[d] & K^2 \arrow[dl] \arrow[dr] & K \arrow[d] \\ K^2 & & K^2 \end{tikzcd} \qquad V'_{(47)}: \begin{tikzcd} & & K \arrow[dll] \arrow[drr] & & \\ K^2 \arrow[d] \arrow[drr] & & & & K^2 \arrow[dll] \arrow[d] \\ K & & K^2 & & K \end{tikzcd}$$

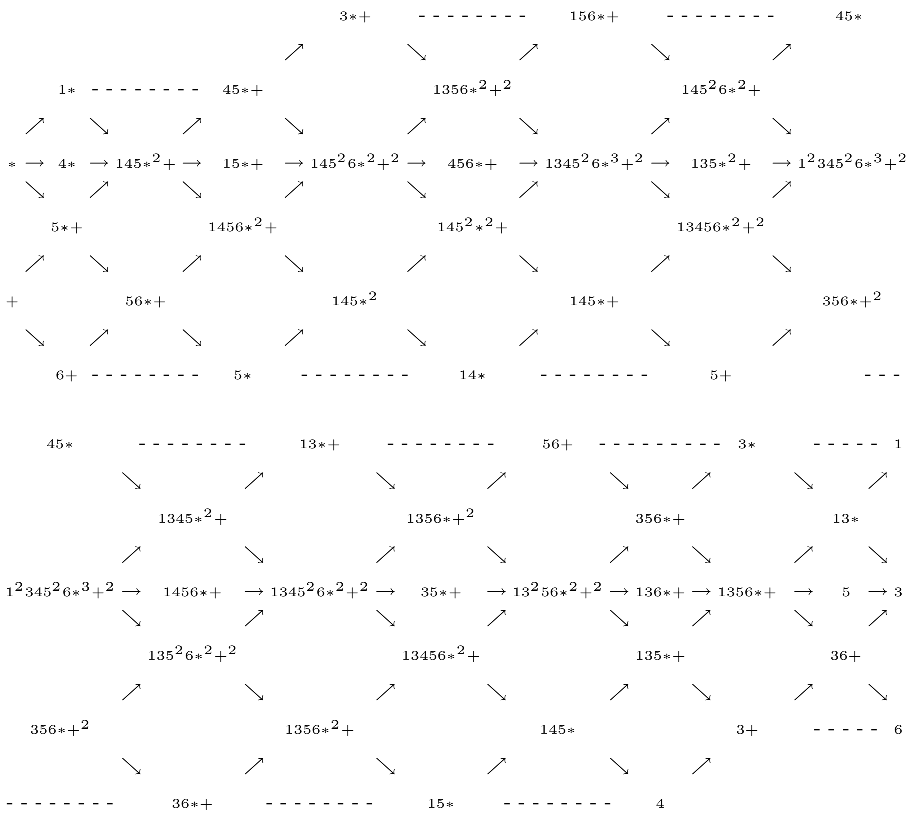
satisfy the required condition.

In order to construct the remaining 17 representations $V'_{(j)}$ we shall look at the Auslander–Reiten quiver $\Gamma(\operatorname{prin} K\mathcal{P}'_j)$. We shall give a detailed proof only in three cases $j = 11$, $j = 12$ and $j = 19$. The proof in the remaining cases is analogous, and we leave it to the reader.

Case $j = 11$. Assume that the vertices of $\mathcal{P}_{11}$ are enumerated as follows:

$$\mathcal{P}_{11}: \begin{tikzcd} 1 \arrow[d] & & 3 \arrow[dl] \arrow[d] & \\ 2 \arrow[dr] & 4 \arrow[d] & 5 \arrow[dl] \arrow[dr] & 6 \arrow[d] \\ & * & & + \end{tikzcd}$$

and $\mathbf{a} = 2$. Then $\mathcal{P}'_{11} = \mathcal{P}_{11} \setminus \{2\}$ is of finite prinjective type and the Auslander–Reiten quiver $\Gamma(\operatorname{prin} K\mathcal{P}'_{11})$ coincides with its preprojective component and looks as follows (see [25], Example 4.17]):

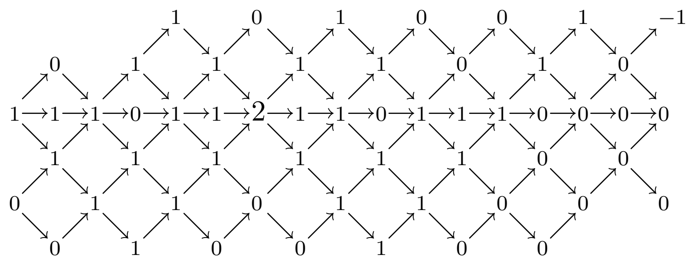


Here we write the coordinate vectors $v = \mathbf{cdn}\, V$ instead of indecomposable modules V, and we use the exponential notation of coordinate vectors introduced in [22, 11.88], that is, the vector $v = \mathbf{cdn}\, V = (v_1, \dots, v_t) \in \mathbb{N}^{J'}$ is written in the form

$$v = \mathbf{cdn}\, V = 1^{v_1} 2^{v_2} \dots t^{v_t}$$

where we omit j^{v_j} if $v_j = 0$, and we set $j^{v(j)} = j$ if $v(j) = 1$.

Note that the additive function (4.5) on $\Gamma(\operatorname{prin} K\mathcal{P}'_j)$ is defined by the formula $\widehat{\ell}_a(v) = v(*) - v(1)$ and therefore the value diagram of the function $\widehat{\ell}_a(-)$ is



It follows form the above diagram that there is a unique module V' in $\mathrm{prin}(K\mathcal{P}'_j)$ such that $\widehat{\ell}_a(\mathbf{cdn}\,V') = 2$ and $\mathbf{cdn}\,V' = 1^2345^26{*}^3{+}^2$. We take for $V'_{(11)}$ the module V'. Its coordinate vector is just the vector $v'_{(11)}$ shown in the table above.

Case $j = 12$. Suppose that $\mathcal{P}_{12}$ and the point a are as in Tables 4.7. Let us enumerate the vertices of $\mathcal{P}'_{12} = \mathcal{P}_{12} \setminus \{\mathbf{a}\}$ as follows:

$$\mathcal{P}'_{12}: \quad \begin{array}{ccccc} & & 5 & & 4 \\ & & \downarrow & & \downarrow \\ 6 & & 1 & & 3 \\ \downarrow & \swarrow & & \searrow & \downarrow \\ + & & & & * \end{array}$$

Then $\mathcal{P}'_{12}$ is of finite prinjective type, $\widehat{\ell}_a(v) = v(+) - v(6)$ and the beginning of the Auslander–Reiten quiver $\mathit{\Gamma}(\mathrm{prin}\,K\mathcal{P}'_{12})$ looks as follows:

$$\begin{array}{ccccccccccccccccc}
 & & & & 4* & \text{- - - - - - - -} & & & 1+ & \text{- - - - - - - -} & & & 56{*}{+} & & \text{- - - -} & & \cdots \\
 & & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & 3* & & & & 14{*}{+} & & & & 156{*}{+}^2 & & & & 1356{*}^2{+} & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
* & & & & 13{*}{+} & & & & 1456{*}^2{+}^2 & & & & 1^2356{*}^2{+}^2 & & & & \cdots \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & 1{*}{+} & \to & 5{*}{+} & \to & 1356{*}^2{+}^2 & \to & 136{*}{+} & \to & 1^23456{*}^3{+}^2 & \to & 145{*}^2{+} & \to & 1^2345^26{*}^3{+}^3 & & \cdots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
+ & & & & 16{*}{+} & & & & 135{*}^2{+} & & & & 13456{*}^2{+}^2 & & & & \cdots \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & 6+ & & \text{- - - - - - - -} & & 1* & & \text{- - - - - - - -} & & 35{*}{+} & & \text{- - - - - - - -} & & 146{*}{+} & \cdots &
\end{array}$$

The value diagram of the additive function $\widehat{\ell}_a(-)$ is

$$\begin{array}{cccccccccccccccccc}
 & & 0 & & 1 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & \\
 & 0 & & 1 & & 1 & & 0 & & 1 & & 1 & & 0 & & 0 & & 0 \\
0 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 0 & & 0 & \\
 & 1\to & 1\to & 1\to & 0\to & 1\to & 1\to & 2\to & 1\to & 1\to & 0\to & 1\to & 1\to & 1\to & 0\to & 0\to & 0\to & 0 \\
1 & & 0 & & 1 & & 1 & & 1 & & 1 & & 0 & & 1 & & 0 & \\
 & 0 & & 0 & & 1 & & 0 & & 1 & & 0 & & 0 & & 1 & & -1
\end{array}$$

It follows from the above diagram that there is a unique module V' in $\mathit{\Gamma}(\mathrm{prin}\,K\mathcal{P}'_j)$ such that $\widehat{\ell}_a(\mathbf{cdn}\,V') = 2$. Note that $\mathbf{cdn}\,V' = 1^2345^26{*}^3{+}^3$. We take for $V'_{(12)}$ the module V'. Its coordinate vector is just the vector $v'_{(12)}$ shown in the table above.

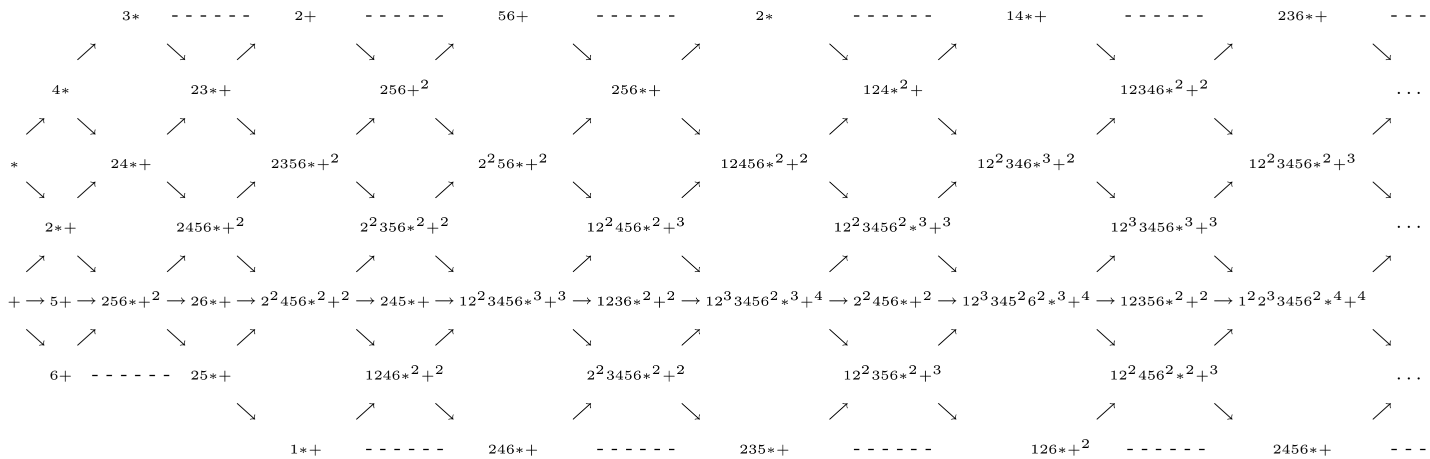
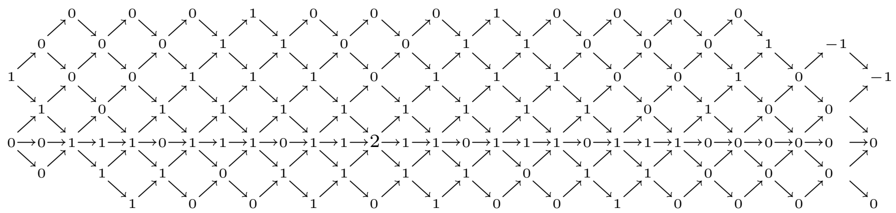


Fig. 1. The Auslander–Reiten quiver of $\mathrm{prin}(K\mathcal{P}'_{19})$

Case $j = 19$. Suppose that $\mathcal{P}_{19}$ and the point a are as in Tables 4.7. Let us enumerate the vertices of $\mathcal{P}'_{19} = \mathcal{P}_{19} \setminus \{\mathbf{a}\}$ as follows:

$$\mathcal{P}'_{19}: \quad \begin{array}{cccc} 3 & 1 & & \\ \downarrow & \downarrow\searrow & & \\ 4 & 2 & 5 & 6 \\ \downarrow\swarrow & \searrow & \downarrow\swarrow & \\ * & & + & \end{array}$$

Then $\mathcal{P}'_{19}$ is of finite prinjective type, $\widehat{\ell}_a(v) = v(*) - v(3) - v(4)$ and the beginning of the Auslander–Reiten quiver $\Gamma(\operatorname{prin} K\mathcal{P}'_{19})$ looks as in Figure 1. The value diagram of the additive function $\widehat{\ell}_a(-)$ is



It follows that there is a unique module V' in the $\Gamma(\operatorname{prin} K\mathcal{P}'_{19})$ such that $\widehat{\ell}_a(\mathbf{cdn}\, V') = 2$. Note that $\mathbf{cdn}\, V' = 1^2 2^3 3456^2 *^4 +^4$. We take for $V'_{(19)}$ the module V'. Its coordinate vector is just the vector $v'_{(19)}$ shown in the table above. This finishes the proof of (b).

(c) Let a and V' be as in (b) and let $v' = \mathbf{cdn}\, V'$. Since $\widehat{\ell}_a(v') = 2$, there exists $p \in \max I$ such that $p \succ a$ and $v'(p) \neq 0$. It follows from the formula (2.4) applied to $J = I'_a$ and $X = V'$ that $(\mathbf{cdn}\, V)(p) = v'(p) \neq 0$ and therefore $V_p \neq 0$. Moreover, it follows from Proposition 2.2 that $V = \widehat{T}_a(V')$ is indecomposable and $(\mathbf{cdn} V)(a) = 0$. Then according to Lemma 4.2 the map h_a is injective, $\operatorname{Hom}_{KI}(S(a), V) = 0$ and $\dim_K \operatorname{Ext}^1_{KI}(S(a), V) = \ell_a(\mathbf{cdn}\, V) = \widehat{\ell}_a(v') = 2$. Then (iii) follows.

(i) Since $S(a)$ is simple, $\operatorname{End}(S(a)) \cong K$. The poset I'_a is of finite prinjective type, because I is a minimal poset of infinite prinjective type. It follows from Corollary 3.2 of [23] that the indecomposable KI'_a-module is preprojective and $\operatorname{End}(V') \cong K$. Then Proposition 2.2 yields $\operatorname{End}(V) \cong \operatorname{End}(V') \cong K$ and (i) follows.

(ii) Since it was shown above that $\operatorname{Hom}_{KI}(S(a), V) = 0$, it remains to prove that $\operatorname{Hom}_{KI}(V, S(a)) = 0$. For this purpose we note that the prinjectivity of V implies that the restriction $V^- = Ve^-$ of V to I^- is a projective module. By applying formula (2.1) to $X = V$ we get $V^- \cong \bigoplus_{a \neq j \in I^-} (e_j KI^-)^{v(j)}$. Hence we get

$$\begin{aligned}\mathrm{Hom}_{KI}(V, S(a)) &= \mathrm{Hom}_{KI^-}(V^-, S(a))\\ &\cong \bigoplus_{a\neq j\in I^-} (\mathrm{Hom}_{KI^-}(e_j KI^-, S(a))^{v(j)}\\ &\cong \bigoplus_{a\neq j\in I^-} (S(a)e_j)^{v(j)} = 0.\end{aligned}$$

This finishes the proof of Theorem 4.4. ■

COROLLARY 4.6. *If I is a finite poset of infinite prinjective type then for any field K there exists a full and faithful exact functor*

$$T : \mathrm{mod}\begin{pmatrix} K & K^2\\ 0 & K\end{pmatrix} \to \mathrm{prin}(KI).$$

Proof. Let I be a finite poset of infinite prinjective type. It follows from Theorem 3.1 of [23] that I contains as a full peak subposet a poset J isomorphic to one of the critical posets $\mathcal{P}_1, \dots, \mathcal{P}_{110}$ listed in Section 5 of [23]. In view of Proposition 2.2 it is sufficient to prove the corollary for the critical posets $\mathcal{P}_1, \dots, \mathcal{P}_{110}$. Moreover, in view of Lemma 2.8, it is sufficient to prove the corollary for each of the peak-irreducible posets from the 28 critical posets among $\mathcal{P}_1, \dots, \mathcal{P}_{110}$.

Assume that I is one of the 28 peak-irreducible critical posets in Tables 4.7. It follows from Theorem 4.4 that there exists a maximal point $a \in I^-$ and an indecomposable module Vinprin(KI) such that the modules $S(a)$ and V satisfy the conditions (i)–(iii) in Theorem 4.4(c). By applying Proposition 3.1 to $\Lambda = KI$, $V = S(a)$ and $U = V$ we get a full and faithful exact functor

$$T_{S(a),V} : \mathrm{mod}\begin{pmatrix} K & K^2\\ 0 & K\end{pmatrix} \to \mathrm{mod}(KI).$$

Since the modules $S(a)$ and V are prinjective and the subcategory $\mathrm{prin}(KI)$ of $\mathrm{mod}(KI)$ is closed under forming extensions, by the final statement of Proposition 3.1 the image of $T_{S(a),V}$ is contained in $\mathrm{prin}(KI)$. This finishes the proof. ■

We finish this section by presenting a list of minimal peak-irreducible posets of infinite prinjective type used in the proof of Theorem 4.4.

Tables 4.7

Minimal peak-irreducible posets of infinite prinjective type

In each of the diagrams $\mathcal{P}_j$ below we denote by $\mathbf{a}$ an element satisfying the following conditions stated in Theorem 4.4 and its proof.

(a1) $\mathbf{a} \in \mathcal{P}_j^- = \mathcal{P}_j \setminus \max\mathcal{P}_j$ and $\mathbf{a}$ is a maximal element in $\mathcal{P}_j^-$.

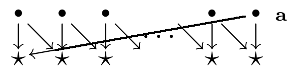
(a2) The subposet $\mathbf{a}^\nabla = \{j \in \mathcal{P}_j : j \preceq \mathbf{a}\}$ of $\mathcal{P}_j^-$ is linearly ordered and each element of $\mathbf{a}^\nabla$ is incomparable with any element of $\mathcal{P}_j^- \setminus \mathbf{a}^\nabla$.

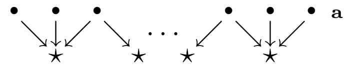
(a3) The poset $\mathcal{P}'_j = \mathcal{P}_j \setminus \{\mathbf{a}\}$ is sincere of finite prinjective type and there exists an indecomposable prinjective $K\mathcal{P}'_j$-module $V'_{(j)}$ such that the additive function

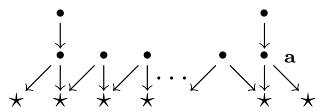
$$\widehat{\ell}_{\mathbf{a}}(v') := \sum_{\mathbf{a} \prec p \in \max I} v'(p) - \sum_{i \prec \mathbf{a}} v'(i)$$

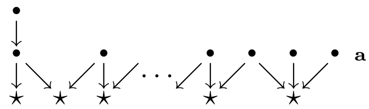
has the value 2 on the coordinate vector $v'_{(j)} = \mathbf{cdn}\, V'_{(j)}$ (see (4.5)).

Part 1. Peak-irreducible critical posets of type $\widetilde{\mathbb{A}}^*_n$ and $\widetilde{\mathbb{D}}^*_n$

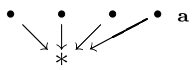
$\mathcal{P}_{1,n+1}$: 

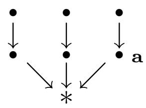
$\mathcal{P}'_{2,n+1}$: 

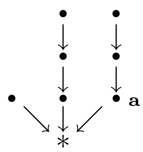
$\mathcal{P}'_{3,n+1}$: 

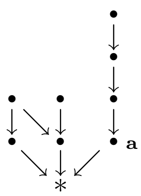
$\mathcal{P}''_{3,n+1}$: 

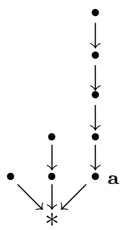
Part 2. One-peak enlargements $\mathcal{P}_4$–$\mathcal{P}_8$ of Kleiner's posets

$\mathcal{P}_4$: 

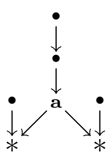
$\mathcal{P}_5$: 

$\mathcal{P}_6$: 

$\mathcal{P}_7$: 

$\mathcal{P}_8$: 

Part 3. Peak-irreducible critical posets $\mathcal{P}_j$, where
$j \in \{10, 11, 12, 15, 19, 20, 21, 23, 25, 27, 29, 31, 47, 48, 49, 100, 106, 108, 110\}$



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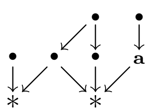

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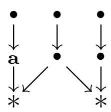

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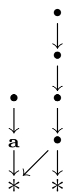

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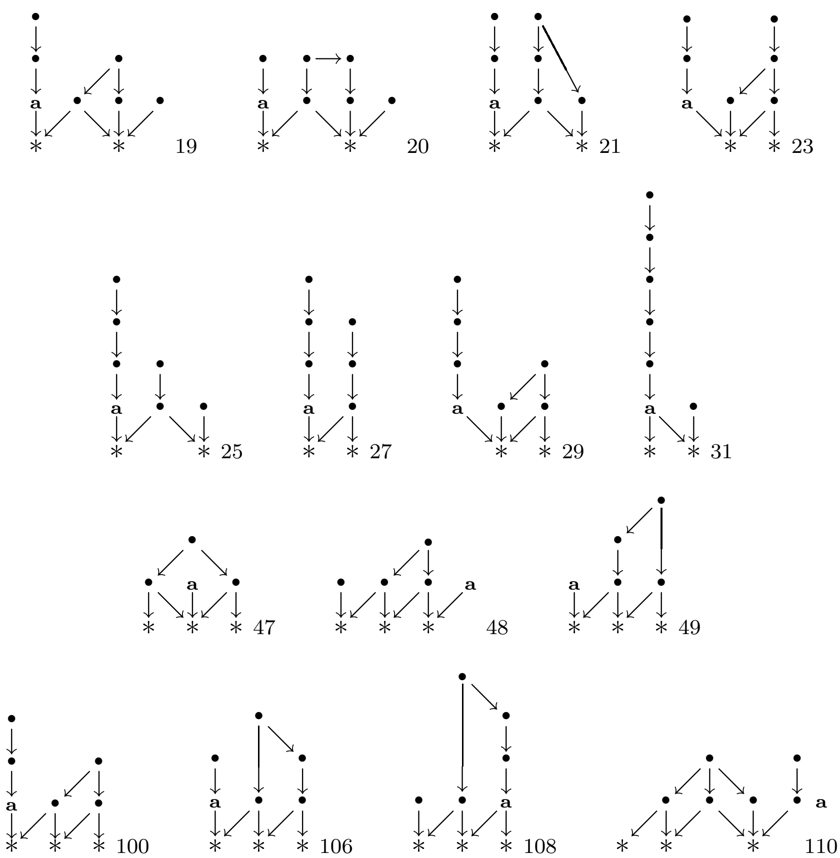

5. Embeddings of Kronecker modules over commutative rings. The main aim of this section is to show how a full faithful exact functor

$$T : \mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}} \begin{pmatrix} R & R^2 \\ 0 & R \end{pmatrix} \to \mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(RI) \tag{5.1}$$

required in Theorem 1.7(e) can be constructed for any commutative ring R. Although a complete construction is given only in the case where R is a commutative K-algebra, we give an idea of this construction for R arbitrary.

Assume that I is a finite poset of infinite prinjective type. We shall construct the Kronecker embedding functor (5.1) by applying Proposition 3.1. For this purpose it is sufficient to find a pair of modules U and V in $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(RI)$ satisfying the conditions (i)–(iii) of Proposition 3.1. Since I contains as a peak subposet a poset J isomorphic to one of the 110 critical posets $\mathcal{P}_1, \ldots, \mathcal{P}_{110}$ listed in [23, Section 5], it is sufficient to find such a pair of modules U and V in $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(RJ)$ for J critical, and to construct a full faithful exact functor $\widehat{T}_J : \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(RJ) \to \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(RI)$. The functor $\widehat{T}_J$ is constructed in Proposition 5.2 below by a generalization of Proposition 2.2.

Next we extend Lemma 2.8 from prinjective modules to propartite RI-modules in Lemma 5.4 below. This will reduce the problem to the case of J being one of the 28 peak-irreducible posets listed in Tables 4.7.

In this case we need an extension of Theorem 4.4 to propartite RI-modules. We recall that for each of those peak-irreducible posets J we have constructed in Theorem 4.4 an element $a \in J \setminus \max J$ and a pair of indecomposable modules $S(a)$ and V in $\operatorname{prin}(KJ)$ satisfying the conditions (i)–(iii) of Theorem 4.4(c).

From Proposition 3.1 and the discussion above it follows that, in order to construct a Kronecker embedding (5.1) for $I = J$, it is sufficient to construct a pair of indecomposable modules $\widehat{S}(a)$ and $\widehat{V}$ in $\operatorname{mod}_{\mathrm{pr}}^{\mathrm{pr}}(RJ)$ satisfying the natural extensions of the conditions (i)–(iii) of Theorem 4.4(c), where J is one of the peak-irreducible posets listed in Tables 4.7.

We can take for $\widehat{S}(a)$ the unique KJ-module such that $\widehat{S}(a)e_a = R$ and $\widehat{S}(a)e_j = 0$ for all $j \neq a$. If $J = \mathcal{P}_j$ is any of the 11 posets $\mathcal{P}_{1,n+1}$, $\mathcal{P}'_{2,n+1}$, $\mathcal{P}'_{3,n+1}$, $\mathcal{P}''_{3,n+1}$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_{10}$, $\mathcal{P}_{47}$ we can take for $\widehat{V}$ the R-form of the corresponding K-form $V_{(j)}$ constructed and listed in the proof of Theorem 4.4. In this case the extensions of the conditions (i)–(iii) are satisfied, because the arguments given in the proof of Lemma 4.2 generalize to our situation. However, the isomorphism $\operatorname{End}_{RJ}(\widehat{V}) \cong R$ has to be checked directly using the definition of $\widehat{V}$. The same arguments apply to the remaining 17 peak-irreducible posets $\mathcal{P}_j$, $j \in \{11, 12, 15, 19, 20, 21, 23, 25, 27, 29, 31, 48, 49, 100, 106, 108, 110\}$, if we were able to define $\widehat{V}_{(j)}$ in such a way that $\operatorname{End}_{R\mathcal{P}_j}(\widehat{V}_{(j)}) \cong R$ for each of the remaining 17 posets $\mathcal{P}_j$.

At present we can do it only in the case where R is a K-algebra. In this situation we set $\widehat{V}_{(j)} = V_{(j)} \otimes_K R$ and we apply Corollary 5.7 below.

First we extend Proposition 2.2 to propartite AI-modules as follows.

PROPOSITION 5.2. *Let A be a ring with an identity element, and let $J \subseteq I$, e_J, $e_{J\cup I^-}$ be idempotents of the incidence A-algebra AI defined as in Proposition* 2.2 (*see also below*). *Then there exist an A-algebra isomorphism $e_J(AI)e_J \cong AJ$ and the subposet induction functor* (see [22, Section 5.3], [12, Proposition 2.4])

$$\widehat{T}_J = (-) \otimes_{AJ} e_J(AI)e_{J\cup I^-} : \operatorname{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(AJ) \to \operatorname{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(AI) \tag{5.3}$$

with the following properties.

(a) *$\widehat{T}_J$ is full, faithful and exact.*
(b) *If X is a finitely generated module then so is $\widehat{T}_J(X)$.*
(c) *If X is a free A-module then so is $\widehat{T}_J(X)$.*

Proof. We assume that $J \subseteq I$ is a peak subposet. We set $I^- = I \setminus \max I$ and $J^- = J \setminus \max J$. For $i \preceq j$ we denote by $e_{ij} \in AI$ the matrix having

1 at the i-j-th position and zeros elsewhere. Given j in I we denote by $e_j = e_{jj}$ the standard primitive idempotent corresponding to j. We define the following idempotents of AI:

$$\begin{aligned} e_J &= \sum_{j\in J} e_j, & e_{J^-} &= \sum_{j\in J^-} e_j, \\ e_{I^-} &= \sum_{j\in I^-} e_j, & e_J^+ &= e_{J\cup I^-} = \sum_{j\in J\cup I^-} e_j, \\ e_* &= \sum_{p\in \max I} e_p, & e'_* &= \sum_{p\in \max J} e_p. \end{aligned}$$

The A-algebra isomorphism $AJ \cong e_J(AI)e_J$ follows easily.

First we shall show that $\widehat{T}_J(X)$ is a right AI-module for any AJ-module Y. For this we note that the map $AI \to e_J^+(AI)e_J^+ \cong A(J\cup I^-)$, $\lambda \mapsto e^+\lambda e^+$, is a surjective A-algebra homomorphism, because AI has the triangular form (1.1). It follows that any right $e_J^+(AI)e_J^+$-module is a right AI-module via $AI \to e_J^+(AI)e_J^+$. In particular, $\widehat{T}_J(X)$ is a right AI-module as required.

Following (1.5), the A-algebras AI and AJ can be viewed as bipartite A-algebras

$$AI = \begin{pmatrix} AI^- & M \\ 0 & B \end{pmatrix}, \quad AJ = \begin{pmatrix} AJ^- & M' \\ 0 & B' \end{pmatrix}$$

where $B = e_*(AI)e_* = A \times \ldots \times A$ ($|\max I|$ times), $B' = e'_*(AJ)e'_* = A \times \ldots \times A$ ($|\max J|$ times), the free A-modules

$$M = \bigoplus_{p\in\max I} \bigoplus_{\substack{i\prec p \\ i\in I^-}} e_{ip}R, \qquad M' = \bigoplus_{p\in\max J} \bigoplus_{\substack{i\prec p \\ i\in J^-}} e_{ip}R$$

are viewed as an AI^--B-bimodule and an AJ^--B'-bimodule respectively in the obvious way, and multiplication is matrix multiplication. Note that $AI^- \cong e_{I^-}(AI)e_{I^-}$ and $AJ^- \cong e_{J^-}(AJ)e_{J^-}$.

We recall that a right AI-module $Y = (Y', Y''_B, \psi)$ is *propartite* (that is, Y is in $\mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(AI)$) if and only if $Y' = Ye_{I^-}$ is a projective AI^--module and $Y''_B = Ye_*$ is a projective B-module. Similarly a right AJ-module $X = (X', X''_{B'}, \varphi)$ is propartite (that is, X is in $\mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(AJ)$) if and only if $X' = Xe_{J^-}$ is a projective AJ^--module and $X''_B = Ye'_*$ is a projective B'-module.

First we shall show that the AI-module $\widehat{T}_J(X)$ is propartite if the AJ-module X is propartite. For this purpose we prove the following two statements:

(a1) For any AJ-module X there exists a natural B-module isomorphism

$$\widehat{T}_J(X)e_* \cong Xe'_*$$

where the B'-module Xe'_* is viewed as a B-module via the natural direct summand projection $B \to B'$. If the B'-module Xe'_* is projective then $\widehat{T}_J(X)e_*$ is a projective B-module.

(a2) For any $j \in J$ there exist natural AI^--module isomorphisms

$$\widehat{T}_J(e_jAJ)e_{I^-} \cong \begin{cases} e_jAI^- & \text{for } j \in J^-, \\ 0 & \text{for } j \in \max J, \end{cases}$$

along the A-algebra isomorphism $e_{I^-}(AI)e_{I^-} \cong AI^-$.

Since $e_J^+ e_* = e_J^+ e'_* = e_J e'_*$ and $e_J(AI)e_J \cong AJ$, we have

$$\begin{aligned} \widehat{T}_J(X)e_* &= (X \otimes_{AJ} e_J(AI)e_J^+)e_* = X \otimes_{AJ} e_J(AI)e_J e'_* \\ &= (X \otimes_{AJ} e_J(AI)e_J)e'_* \cong (X \otimes_{AJ} AJ)e'_* \cong Xe'_* \end{aligned}$$

and (a1) follows.

For the proof of (a2) we note that $e_J^+ e_{I^-} = e_{I^-}$ and therefore $\widehat{T}_J(e_jAJ)e_{I^-} = (e_jAJ \otimes_{AJ} e_J(AI)e_J^+)e_{I^-} \cong (e_jAI)e_J^+ e_{I^-} = (e_jAI)e_{I^-}$.

If $j \in J^-$ then $e_je_{I^-} = e_j$. Hence $(e_jAI)e_{I^-} = e_j(e_{I^-}AIe_{I^-}) \cong e_jAI^-$ and the first isomorphism in (a2) follows. If $j \in \max J$ then $(e_jAI)e_{I^-} = (e_je'_*AIe'_*)e_{I^-} = 0$. This finishes the proof of (a2).

Now we show that for any propartite AJ-module X the AI-module $Y = \widehat{T}_J(X)$ is propartite.

It follows from (a1) that the B-module $\widehat{T}_J(X)e_*$ is projective. It remains to show that the AI^--module $\widehat{T}_J(X)e_{I^-}$ is projective. This follows from (a2) for $X = e_jAJ$; hence for any AJ-projective module X, because any such X is a summand of a direct sum of modules of the form e_jAJ, $j \in J$. If X is an arbitrary propartite AJ-module then according to [26, Proposition 3.7], X admits an AJ-projective resolution

$$(*) \qquad 0 \to P_1 \xrightarrow{\eta_1} P_0 \xrightarrow{\eta_0} Y \to 0$$

where P_1 is a projective B'-module and $P_1e_{J^-} = 0$. For this we note that the bimodule module M' in the matrix form of AJ above is projective when viewed as a right B'-module.

By applying the functor $\widehat{T}_J$ from (5.3) to the short exact sequence $(*)$ we derive the four-term exact sequence

$$(**) \quad 0 \to \mathrm{Tor}_1^{AJ}(X, e_J(AI))e_J^+ \to \widehat{T}_J(P_1) \xrightarrow{\widehat{T}_J(\eta_1)} \widehat{T}_J(P_0) \xrightarrow{\widehat{T}_J(\eta_0)} \widehat{T}_J(X) \to 0.$$

Since $e_{I^-} \in AI$ is an idempotent and $P_1e_{J^-} = 0$, by (a2) the module $\widehat{T}_J(P_1)e_{I^-}$ is zero, and from $(**)$ we conclude that $\mathrm{Tor}_1^{AJ}(X, e_J(AI))e_J^+e_{I^-}$ is zero and we derive the isomorphism $\widehat{T}_J(P_1)e_{I^-} \cong \widehat{T}_J(X)e_{I^-}$ of KI^--modules. It follows from the discussion above that $\widehat{T}_J(X)e_{I^-}$ is a projective KI^--module and consequently $\widehat{T}_J(X)$ is in $\mathrm{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(AI)$ as we required.

Finally, we show that $\widehat{T}_J$ is exact. Since $\mathrm{Tor}_1^{AJ}(X, e_J(AI))e_J^+e_{I^-}$ is zero and $1 = e_{I^-} + e_*$, the exactness of $\widehat{T}_J$ will follow if we show that

$\mathrm{Tor}_1^{AJ}(X, e_J(AI))e_J^+ e_*$ is zero, because this will imply that the left hand term in $(**)$ is zero.

To see this we apply $(**)$ and we note that according to (a1) the sequence

$$0 \to \widehat{T}_J(P_1)e_* \xrightarrow{\widehat{T}_J(\eta_1)e_*} \widehat{T}_J(P_0)e_* \xrightarrow{\widehat{T}_J(\eta_0)e_*} \widehat{T}_J(X)e_* \to 0$$

is isomorphic to the sequence

$$0 \to P_1 e'_* \to P_0 e'_* \to X e'_* \to 0$$

which is exact, because $(*)$ is exact and e'_* is an idempotent of AJ.

The statement (a) immediately follows from the natural isomorphism $\mathrm{Hom}_{AI}(\widehat{T}_J(X), \widehat{T}_J(Y)) \cong \mathrm{Hom}_{AJ}(X, Z)$ for X, Z in $\mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(AJ)$, which can easily be obtained as follows (see also the proof of Proposition 2.4 of [12]):

$$\begin{aligned}
&\mathrm{Hom}_{AI}(\widehat{T}_J(X), \widehat{T}_J(Z)) \\
&\qquad \cong \mathrm{Hom}_{AI}(X \otimes_{AJ} e_J(AI)e_J^+, Z \otimes_{AJ} e_J(AI)e_J^+) \\
&\qquad \cong \mathrm{Hom}_{AJ}(X, \mathrm{Hom}_{AI}(e_J(AI)e_J^+, Z \otimes_{AJ} e_J(AI)e_J^+)) \\
&\qquad \cong \mathrm{Hom}_{AJ}(X, \mathrm{Hom}_{e^+(AI)e_J^+}(e_J(AI)e_J^+, Z \otimes_{AJ} e_J(AI)e_J^+)) \\
&\qquad \cong \mathrm{Hom}_{AJ}(X, Z \otimes_{AJ} e_J(AI)e_J) \\
&\qquad \cong \mathrm{Hom}_{AJ}(X, Z).
\end{aligned}$$

Since the statements (b) and (c) follow easily the proof is complete. ∎

Next we extend Lemma 2.8 to propartite AI-modules as follows.

LEMMA 5.4. *Let I be a peak-reducible poset with a reducible peak q, and let A be a ring with an identity element. In the notation of Section* 1 *the formula* (2.7) *defines an additive functor*

$$\widetilde{\mathbf{G}}_q : \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(AS_qI) \to \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(AI) \tag{5.5}$$

which is full, faithful and exact. Moreover, *if X is a projective A-module then $\widetilde{\mathbf{G}}_q(X)$ is a projective A-module.*

Proof. We shall use the notation introduced in Section 2. In particular we view any propartite module X in $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(AS_qI)$ as a system $X = (X_i, {}_jh_i)_{i,j \in S_qI, i \prec j}$, where X_i is the A-module Xe_i and ${}_jh_i : X_i \to X_j$ is the A-module homomorphism defined by multiplication by $e_{ji} \in AS_qI$. We define $\widetilde{\mathbf{G}}_q(X) = (\overline{X}_i, {}_j\overline{h}_l)_{i,j,l \in I, l \prec j}$ by the formula (2.7).

In order to show that $\widetilde{\mathbf{G}}_q(X)$ is propartite we need to show that ${}_jh_i : X_i \to X_j$ is a splittable A-module monomorphism for every pair $i \prec j$ in the chain

$$C = \{c \to b_1cq \to \ldots \to b_mcq \to cq\}$$

contained in S_qI. For this purpose we note that the incidence A-algebra of C has the form

$$AC = \begin{pmatrix} A & A & \dots & A \\ 0 & A & \dots & A \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix}.$$

Since X is propartite and there is no relation $t \prec c$ with $t \notin C$ and $c \in C$, the restriction Y of X to C is a projective AC-module. It follows that the AC-module $Y = (X_i, {}_jh_i)_{i,j\in C}$ has the following property: *"${}_jh_i : X_i \to X_j$ is a splittable A-module monomorphism for every pair $i \prec j$ in C"* because this property is enjoyed by the AC-module AC; hence also by every free right AC-module and by every projective right AC-module. In particular, Y has this property.

Hence we easily conclude that the module $\widetilde{\mathbf{G}}_q(X)$ is propartite. Moreover, this allows us to prove the lemma by repeating the arguments used in the proof of Theorem 2.15 of [14]. This is an extension of the case where A is a field to the case where A is a ring. We leave it to the reader. ∎

From now on we assume in this section that A is a K-algebra and I is a finite poset. Homological connections between AI-modules and KI-modules are discussed in the following simple lemma.

LEMMA 5.6. *Let A be an algebra over the field K. If X, Y are modules in* $\mathrm{Mod}(KI)$ *and the projective dimension of X is ≤ 1, then there exist functorial isomorphisms*

(a) $\mathrm{Hom}_{AI}(X \otimes_K A, Y \otimes_K A) \cong \mathrm{Hom}_{KI}(X, Y) \otimes_K A$,

(b) $\mathrm{Ext}^1_{AI}(X \otimes_K A, Y \otimes_K A) \cong \mathrm{Ext}^1_{KI}(X, Y) \otimes_K A$.

In particular, if the module X is prinjective then (a) *and* (b) *holds.*

P r o o f. Let Y and X be modules in $\mathrm{Mod}(KI)$. If the projective dimension of X is ≤ 1 then there exists an exact sequence

$$0 \to P_1 \to P_0 \to X \to 0$$

where P_0 and P_1 are projective KI-modules. Tensoring the sequence over the field K we derive the exact sequence of AI-modules

$$0 \to P_1 \otimes A \to P_0 \otimes A \to X \otimes A \to 0$$

where $P_0 \otimes A$ and $P_1 \otimes A$ are projective AI-modules. The obvious Hom-Ext-sequences yield the commutative diagrams

$$\begin{array}{ccccc}
0 \to \operatorname{Hom}_{KI}(X,Y)\otimes A & \to & \operatorname{Hom}_{KI}(P_0,Y)\otimes A & \to & \operatorname{Hom}_{KI}(P_1,Y)\otimes A \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_1} \\
0 \to \operatorname{Hom}_{AI}(X\otimes A, Y\otimes A) & \to & \operatorname{Hom}_{AI}(P_0\otimes A, Y\otimes A) & \to & \operatorname{Hom}_{AI}(P_1\otimes A, Y\otimes A)
\end{array}$$

$(*)$

$$\begin{array}{ccccc}
\operatorname{Hom}_{KI}(P_0,Y)\otimes A & \to & \operatorname{Hom}_{KI}(P_1,Y)\otimes A & \to & \operatorname{Ext}^1_{KI}(X,Y)\otimes A \to 0 \\
\downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\psi} \\
\operatorname{Hom}_{AI}(P_0\otimes A, Y\otimes A) & \to & \operatorname{Hom}_{AI}(P_1\otimes A, Y\otimes A) & \to & \operatorname{Ext}^1_{AI}(X\otimes A, Y\otimes A) \to 0
\end{array}$$

where the vertical homomorphisms φ, φ_0 and φ_1 are defined by the formula $f\otimes r \mapsto (x\otimes\lambda \mapsto f(x)\otimes\lambda r)$. Since P_0 and P_1 are projective modules, obviously φ_0 and φ_1 are isomorphisms. It follows that φ is an isomorphism, and there exists a unique isomorphism ψ making the remaining square in $(*)$ commutative. Hence the statements (a) and (b) follow.

For the final statement we recall from [23] or from Section 1 that every prinjective KI-module has the projective dimension 0 or 1. ∎

Theorem 4.4 and Lemma 5.6 immediately yield

Corollary 5.7. *Let A be a K-algebra and let V, $S(a)$ be the prinjective KI-modules chosen as in Theorem* 4.4(c). *Then $\widehat{V} = V\otimes_K A$ and $\widehat{S}(a) = S(a)\otimes_K A$ are propartite AI-modules, they are A-free and there exist isomorphisms*

(a) $\operatorname{End}_{AI}(S(a)\otimes_K A) \cong \operatorname{End}_{AI}(V\otimes_K A) \cong A$,
(b) $\operatorname{Hom}_{AI}(S(a)\otimes_K A, V\otimes_K A) = \operatorname{Hom}_{AI}(V\otimes_K A, S(a)\otimes_K A) = 0$,
(c) $\operatorname{Ext}^1_{AI}(S(a)\otimes A, V\otimes A) \cong A^2$. ∎

The discussion presented at the beginning of this section and the results of Proposition 5.2, Lemma 5.6 and Corollary 5.7 can be summarized as follows.

Theorem 5.8. *Let I be a poset of infinite prinjective type and let A be a K-algebra. Then there exist full, faithful and exact functors*

$$T : \operatorname{Mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & A^2 \\ 0 & A\end{pmatrix} \to \operatorname{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(AI), \quad T' : \operatorname{mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & A^2 \\ 0 & A\end{pmatrix} \to \operatorname{mod}^{\mathrm{pr}}_{\mathrm{pr}}(AI).$$

If X is a projective A-module then so are $T(X)$ and $T'(X)$. ∎

6. Proof of main theorems

6.1. *Proof of Theorem 1.7.* In view of Theorem 3.1 of [23] the proof of Theorem 1.7 reduces to the proof of the equivalences (a)⇔(d) and (d)⇔(e). The implication (a)⇒(d) follows from Corollary 4.6 and the implication (a)⇒(e) is a consequence of Theorem 5.8. Since the category of Kronecker modules over a field is of infinite representation type (see [18] and [22, Example 1.5]), the implications (e)⇒(d)⇒(a) follow and the proof of Theorem 1.7 is complete. ■

6.2. *Proof of Theorem 1.8.* Let K be a field, let A be a K-algebra and let

$$\Gamma_2(K) = \begin{pmatrix} K & K^2 \\ 0 & K \end{pmatrix}$$

be the Kronecker K-algebra. It follows from [10, Theorem 1.2] that for any K-algebra A generated by at most λ elements, where λ is an infinite cardinal number, there exists a direct system

$$\mathbb{G} = \{\mathbb{G}_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$$

of K-linear additive functors $\mathbb{G}_\beta : \mathrm{Mod}(A) \to \mathrm{Mod}(\Gamma_2(K))$ connected by functorial morphisms $u_{\beta\gamma} : \mathbb{G}_\beta \to \mathbb{G}_\gamma$ satisfying the following conditions:

(i) If M is a module in $\mathrm{Mod}(A)$ which is A-free, then the Kronecker module $\mathbb{G}_\beta(M) = (M'_\beta, M''_\beta, \varphi'_\beta, \varphi''_\beta)$ is A-free for all $\beta \subseteq \lambda$.

(ii) If M and N are modules in $\mathrm{Mod}(A)$ then

$$\mathrm{Hom}_{\Gamma_2(K)}(\mathbb{G}_\beta(M), \mathbb{G}_\gamma(N)) = 0 \quad \text{if } \beta \not\subseteq \gamma,$$

and the natural A-homomorphism

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\Gamma_2(A)}(\mathbb{G}_\beta(M), \mathbb{G}_\gamma(N)), \quad f \mapsto u_{\beta\gamma}(N) \circ \mathbb{G}_\beta(f),$$

is an isomorphism for all $\beta \subseteq \gamma \subseteq \lambda$.

On the other hand, if I is a finite poset of infinite prinjective type then according to Theorem 1.7 there exists a full, faithful and exact functor

$$T : \mathrm{Mod}(\Gamma_2(K)) \to \mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(KI).$$

Note that $\mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(\Gamma_2(K)) = \mathrm{Mod}(\Gamma_2(K))$. In view of Theorem 5.8 and its proof, it is easy to see that the direct system $\mathbb{F} = \{\mathbb{F}_\beta, v_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ of the K-linear composed functors $\mathbb{F}_\beta = \mathbb{G}_\beta \circ T : \mathrm{Mod}(A) \to \mathrm{Mod}(KI)$ connected by the induced functorial morphisms $v_{\beta\gamma}(-) = T(u_{\beta\gamma}(-)) : \mathbb{F}_\beta(-) \to \mathbb{F}_\gamma(-)$ satisfies the conditions required in Theorem 1.8. ■

It seems to us that following the idea explained at the beginning of Section 5 one can solve the following problem, which extends Theorem 5.8 from K-algebras A to arbitrary rings A with an identity element.

Problem 6.3. *Prove that for every ring A with an identity element and for every poset I of infinite prinjective type there exist full, faithful and exact functors*

$$T : \mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & A^2 \\ 0 & A \end{pmatrix} \to \mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(AI), \quad T' : \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & A^2 \\ 0 & A \end{pmatrix} \to \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(AI).$$

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