COLLOQUIUM MATHEMATICUM

VOL. 75 1998 NO. 2

THE IMAGINARY CYCLIC SEXTIC FIELDS WITH CLASS NUMBERS EQUAL TO THEIR GENUS CLASS NUMBERS

BY

STÉPHANE LOUBOUTIN (CAEN)

It is known that there are only finitely many imaginary abelian number fields with class numbers equal to their genus class numbers. Here, we determine all the imaginary cyclic sextic fields with class numbers equal to their genus class numbers.

Introduction. Let K be an abelian number field of degree $n_K=[K:Q]$. The *narrow genus field* G_K of K is the maximal abelian number field containing K and unramified above K at all the finite places. According to class field theory, G_K is a subfield of the narrow Hilbert class field H_K^+ of K and the degree $g_K=[G_K:K]$ divides the narrow class number h_K^+ of K. When the group X_K of Dirichlet characters associated with an abelian number field K is given we can easily compute the degree of G_K and the genus class number g_K of K: we have

$$g_K=\frac{1}{n_K}\prod_p e_p$$

where this product ranges over all the rational primes p which are ramified in K and where e_p denotes the index of ramification of p in the extension K/Q (see Chapter 3 in [Wa]). Note that if K is imaginary then G_K is an imaginary abelian number field and G_K/K is unramified at all the places. In particular, if K is imaginary then g_K divides h_K, the class number of K. In [Lou 4] we proved that there are only finitely many imaginary abelian number fields such that their class numbers h_N are equal to their genus class numbers g_N and proved that apart from the quadratic and bicyclic quadratic ones, one can find an effective upper bound on their conductors. The aim of the present paper is to determine all the imaginary cyclic sextic fields with class numbers equal to their genus class numbers:

1991 *Mathematics Subject Classification*: Primary 11R20, 11R29; Secondary 11R42.

Key words and phrases: genus field, relative class number, class number, sextic number field.

THEOREM 1. *There are* 32 *imaginary cyclic sextic fields with class numbers equal to their genus class numbers. Their conductors are less than or equal to* 247 *and their class numbers are less than or equal to* 6. *They are listed in Table* 1 *of Section* 3.

To prove this theorem we first use lower bounds on relative class numbers of imaginary cyclic sextic fields (see Theorem 5). Second, according to Proposition 3, relative class numbers of imaginary cyclic sextic fields with class numbers equal to their genus class numbers cannot be that large. Hence, we will thirdly get upper bounds on the conductors of the imaginary cyclic sextic fields with class numbers equal to their genus class numbers (see Theorems 5 and 6). Fourth, we will compute the relative class numbers of all the possible imaginary cyclic sextic fields of conductors less than or equal to these upper bounds. This will provide us with a short list of imaginary cyclic sextic fields of conductors less than or equal to this bound and with relative class numbers satisfying the necessary condition of Proposition 3. Finally, we will determine all the fields of this small list which are such that their class numbers are equal to their genus class numbers, which will complete the proof of Theorem 1.

LEMMA 2. *Let N be an imaginary abelian number field with maximal real subfield N^+. Let t denote the number of rational primes p such that all the ideals of N^+ above p are ramified in the quadratic extension N/N^+ If $h_N = g_N$ then*

$$h_N^- = 2^{t-1+\varepsilon},$$

where $\varepsilon = 0$ or 1 according as G_{N^+}, the narrow genus field of N^+, is real or imaginary.

Proof. Since $h_N = g_N$, we have $H_N = G_N$. Let $H_{N^+}^+$ denote the narrow Hilbert class field of N^+. Since $H_{N^+}^+/N^+$ is an abelian extension unramified at all the finite places and since N is imaginary, it follows that $H_{N^+}^+N/N$ is an abelian extension unramified at all the places. Therefore, we have $H_{N^+}^+N \subseteq H_N = G_N$ and $H_{N^+}^+$ is an abelian field. Hence, $H_{N^+}^+ = G_{N^+}$.

Now, if G_{N^+} is real we have $H_{N^+} = G_{N^+}$ and we get

$$h_{N^+} = [H_{N^+} : N^+] = [G_{N^+} : N^+] = \frac{1}{n}\prod_p e_p^+,$$

which together with $h_N = g_N = \frac{1}{2n}\prod_p e_p$ yields

$$h_N^- = \frac{1}{2}\prod_p (e_p/e_p^+)$$

and the desired result.

In the same way, if G_{N^+} is imaginary and if we let $G_{N^+}^+$ denote its maximal real subfield then $H_{N^+} = G_{N^+}^+$ and we get

$$h_{N^+} = [H_{N^+} : N^+] = [G_{N^+}^+ : N^+] = \frac{1}{2}[G_{N^+} : N^+] = \frac{1}{2n}\prod_p e_p^+,$$

which together with $h_N = g_N = \frac{1}{2n}\prod_p e_p$ yields

$$h_N^- = \prod_p (e_p/e_p^+)$$

and the desired result. ∎

1. Imaginary cyclic sextic fields. Let N denote a cyclic sextic field, w_N its number of roots of unity, f_N its conductor, χ_N any of the two primitive Dirichlet characters of order 6 associated with N, h_N its class number, N^+ its cyclic cubic subfield, f_{N^+} the conductor and h_{N^+} the class number of its cyclic cubic subfield, k its quadratic subfield, f_k the conductor and h_k the class number of its quadratic subfield. Then N is real or imaginary according as $\chi_N(-1) = +1$ or $\chi_N(-1) = -1$. From now on, we assume N is imaginary. Then k is imaginary, h_{N^+} divides h_N and $h_N^- = h_N/h_{N^+}$ is the relative class number of N. Theorem 4.17 of [Wa] yields

$$(1) \qquad h_N^- = h_k \frac{w_N}{w_k}|\tau_\chi|^2 \quad \text{with} \quad \tau_\chi = \frac{1}{2f_N}\sum_{x=1}^{f_N-1} x\chi_N(x).$$

Note that h_k always divides h_N^- and that $N \neq Q(\zeta_7), Q(\zeta_9)$ implies $w_N = w_k$ (see [Lou 1]), whereas $N = Q(\zeta_7), Q(\zeta_9)$ yields $h_N = g_N = 1$.

PROPOSITION 3. *Let N be an imaginary cyclic sextic field, let k be its imaginary quadratic subfield and let r_k be the number of prime divisors of f_k, the conductor of k. If $h_N = g_N$ then*

$$h_N^- = 2^{r_k-1} = g_k = h_k.$$

In particular, if $N \neq Q(\zeta_7), Q(\zeta_9)$ then $|\tau_\chi|^2$ is a positive integer and $h_N = g_N$ implies

$$|\tau_\chi|^2 = 1.$$

Proof. If $h_N = g_N$ then $h_k = g_k$. However, $g_k = 2^{r_k-1}$, and we get the desired results. ∎

Now we explain in detail how we determine all the imaginary cyclic sextic fields with conductors less than or equal to a precribed upper bound, and how we compute their relative class numbers, i.e., how we compute the values of their associated sextic characters. Using the factorization $\chi = \prod_{p|f_N}\chi_p$ corresponding to the decomposition $f = \prod_{p|f} p^{\nu_p(f_N)}$ and noticing that a primitive cubic character of conductor a p-power has conductor 9 or and odd prime $p \equiv 1 \pmod 6$, we get:

PROPOSITION 4. *Let f_N be the conductor and χ_N be any of the two conjugate primitive sextic characters associated with an imaginary cyclic sextic field N. Then $f_N = f_2 f_3$ where*

$$f_2 = \prod_{i=1}^{r} p_i$$

with $r \geq 1$ and $p_i = 4, 8$ or $p_i \equiv 1 \pmod 2$ a prime,

$$f_3 = \prod_{j=1}^{s} q_j$$

with $s \geq 1$ and $q_j = 9$ or $q_j \equiv 1 \pmod 6$ a prime; further,

$$\chi_N = \chi_2^- \chi_3^+$$

where χ_2^- is an odd primitive quadratic character modulo f_2 and χ_3^+ is an even primitive cubic character modulo f_3. Moreover, $f_k = f_2$ and $f_{N^+} = f_3$. Finally, we have

$$g_N = 2^{r-1} 3^{s-1}. \tag{2}$$

Now, we explain how to compute the values taken on by χ_3^+.

For any prime $q \equiv 1 \pmod 6$ let $\chi_q^{(3)}$ denote any one of the two conjugated characters of order 3 and conductor q. Note that $\chi_q^{(3)}(-1) = +1$. For numerical computation, whenever $q \equiv 1 \pmod 6$ is prime we chose for $\chi_q^{(3)}$ the cubic character of conductor q defined by means of $\chi_q^{(3)}(a_q) = j = \exp(2\pi i/3)$ where

$$a_q = \min\{a \geq 1 : a^{(q-1)/3} \not\equiv 1 \pmod q\}.$$

Setting

$$b_q = a_q^{(q-1)/3}$$

we can easily compute $\chi_q^{(3)}(n)$ for we have

$$\chi_q^{(3)}(n) = j^k \Leftrightarrow k = \min\{k \in \{0,1,2\} : n^{(q-1)/3} \equiv b_q^k \pmod q\}.$$

Now, we let $\chi_9^{(3)}$ be the even cubic character of conductor 9 defined by $\chi_9^{(3)}(2) = j = \exp(2\pi i/3)$ (note that 2 generate the multiplicative cyclic group $(\mathbb{Z}/9\mathbb{Z})^*$). We have the following table of values of $\chi_9^{(3)}$:

n	1	2	4	5	7	8
$\chi_9^{(3)}(n)$	1	j	j^2	j^2	j	1

Since $\chi_9^{(3)}$ and $\overline{\chi_9^{(3)}}$ are the only even primitive cubic characters of 3-power conductors and since $\overline{\chi} = \chi_2\overline{\chi}_3$ and $\chi = \chi_2\chi_3$ are associated with the same cyclic sextic field, we may assume that

$$\chi_3^+ = \prod_{q|f_3} (\chi_q^{(3)})^{e_p}$$

with $e_q \in \{1, 2\}$, and $e_9 = 1$ if 9 divides f_2, and any one of the e_q's is equal to one if 9 does not divide f_3.

Proposition 4, formula (1) and the previous description of χ_3 make it easy to determine all the imaginary cyclic sextic fields of conductors f_N less than or equal to a prescribed upper bound, and to compute their relative class numbers.

The aim of the next section is to get a reasonable upper bound on the conductors of the imaginary cyclic sextic fields whose class numbers h_N are equal to their genus numbers g_N.

2. Lower bounds on relative class numbers of imaginary cyclic sextic fields

THEOREM 5. *Let N be an imaginary cyclic sextic field of conductor f_N and imaginary quadratic subfield k. Assume $f_N > 5 \cdot 10^5$. We have*

$$h_N^- \geq h_k \frac{f_N}{7300 \log^2(f_N/\pi)}. \tag{3}$$

In particular, if $h_N = g_N$ then $f_N \leq 1.3 \cdot 10^6$.

Proof. The lower bound (3) follows from [Lou 1]. Now, according to Proposition 3, if $h_N = g_N$ then $h_N^- = h_k$ and using (3) we get $7300 \geq f_N/\log^2(f_N/\pi)$, which yields the desired bound $f_N \leq 1.3 \cdot 10^6$. ∎

Now, when using formula (1), the time required to compute on a micro-computer the relative class numbers of all the imaginary cyclic sextic fields of conductors less than or equal to a prescribed upper bound B goes to infinity at least quadratically with B. Therefore, we will now explain how we can get a much better bound on the conductors of the imaginary cyclic sextic fields N such that $h_N = g_N$. Assume then that $h_N = g_N$. First, we have $h_k = g_k$ (Proposition 3) and $f_k \leq f_N \leq 1.3 \cdot 10^6$ (Theorem 5). Now, since $h_k = g_k$, the ideal class group of k has exponent ≤ 2 and if p is any prime which splits in k, say $(p) = \mathcal{P}\mathcal{P}'$, then $\mathcal{P}^2$ is principal and there exists an algebraic integer $\alpha = (x + y\sqrt{-f_k}\,)/2 \in k$ such that $(\alpha) = \mathcal{P}^2$. Moreover, one can easily see that y is not equal to zero. Taking norms, we get $p^2 = (x^2 + f_k y^2)/4 \geq f_k/4$. Therefore, if $h_N = g_N$ then $f_k \leq 1.3 \cdot 10^6$ and all the rational primes p less than $\sqrt{f_k/4}$ do not split in k. Now, one can easily check that there are only 65 imaginary quadratic fields $k = Q(\sqrt{-f_k}\,)$ with conductors $f_k \leq 1.3 \cdot 10^6$ such that $p \leq \sqrt{f_k/4}$ implies $(-f_p/p) \neq +1$ (Kronecker's symbol), the largest one being $f_k = 5460$. We note that all these 65 imaginary quadratic fields are such that $h_k = g_k = 2^{r_k - 1}$. Second, according to [Low] the Dedekind zeta functions of these 65 imaginary quadratic fields satisfy $\zeta_k(s) < 0$, $0 < s < 1$. Since

$$\zeta_N(s) = \zeta_k(s) L(s,\chi_N) L(s,\chi_N^5) L(s,\chi_{N^+}) L(s,\chi_{N^+}^2),$$

we see that $0 < s < 1$ yields

$$\zeta_N(s) = \zeta_k(s) |L(s,\chi_N)|^2 |L(s,\chi_{N^+})|^2.$$

In particular, if $h_N = g_N$ then $\zeta_N(s) < 0$, $0 < s < 1$, and according to [Lou 2] and [Lou 3], we get

$$h_N^- \geq \varepsilon_N \frac{w_N}{e\pi^3} \cdot \frac{f_N \sqrt{f_k}}{(\log f_{N^+} + 0.05)^2 \log d_N} \geq \eta_N \frac{w_k \sqrt{f_k}}{5e\pi^3} \cdot \frac{f_N}{(\log f_N + 0.05)^3}. \tag{4}$$

where

$$\varepsilon_N = 1 - (6\pi e^{1/3}/d_N^{1/6}) \quad \text{and} \quad \eta_N = 1 - (6\pi e^{1/3}/\sqrt{f_N})$$

(note that $f_N^3 \leq d_N \leq f_N^5$). Since $h_N^- = h_k = 2^{r_k - 1}$ and since we clearly have

$$2^{1-r_k} w_k \sqrt{f_k} \geq \sqrt{15}$$

(and equality holds if $f_k = 15$), we get

$$1 \geq \eta_N \frac{\sqrt{15}}{5e\pi^3} \cdot \frac{f_N}{(\log f_N + 0.05)^3}, \tag{5}$$

which yields:

THEOREM 6. *Let N be an imaginary cyclic sextic field of conductor f_N. If $h_N = g_N$ then $f_N \leq 220000$ and the imaginary quadratic subfield k of N is one of the* 65 *imaginary quadratic fields k of conductor $f_k \leq 5460$ such that $h_k = g_k$.*

3. Conclusion and table. According to numerical computations based on Proposition 4 and on the results of Section 1, there are 426451 imaginary cyclic sextic fields N of conductors $f_N \leq 220000$. Moreover, only 94569 out of them are such that their imaginary quadratic subfields k satisfy $h_k = g_k$. Now, thanks to the computation of τ_χ for all these 94569 imaginary sextic fields, we find that the necessary condition $|\tau_\chi|^2 = 1$ for $h_N = g_N$ (see Proposition 3) is satisfied for only 32 out of them: those listed in Table 1. These computations were done by using Pr. Y. Kida's UBASIC and (1). In particular, we did not use PARI and we do not have to assume any Riemann hypothesis to warrant the results of these computations. For all these 32 fields we used [Gra] to determine the class numbers h_{N^+} of their maximal real subfields N^+ (note that N^+ is a cyclic cubic field). Here again, we do not have to assume any Riemann hypothesis to warrant these class number determinations. Since our table yields $h_N = g_N$ for these 32 fields N (use (2)), Theorem 1 is proved.

To conclude, we note that, thanks to very good lower bounds for relative class numbers of non-quadratic imaginary cyclic fields of 2-power degree, we were able to determine in [Lou 5] all the non-quadratic imaginary cyclic fields of 2-power degree with class numbers equal to their genus class numbers. Here, in Section 2, we had to be a little more clever to get a reasonable bound on the conductors of the imaginary cyclic sextic fields with class numbers equal to their genus class numbers.

Table 1

Case	f_N	$f_k = f_2$	$f_{N^+} = f_3$	g_N	h_N^-	$P_{N^+}(X)$	h_{N^+}	h_N
1	19	19	191	1	1	X^3+X^2-6X-7	1	1
2	21	3	7	1	1	X^3+X^2-2X-1	1	1
3	28	4	7	1	1	X^3+X^2-2X-1	1	1
4	35	$35=5\cdot 7$	7	2	2	X^3+X^2-2X-1	1	2
5	36	4	9	1	1	X^3-3X+1	1	1
6	39	3	13	1	1	X^3+X^2-4X+1	1	1
7	43	43	43	1	1	$X^3+X^2-14X-1$	1	1
8	45	$15=3\cdot 5$	9	2	2	X^3-3X+1	1	2
9	52	$52=4\cdot 13$	13	2	2	X^3+X^2-4X+1	1	2
10	56	8	7	1	1	X^3+X^2-2X-1	1	1
11	63	3	$63=9\cdot 7$	3	1	$X^3-21X+35$	3	3
12	63	3	$63=9\cdot 7$	3	1	$X^3-21X-28$	3	3
13	63	7	9	1	1	X^3-3X+1	1	1
14	63	7	$63=9\cdot 7$	3	1	$X^3-21X+35$	3	3
15	67	67	67	1	1	$X^3+X^2-22X+5$	1	1
16	72	$24=8\cdot 3$	9	2	2	X^3-3X+1	1	2
17	76	4	19	1	1	X^3+X^2-6X-7	1	1
18	77	11	7	1	1	X^3+X^2-2X-1	1	1
19	84	$84=4\cdot 3\cdot 7$	7	4	4	X^3+X^2-2X-1	1	4
20	91	7	13	1	1	X^3+X^2-4X+1	1	1
21	91	7	$91=7\cdot 13$	3	1	$X^3-X^2-30X-27$	3	3
22	91	$91=7\cdot 13$	$91=7\cdot 13$	6	2	$X^3-X^2-30X-27$	3	6
23	91	$91=7\cdot 13$	$91=7\cdot 13$	6	2	$X^3-X^2-30X+64$	3	6
24	93	3	31	1	1	$X^3+X^2-10X-8$	1	1
25	104	8	13	1	1	X^3+X^2-4X+1	1	1
26	105	$15=3\cdot 5$	7	2	2	X^3+X^2-2X-1	1	2
27	117	3	$117=9\cdot 13$	3	1	$X^3-39X+26$	3	3
28	129	3	43	1	1	$X^3+X^2-14X-1$	1	1
29	133	7	$133=7\cdot 19$	3	1	$X^3-X^2-44X-69$	3	3
30	171	19	$171=9\cdot 19$	3	1	$X^3-57X+152$	3	3
31	217	7	$217=7\cdot 31$	3	1	$X^3-X^2-72X+225$	3	3
32	247	19	$247=13\cdot 19$	3	1	$X^3-X^2-82X+64$	3	3

REFERENCES

[Gra] M. N. Gras, *Méthodes et algorithmes pour le calcul numérique du nombre de classes et des unités des extensions cubiques cycliques de Q*, J. Reine Angew. Math. 277 (1975), 89–116.

[Lou 1] S. Louboutin, *Minoration au point 1 des fonctions L et détermination des corps sextiques abéliens totalement imaginaires principaux*, Acta Arith. 62 (1992), 109–124.

[Lou 2] —, *Majorations explicites de* $|L(1,\chi)|$, C. R. Acad. Sci. Paris Sér. I Math. 316 (1993), 11–14.

[Lou 3] —, *Lower bounds for relative class numbers of CM-fields*, Proc. Amer. Math. Soc. 120 (1994), 425–434.

[Lou 4] —, *A finiteness theorem for imaginary abelian number fields*, Manuscripta Math. 91 (1996), 343–352.

[Lou 5] —, *The nonquadratic imaginary cyclic fields of* 2*-power degrees with class numbers equal to their genus numbers*, Proc. Amer. Math. Soc., to appear.

[Low] M. E. Low, *Real zeros of the Dedekind zeta function of an imaginary quadratic field*, Acta Arith. 14 (1968), 117–140.

[Miy] I. Miyada, *On imaginary abelian number fields of type* $(2, 2, \ldots, 2)$ *with one class in each genus*, Manuscripta Math. 88 (1995), 535–540.

[PK] Y.-H. Park and S.-H. Kwon, *Determination of all imaginary abelian sextic number fields with class number* ≤ 11, Acta Arith., to appear.

[Wa] L. C. Washington, *Introduction to Cyclotomic Fields*, Grad. Texts in Math. 83, Springer, 1982.

[Yam] K. Yamamura, *The determination of the imaginary abelian number fields with class-number one*, Math. Comp. 62 (1994), 899–921.

Département de Mathématiques
Université de Caen, UFR Sciences
14032 Caen Cedex, France
E-mail: loubouti@math.unicaen.fr

Received 3 April 1997;
revised 14 May 1997