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HEREDITARILY WEAKLY CONFLUENT INDUCED MAPPINGS ARE HOMEOMORPHISMS

BY

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For a given mapping f between continua we consider the induced mappings between the corresponding hyperspaces of closed subsets or of subcontinua. It is shown that if either of the two induced mappings is hereditarily weakly confluent (or hereditarily confluent, or hereditarily monotone, or atomic), then f is a homeomorphism, and consequently so are both the induced mappings. Similar results are obtained for mappings between cones over the domain and over the range continua.

1. Introduction. For a compact metric space X we denote by 2^X and $C(X)$ the hyperspaces of all nonempty closed and of all nonempty closed connected subsets of X, respectively. Given a mapping $f : X \to Y$ between continua, we let $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ denote the induced mappings. Let $\mathfrak{M}_i$, where $i \in \{1, 2, 3\}$, be some three classes of mappings between continua. A general problem is to find all interrelations between the following three statements:

(1.1) $f \in \mathfrak{M}_1$;
(1.2) $C(f) \in \mathfrak{M}_2$;
(1.3) $2^f \in \mathfrak{M}_3$.

For particular results concerning this problem for various classes $\mathfrak{M}_i$ of mappings like open, monotone, confluent and some others, see [2], [3], [5]–[10], [12]. In the present paper we prove that if either of the two induced mappings is atomic, or hereditarily monotone, or hereditarily confluent, or hereditarily weakly confluent, then f is a homeomorphism (and consequently the induced mappings are also homeomorphisms). The final part of the paper deals with mappings between cones over continua X and Y. Namely, if (instead of the induced mappings between hyperspaces) a mapping between

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cones over the domain and over the range continua is considered, then the same conclusion is true when the mapping is either hereditarily joining or hereditarily atriodic.

2. Preliminaries. All spaces considered in this paper are assumed to be metric. A *mapping* means a continuous function. A *continuum* means a compact connected space. Given a continuum X with a metric d, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A,B) = \max\{\sup\{d(a,B) : a \in A\}, \sup\{d(b,A) : b \in B\}\}$$

(equivalently: with the Vietoris topology; see e.g. [12, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X, and by $F_1(X)$ the hyperspace of the singletons of X. The reader is referred to Nadler's book [12] for information on the structure of the hyperspaces. In particular, the following facts are well known (see [12, Theorem (1.13), p. 65]).

2.1. Fact. *For each continuum X the hyperspaces 2^X and $C(X)$ are arcwise connected.*

2.2. Fact. *For each continuum X the hyperspace $C(X)$ is a subcontinuum of 2^X.*

The next property of hyperspaces is a consequence of the definitions.

2.3. Proposition. *Let K be a nonempty closed subset of a continuum X. Then $2^K \subset 2^X$ and $C(K) \subset C(X)$.*

By an *order arc* in 2^X we mean an arc Φ in 2^X such that if $A, B \in \Phi$, then either $A \subset B$ or $B \subset A$. The following facts are known (see [12, Theorem (1.8), p. 59 and Lemma (1.11), p. 64]).

2.4. Fact. *Let $A, B \in 2^X$ with $A \neq B$. Then there exists an order arc in 2^X from A to B if and only if $A \subset B$ and each component of B intersects A.*

2.5. Fact. *If an order arc Φ in 2^X begins with $A \in C(X)$, then $\Phi \subset C(X)$.*

Given a mapping $f : X \to Y$ between continua, we consider the *induced mappings* $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ defined by

$$2^f(A) = f(A) \quad \text{for } A \in 2^X, \qquad C(f)(A) = f(A) \quad \text{for } A \in C(X).$$

Thus, by Fact 2.2, the following is obvious.

2.6. Fact. *For any continua X and Y and for every mapping $f : X \to Y$ we have $2^f|C(X) = C(f)$.*

The proof of the following fact is straightforward.

2.7. FACT. *Let $f : X \to Y$ be a mapping between continua. If K is a nonempty subcontinuum of X, then $2^{f|K} = 2^f|2^K$ and $C(f|K) = C(f)|C(K)$.*

3. Some hereditary classes of mappings. Given a class $\mathfrak{M}$ of mappings between continua, a mapping $f : X \to Y$ between continua is said to be *hereditarily* $\mathfrak{M}$ provided that for each subcontinuum K of X the partial mapping $f|K : K \to f(K) \subset Y$ is in $\mathfrak{M}$.

Recall that a mapping $f : X \to Y$ between continua is said to be *monotone* if the inverse image of each point of Y is connected. It is known that if f is monotone, then so is $C(f)$ (see [10, Theorem 1.1, p. 121]; compare [6, Theorem 3.3, p. 4]). Therefore, as a consequence of Proposition 2.3 and Fact 2.7 we have the following result.

3.1. PROPOSITION. *If a surjective mapping $f : X \to Y$ between continua is hereditarily monotone, then for each subcontinuum K of X the partial induced mapping*

$$C(f)|C(K) : C(K) \to C(f)(C(K)) \subset C(Y)$$

is monotone.

A natural question arises about possible inverse implications, that is, about implications from the hereditary monotonicity of either 2^f or $C(f)$ to properties of f. A partial answer to this question is known. Namely, under an additional assumption that $C(f)$ is atomic, it can be deduced that f must be a homeomorphism (see [5, p. 2]). However, a much stronger result can be obtained.

Recall that a mapping $f : X \to Y$ from a continuum X onto Y is said to be *atomic* provided that, for each subcontinuum K of X, either $f(K)$ is degenerate, or $f^{-1}(f(K)) = K$. This notion was introduced by R. D. Anderson [1] to describe special decompositions of continua. Soon, atomic mappings turned out to be important tools in continuum theory and proved to be interesting in themselves; several properties of these mappings have been discovered, e.g. in [4], [5] and [11]. The following two facts on atomic mappings are known. The first is an immediate consequence of the definitions; for the proof of the second, see [4, Theorem 1, p. 49] and [11, (4.14), p. 17].

3.2. FACT. *Every atomic mapping of a continuum is hereditarily atomic.*

3.3. FACT. *Every atomic mapping of a continuum is hereditarily monotone.*

The stronger result mentioned above consists in replacing the atomicity or hereditary monotonicity of $C(f)$ by a much weaker property, namely by the hereditary weak confluence. Recall that a mapping $f : X \to Y$ between continua is said to be *weakly confluent* if for each subcontinuum Q of Y there is a component K of $f^{-1}(Q)$ for which $f(K) = Q$. If the equality is true for each component K of $f^{-1}(Q)$, then f is said to be *confluent*.

The following table illustrates inclusions between these and some other classes of mappings used throughout the paper.

atomic	$\Rightarrow$	monotone				
		$\Downarrow$				
open	$\Rightarrow$	confluent	$\Rightarrow$	semi-confluent	$\Rightarrow$	weakly confluent
				$\Downarrow$		$\Downarrow$
				joining		atriodic

3.4. THEOREM. *Let $f : X \to Y$ be a surjective mapping between continua. If the induced mapping $C(f) : C(X) \to C(Y)$ is hereditarily weakly confluent, then f is a homeomorphism.*

P r o o f. Suppose that f is not a homeomorphism. Then there are points $a, b \in X$ such that $f(a) = f(b)$. We claim that

(3.5) for any $A, B \in C(X)$ with $a \in A$, $b \in B$ and $A \cap B = \emptyset$ we have either $f(A) \subset f(B)$ or $f(B) \subset f(A)$.

Indeed, if (3.5) does not hold, then there are disjoint subcontinua A and B of X with $a \in A$ and $b \in B$ for which neither of the two inclusions of (3.5) is true. By Facts 2.4 and 2.5 there are four order arcs in $C(X)$: $\mathcal{A}_1$ from $\{a\}$ to A, $\mathcal{A}_2$ from A to X, $\mathcal{B}_1$ from $\{b\}$ to B, and $\mathcal{B}_2$ from B to X. Since $A \cap B = \emptyset$ we have $\mathcal{A}_1 \cap \mathcal{B}_1 = \mathcal{A}_1 \cap \mathcal{B}_2 = \mathcal{A}_2 \cap \mathcal{B}_1 = \emptyset$. By assumption we have $f(A) \in C(f)(\mathcal{A}_1) \setminus C(f)(\mathcal{B}_1)$, and similarly $f(B) \in C(f)(\mathcal{B}_1) \setminus C(f)(\mathcal{A}_1)$. Put, for shortness, $\mathcal{L} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_2$. Striving for a contradiction, we will show that

(3.6) the partial mapping $C(f)|\mathcal{L}$ is not weakly confluent.

In fact, let $\mathcal{U}$ be an open connected subset of the arc $C(f)(\mathcal{B}_1)$ disjoint from $C(f)(\mathcal{A}_1 \cup \mathcal{A}_2)$. Then $\mathcal{Q} = C(f)(\mathcal{L}) \setminus \mathcal{U}$ is a subcontinuum of $C(f)(\mathcal{L})$, and no component of $(C(f)|\mathcal{L})^{-1}(\mathcal{Q})$ is mapped onto $\mathcal{Q}$ under $C(f)$. This shows (3.6), which is our required contradiction; thereby claim (3.5) is shown.

The following is a consequence of (3.5).

(3.7) Let $P, Q \in C(X)$ with $P \cap Q = \emptyset$. If $\mathcal{A}$ and $\mathcal{B}$ are order arcs in $C(X)$ from $\{a\}$ to P and from $\{b\}$ to Q respectively, where $f(a) = f(b)$, then either $C(f)(\mathcal{A}) \subset C(f)(\mathcal{B})$ or $C(f)(\mathcal{B}) \subset C(f)(\mathcal{A})$.

Since the mapping f is not constant, there is a point $c \in X$ such that $f(c) \neq f(a) = f(b)$. Denote by $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ order arcs in $C(X)$ from the singletons $\{a\}$, $\{b\}$ and $\{c\}$, respectively, to X. By (3.7) the intersection $C(f)(\mathcal{A}) \cap C(f)(\mathcal{B})$ contains an order arc from $\{f(a)\}$ to a continuum $T \in C(Y)$. Choose continua K and L in X such that $K \in \mathcal{A} \setminus \mathcal{B}$ and $L \in \mathcal{B} \setminus \mathcal{A}$

with $K \cap L = \emptyset$ and $f(K) \subsetneq f(L) \subsetneq T$. Let $\mathcal{U}$ be an open neighborhood of K in $\mathcal{A}$ and $\mathcal{V}$ be an open neighborhood of L in $\mathcal{B}$ such that

$$\mathcal{U} \cap \mathcal{B} = \emptyset = \mathcal{V} \cap \mathcal{A}, \quad C(f)(\mathcal{U}) \cap C(f)(\mathcal{V}) = \emptyset,$$

and $C(f)(\mathcal{V}) \subset C(f)(\mathcal{A})$. Note that, since $f(K) \subsetneq f(L)$, we have $C(f)(\mathcal{U}) \subset C(f)(\mathcal{B})$. Putting

$$\mathcal{P} = F_1(X) \cup (\mathcal{A} \setminus \mathcal{U}) \cup (\mathcal{B} \setminus \mathcal{V}) \cup \mathcal{C}$$

we see that $\mathcal{P}$ is a subcontinuum in $C(X)$. Observe that by the definitions of $\mathcal{U}$ and $\mathcal{V}$ we have

$$C(f)(\mathcal{P}) = F_1(Y) \cup C(f)(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}).$$

We will show that $C(f)|\mathcal{P}$ is not weakly confluent. Let C be a subcontinuum of X such that $C \in \mathcal{C}$, $f(C)$ is nondegenerate, and $f(C) \cap \{f(a)\} = \emptyset$. Put $\mathcal{S} = \{S \in C(f)(\mathcal{C}) : \{f(c)\} \subsetneq S \subsetneq f(C)\}$. Thus $\mathcal{S}$ is an open connected subset of the arc $C(f)(\mathcal{C})$ and $\mathcal{S} \cap C(f)(\mathcal{A} \cap \mathcal{B}) = \emptyset$. Define

$$\mathcal{Q} = F_1(Y) \cup C(f)((\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) \setminus \mathcal{S}).$$

Then $\mathcal{Q}$ is a subcontinuum of $C(f)(\mathcal{P})$ and $(C(f)|\mathcal{P})^{-1}(\mathcal{Q})$ has two components: one, $\mathcal{K}_1$, containing $F_1(X)$, and the other, $\mathcal{K}_2$, containing X. Observe that $Y \notin C(f)(\mathcal{K}_1)$, and $\{f(a)\} \notin C(f)(\mathcal{K}_2)$, so $C(f)|\mathcal{P}$ is not weakly confluent indeed. The proof is thus finished.

3.8. COROLLARY. *Let $f : X \to Y$ be a surjective mapping between continua. If the induced mapping $2^f : 2^X \to 2^Y$ is hereditarily weakly confluent, then f is a homeomorphism.*

Proof. This is a consequence of Theorem 3.4 and Fact 2.6.

Recall one more definition. A mapping $f : X \to Y$ between continua is said to be *semi-confluent* if for each subcontinuum Q of Y and for any two components K_1 and K_2 of $f^{-1}(Q)$ either $f(K_1) \subset f(K_2)$ or $f(K_2) \subset f(K_1)$. The following two facts on this class of mappings will be needed. The first is a consequence of the definitions. For the proof of the second, see [11, Theorem 3.8, p. 13].

3.9. FACT. *Each confluent mapping is semi-confluent.*

3.10. FACT. *Each semi-confluent mapping is weakly confluent.*

3.11. THEOREM. *Let $f : X \to Y$ be a surjective mapping between continua. Then the following conditions are equivalent.*

(1) *f is a homeomorphism*;
(2) *2^f is a homeomorphism*;
(3) *2^f is atomic*;
(4) *2^f is hereditarily monotone*;
(5) *2^f is hereditarily confluent*;

(6) 2^f *is hereditarily semi-confluent*;
(7) 2^f *is hereditarily weakly confluent*;
(8) $C(f)$ *is a homeomorphism*;
(9) $C(f)$ *is atomic*;
(10) $C(f)$ *is hereditarily monotone*;
(11) $C(f)$ *is hereditarily confluent*;
(12) $C(f)$ *is hereditarily semi-confluent*;
(13) $C(f)$ *is hereditarily weakly confluent.*

Proof. Conditions (1), (2) and (8) are well known to be equivalent (see [12, Theorem (0.52), p. 29]). Consider the following two circles of implications:

$$(1)\Rightarrow(3)\Rightarrow(4)\Rightarrow(5)\Rightarrow(6)\Rightarrow(7)\Rightarrow(1),$$
$$(1)\Rightarrow(9)\Rightarrow(10)\Rightarrow(11)\Rightarrow(12)\Rightarrow(13)\Rightarrow(1).$$

Since each homeomorphism is obviously an atomic mapping, the implications (1)⇒(3) and (1)⇒(9) follow from the above mentioned equivalence of (1), (2) and (8). The implications (3)⇒(4) and (9)⇒(10) are consequences of Fact 3.3. Since each monotone mapping is obviously confluent, (4)⇒(5) and (10) ⇒(11) hold. (5)⇒(6) and (11)⇒(12) are consequences of Fact 3.9, and (6) ⇒(7) and (12)⇒(13) follow from Fact 3.10. Finally, (7)⇒(1) and (13)⇒(1) are just Corollary 3.8 and Theorem 3.4, respectively. The proof is complete.

As a consequence of Theorem 3.11 and of the equivalence of the monotonicity of f, 2^f and $C(f)$ (see [6, Theorem 3.3, p. 4]) we get the following result.

3.12. COROLLARY. *Let $f : X \to Y$ be a surjective mapping between continua. If f is atomic but not a homeomorphism, then the induced mappings 2^f and $C(f)$ are monotone but not hereditarily monotone, thus not atomic.*

A mapping $f : X \to Y$ between continua is said to be *joining* if for each subcontinuum Q of Y and for any two components K_1 and K_2 of $f^{-1}(Q)$ we have $f(K_1) \cap f(K_2) \neq \emptyset$. Then we have the following fact ([11, (3.4), p. 13]).

3.13. FACT. *Each semi-confluent mapping is joining.*

However, the mapping $\sin : [-3\pi/4, 3\pi/4] \to [-1, 1]$ is weakly confluent but not joining.

Further, f is said to be *atriodic* if for each subcontinuum Q of Y there are two components K_1 and K_2 of $f^{-1}(Q)$ such that $f(K_1) \cup f(K_2) = Q$ and for each component K of $f^{-1}(Q)$ either $f(K) = Q$, or $f(K) \subset f(K_1)$, or $f(K) \subset f(K_2)$. So, by the definitions, the next fact follows (compare [11, (3.5), p. 13]).

3.14. FACT. *Each weakly confluent mapping is atriodic.*

3.15. THEOREM. *Let X be a continuum and let $p, q \in X$. Put $Y = X/\{p, q\}$, and define $f : X \to Y$ to be the quotient mapping. Then f is both joining and atriodic.*

Proof. Take $Q \in C(Y)$ and note that $f^{-1}(Q)$ has at most two components: the one containing p and the one containing q. Their images have the point $\{p, q\}$ of Y in common, and the union of the images is Q. Thus f is joining and atriodic.

3.16. COROLLARY. *Let X be a continuum and let $p, q \in X$. Put $Y = X/\{p, q\}$, and define $f : X \to Y$ to be the quotient mapping. Then f is hereditarily joining and hereditarily atriodic.*

Let f be a mapping as in Theorem 3.15. Since the induced mapping $C(f)$ identifies the singletons $\{p\}$ and $\{q\}$ only, applying the previous corollary to $C(f)$, we get the next one.

3.17. COROLLARY. *Let X be a continuum and let $p, q \in X$. Put $Y = X/\{p, q\}$, and define $f : X \to Y$ to be the quotient mapping. Then $C(f)$ is hereditarily joining and hereditarily atriodic.*

Consequently, we see that Theorems 3.4 and 3.11 cannot be extended to hereditarily joining or to hereditarily atriodic induced mappings.

4. Cones. For a continuum X we define $\mathrm{cone}(X) = (X \times [0, 1])/(X \times \{0\})$. Points of $\mathrm{cone}(X)$ are denoted by (x, t) for $x \in X$ and $t \in [0, 1]$; in particular, $(x, 0) = (y, 0)$ for every $x, y \in X$. For a mapping $f : X \to Y$ between continua, we consider the mapping $\mathrm{cone}(f) : \mathrm{cone}(X) \to \mathrm{cone}(Y)$ defined by $\mathrm{cone}(f)(x, t) = (f(x), t)$.

Reading a previous version of the paper, the referee asked if an analog of Theorem 3.4 is true if the hyperspaces are replaced by cones. In Theorem 4.1 below we prove a more general result; namely the assumption of the hereditary weak confluence of the induced mapping can be replaced by its being hereditarily joining or hereditarily atriodic. Corollary 3.17 shows that one cannot go so far with the hyperspaces.

4.1. THEOREM. *Let $f : X \to Y$ be a surjective mapping between continua. If the mapping $\mathrm{cone}(f) : \mathrm{cone}(X) \to \mathrm{cone}(Y)$ is either hereditarily joining or hereditarily atriodic, then f is a homeomorphism.*

Proof. Suppose that f is not a homeomorphism. Then there are points $a, b \in X$ such that $f(a) = f(b)$. Choose points $c, d \in X$ such that $f(a), f(c)$ and $f(d)$ are three distinct points in Y. Let

$$\mathcal{C} = (X \times \{0, 1/2, 1\}) \cup (\{c, d\} \times [0, 1]) \cup (\{a\} \times ([0, 1/4] \cup [3/4, 1])) \cup (\{b\} \times [1/4, 3/4]).$$

Thus $\mathcal{C}$ is a subcontinuum of $\operatorname{cone}(X)$. We shall show that $\operatorname{cone}(f)|\mathcal{C}$ is neither joining nor atriodic. To this end consider the continuum

$$\mathcal{Q} = (\{f(a)\} \times [1/8, 7/8]) \cup (X \times \{1/2\}),$$

and note that $(\operatorname{cone}(f)|\mathcal{C})^{-1}(\mathcal{Q})$ has three components:

$$\mathcal{C}_1 = \{a\} \times [1/8, 1/4], \quad \mathcal{C}_2 = \{a\} \times [3/4, 7/8],$$

and

$$\mathcal{C}_3 = (\{b\} \times [1/4, 3/4]) \cup (X \times \{1/2\}).$$

The images of no two of them cover $\mathcal{Q}$, so $\operatorname{cone}(f)|\mathcal{C}$ is not atriodic. Moreover, $\operatorname{cone}(f)(\mathcal{C}_1) \cap \operatorname{cone}(f)(\mathcal{C}_2) = \emptyset$, so $\operatorname{cone}(f)|\mathcal{C}$ is not joining. The proof is finished.

As an application of Theorem 4.1 we have the following analog of Theorem 3.11.

4.2. THEOREM. *Let $f : X \to Y$ be a surjective mapping between continua. Then the following conditions are equivalent.*

(1) f *is a homeomorphism*;
(2) $\operatorname{cone}(f)$ *is a homeomorphism*;
(3) $\operatorname{cone}(f)$ *is atomic*;
(4) $\operatorname{cone}(f)$ *is hereditarily monotone*;
(5) $\operatorname{cone}(f)$ *is hereditarily confluent*;
(6) $\operatorname{cone}(f)$ *is hereditarily semi-confluent*;
(7) $\operatorname{cone}(f)$ *is hereditarily weakly confluent*;
(8) $\operatorname{cone}(f)$ *is hereditarily joining*;
(9) $\operatorname{cone}(f)$ *is hereditarily atriodic.*

Proof. The implications (1)⇒(2)⇒(3) are obvious; (3)⇒(4) follows from Fact 3.3; (4)⇒(5)⇒(6) are consequences of the definitions of the corresponding classes of mappings; (6)⇒(7) results from Fact 3.10, (7)⇒(9) from Fact 3.14, and (9) implies (1) by Theorem 4.1. Thus conditions (1)–(7) and (9) are equivalent. Finally, (6)⇒(8) follows from Fact 3.13, and (8) implies (1) again by Theorem 4.1. The proof is complete.

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