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RANK ADDITIVITY FOR QUASI-TILTED ALGEBRAS OF CANONICAL TYPE

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Given the category $\operatorname{coh}\mathbb{X}$ of coherent sheaves over a weighted projective line $\mathbb{X} = \mathbb{X}(\underline{\boldsymbol{\lambda}}, \underline{\mathbf{p}})$ (of any representation type), the endomorphism ring $\Sigma = \operatorname{End}(\mathcal{T})$ of an arbitrary tilting sheaf—which is by definition an *almost concealed canonical algebra*—is shown to satisfy a rank additivity property (Theorem 3.2). Moreover, this property extends to the representation-infinite *quasi-tilted algebras of canonical type* (Theorem 4.2). Finally, it is demonstrated that rank additivity *does not* generalize to the case of tilting complexes over $\operatorname{coh}\mathbb{X}$ (Example 4.3).

1. Introduction. Throughout, let k denote an algebraically closed field.

Let $\operatorname{coh}\mathbb{X}$ denote the category of coherent sheaves over a weighted projective line $\mathbb{X} = \mathbb{X}(\underline{\boldsymbol{\lambda}}, \underline{\mathbf{p}})$ (in the sense of [1]) attached to a weight sequence $\underline{\mathbf{p}} = (p_1, \ldots, p_t)$ of integers $p_i \geq 1$ and a parameter sequence $\underline{\boldsymbol{\lambda}} = (\lambda_1, \ldots, \lambda_t)$ of pairwise different points $\lambda_i \in \mathbb{P}^1(k)$. The category $\operatorname{coh}\mathbb{X}$ is an abelian k-category, and is hereditary since $\operatorname{Ext}^2(-,-) \equiv 0$; it has a tilting object $\mathcal{T}$ whose endomorphism ring is a canonical algebra Λ in the sense of [12]. Hence the Grothendieck group $\mathrm{K}_0\mathbb{X}$ is finitely generated and free; it is endowed with a bilinear form—the *Euler characteristic*—induced by

$$\langle E, F\rangle = \dim \operatorname{Hom}_k(E,F) - \dim \operatorname{Ext}^1(E,F) \quad \text{for } E, F \in \operatorname{coh}\mathbb{X}.$$

Furthermore, $\operatorname{coh}\mathbb{X}$ is $\mathbb{L}(\underline{\mathbf{p}})$-graded, where $\mathbb{L}(\underline{\mathbf{p}})$ denotes the rank-one abelian group generated by the t elements $\vec{x}_1, \ldots, \vec{x}_t$ subject to the relations

$$p_1\vec{x}_1 = \ldots = p_t\vec{x}_t \quad (=: \vec{c}).$$

Degree shift by the dualizing element $\vec{\omega} := (t-2)\vec{c} - \sum_{i=1}^{t} \vec{x}_i$ serves as the Auslander–Reiten translation τ for $\operatorname{coh}\mathbb{X}$; Serre duality $\mathrm{D}\operatorname{Ext}^1(E,F) \cong \operatorname{Hom}(F, E(\vec{\omega}))$ holds for any $E, F \in \operatorname{coh}\mathbb{X}$. The representation theory of

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$\operatorname{coh}\mathbb{X}$ depends on the number

$$\delta_\omega := (t-2) - \sum_{i=1}^{t} \frac{1}{p_i};$$

namely, $\operatorname{coh}\mathbb{X}$ is of *tame-domestic*, *tubular* or *wild* representation type depending on whether $\delta_\omega < 0, = 0$ or > 0.

The *radical* $\operatorname{rad}\mathrm{K}_0\mathbb{X}$ of the *quadratic form* $q(x) := \langle x, x\rangle$ associated with the Euler characteristic is a direct summand of $\mathrm{K}_0\mathbb{X}$; it is of rank 1 in case $\delta_\omega \neq 0$ and of rank 2 if $\delta_\omega = 0$. In either case, we can choose a rank-one summand $\mathbb{Z}\mathbf{w}$ of $\operatorname{rad}\mathrm{K}_0\mathbb{X}$ and define a linear function

$$\operatorname{rk} : \mathrm{K}_0\mathbb{X} \to \mathbb{Z}, \qquad x \mapsto \langle \mathbf{w}, x\rangle,$$

called the *rank function* on $\mathrm{K}_0\mathbb{X}$. In particular, rk is τ-invariant, since $\tau\mathbf{w} = \mathbf{w}$. Each indecomposable sheaf F in $\operatorname{coh}\mathbb{X}$ is either a sheaf of finite length or a vector bundle, according as $\operatorname{rk}(F) = 0$ or $\operatorname{rk}(F) > 0$.

DEFINITION 1.1. A sheaf $F \in \operatorname{coh}\mathbb{X}$ is called *exceptional* if $\operatorname{Ext}^1(F, F) = 0$ and $\operatorname{End}(F) \cong k$.

Thus any exceptional sheaf is indecomposable; on the other hand, by Lemma 1.2 below, for an *indecomposable* sheaf F the condition $\operatorname{Ext}^1(F, F) = 0$ is also sufficient for being exceptional:

LEMMA 1.2 [4]. *For two indecomposable objects $E, F \in \operatorname{coh}\mathbb{X}$ satisfying $\operatorname{Ext}^1(F, E) = 0$, any non-zero morphism $\varphi : E \to F$ is an epimorphism or a monomorphism.*

LEMMA 1.3. *Let (E, F) be an* exceptional pair, *i.e. E, F exceptional and* $\operatorname{Hom}(F, E) = 0 = \operatorname{Ext}^1(F, E)$. *Then* $\operatorname{Hom}(E, F) = 0$ *or* $\operatorname{Ext}^1(E, F) = 0$.

Proof. Let $u : E \to F$ be a non-zero morphism. If u is a monomorphism, application of $\operatorname{Ext}^1(-, F)$ along u yields $\operatorname{Ext}^1(E, F) = 0$ since $\operatorname{Ext}^1(F, F) = 0$. Otherwise, u is an epimorphism, and application of $\operatorname{Ext}^1(E, -)$ along u again yields $\operatorname{Ext}^1(E, F) = 0$, now since $\operatorname{Ext}^1(E, E) = 0$. ■

DEFINITION 1.4. A sheaf $\mathcal{T} = P_1 \oplus \ldots \oplus P_n \in \operatorname{coh}\mathbb{X}$, where P_i are pairwise non-isomorphic indecomposable, is called a *tilting sheaf* if

(a) $\operatorname{Ext}^1(\mathcal{T}, \mathcal{T}) = 0$;

(b) $\mathcal{T}$ generates $\operatorname{coh}\mathbb{X}$ in the sense of formation of direct sums, extensions, kernels of epimorphisms and cokernels of monomorphisms.

REMARK 1.5. It can be shown in the presence of (a) that condition (b) is equivalent to each of:

(b′) $\mathcal{T}$ generates the derived category $\mathcal{D}^\flat\operatorname{coh}\mathbb{X}$.

(b″) $n = \operatorname{rk}_{\mathbb{Z}}(\mathrm{K}_0\mathbb{X})$. (For this, note that we require $\mathcal{T}$ to be multiplicity-free!)

Denote by $\mathcal{D}^\flat \operatorname{coh}\mathbb{X}$ the *derived category* of $\operatorname{coh}\mathbb{X}$. Each indecomposable object $\mathcal{E} \in \mathcal{D}^\flat \operatorname{coh}\mathbb{X}$ is isomorphic to a stalk complex $E[n]$ (for some $E \in \operatorname{coh}\mathbb{X}$, $n \in \mathbb{Z}$); moreover, the formula

$$\operatorname{Hom}_{\mathcal{D}^\flat \operatorname{coh}\mathbb{X}}(E[n], F[m]) = \operatorname{Ext}^{m-n}_{\operatorname{coh}\mathbb{X}}(E, F) \tag{1.1}$$

reduces the control of morphisms in $\mathcal{D}^\flat \operatorname{coh}\mathbb{X}$ to $\operatorname{coh}\mathbb{X}$.

2. Almost concealed canonical algebras. Let $\mathcal{T} = P_1 \oplus \ldots \oplus P_n$ be a (multiplicity-free) tilting sheaf on $\operatorname{coh}\mathbb{X}$; $P_1, \ldots, P_n$ denote its (pairwise non-isomorphic) exceptional summands; these can be numbered in such a way that $\operatorname{Hom}(P_k, P_l) = 0$ for any pair $k > l$ (cf. Lemma 1.2). We say that $P_1, \ldots, P_n$ constitute an *exceptional sequence*, i.e. $\operatorname{Hom}(P_k, P_l) = 0 = \operatorname{Ext}(P_k, P_l)$ for any $k > l$; cf. [11].

Let $\Sigma := \operatorname{End}(\mathcal{T})$. An algebra arising as the endomorphism ring of a tilting sheaf on $\operatorname{coh}\mathbb{X}$ is called an *almost concealed canonical algebra* (cf. [8]). In the sequel, Σ-modules will be identified with *representations* of the quiver (with relations) underlying $\mathcal{T}$.

By [1], Σ has global dimension at most two; moreover, the categories $\operatorname{coh}\mathbb{X}$ and $\operatorname{mod}\Sigma$ are *derived equivalent*, in particular, we have the following equivalences:

$$\operatorname{Hom}_{\mathbb{X}}(\mathcal{T}, -) : \operatorname{coh}_+ \mathcal{T} \to \operatorname{mod}_+ \Sigma, \qquad F \mapsto \operatorname{Hom}_{\mathbb{X}}(\mathcal{T}, F),$$
$$\operatorname{Ext}^1_{\mathbb{X}}(\mathcal{T}, -) : \operatorname{coh}_- \mathcal{T}[1] \to \operatorname{mod}_- \Sigma, \qquad F[1] \mapsto \operatorname{Ext}^1_{\mathbb{X}}(\mathcal{T}, F).$$

Here,

$$\operatorname{coh}_+ \mathcal{T} := \{F \in \operatorname{coh}\mathbb{X} \mid \operatorname{Ext}^1_{\mathbb{X}}(\mathcal{T}, F) = 0\};$$
$$\operatorname{coh}_- \mathcal{T} := \{F \in \operatorname{coh}\mathbb{X} \mid \operatorname{Hom}_{\mathbb{X}}(\mathcal{T}, F) = 0\};$$

the Σ-modules from $\operatorname{mod}_+ \Sigma$ [$\operatorname{mod}_- \Sigma$ resp.] are called *torsionfree* [*torsion*] modules, and the pair $(\operatorname{mod}_+ \Sigma, \operatorname{mod}_- \Sigma)$ yields a *split torsion theory* for $\operatorname{mod}\Sigma$: each indecomposable Σ-module lies in exactly one of the two subcategroies. We have

$$\operatorname{Ext}^i_{\Sigma}(M, N) \cong \operatorname{Ext}^i_{\mathbb{X}}(M, N)$$

for $M, N \in \operatorname{mod}_+ \Sigma$ or $M, N \in \operatorname{mod}_- \Sigma$ resp.;

$$\operatorname{Ext}^i_{\Sigma}(M, N[1]) \cong \operatorname{Ext}^{i+1}_{\mathbb{X}}(M, N), \qquad \operatorname{Ext}^{i+1}_{\Sigma}(N[1], M) \cong \operatorname{Ext}^i_{\mathbb{X}}(N, M)$$

for $M \in \operatorname{mod}_+ \Sigma$, $N[1] \in \operatorname{mod}_- \Sigma$. In particular, for two indecomposable Σ-modules X, Y, $\operatorname{Ext}^2_{\Sigma}(X, Y) \neq 0$ can only occur for $X \in \operatorname{mod}_- \Sigma$ and $Y \in \operatorname{mod}_+ \Sigma$.

Furthermore,

$$[F] \mapsto [\operatorname{Hom}_{\mathbb{X}}(\mathcal{T}, F)] - [\operatorname{Ext}^1_{\mathbb{X}}(\mathcal{T}, F)]$$

induces an isomorphism $\mathrm{K}_0\mathbb{X} \xrightarrow{\sim} \mathrm{K}_0\Sigma$ between the corresponding Grothendieck groups (cf. [9]).

The exceptional direct summands $P_1, \ldots, P_n$ of $\mathcal{T}$ constitute a complete set of representatives of indecomposable *projective* Σ-modules. Let S_i be the simple Σ-module corresponding to P_i.

LEMMA 2.1. *$[P_1], \ldots, [P_n]$ and $[S_1], \ldots, [S_n]$ are dual bases of $\mathrm{K}_0\mathbb{X}$ with respect to the Euler characteristic, i.e. $\langle P_k, S_l\rangle = \delta_{kl}$ for any $k, l \in \{1, \ldots, n\}$.*

Proof. Note that

$$\mathrm{Hom}(P_i, S_j) \cong \begin{cases} k, & i = j, \\ 0, & i \neq j. \end{cases} \blacksquare$$

We can now easily derive the following well-known relation between (the classes of) projective and simple modules for an algebra Σ. Recall that τ denotes the Auslander–Reiten shift on $\mathrm{coh}\,\mathbb{X}$, and in the sequel (also) the corresponding Coxeter transformation on $\mathrm{K}_0\mathbb{X}$.

We will usually write coordinates of classes in $\mathrm{K}_0\mathbb{X}$ as *column vectors* in $\mathbb{Z}^n$.

PROPOSITION 2.2. *Let $\mathcal{T} = P_1 \oplus \ldots \oplus P_n \in \mathrm{coh}\,\mathbb{X}$ be a tilting sheaf and $\Sigma := \mathrm{End}(\mathcal{T})$.*

(a) *Let S_i denote the ith simple Σ-module (having P_i as its projective cover). The change of base (for $\mathrm{K}_0\mathbb{X}$) from $\mathcal{S} = [S_1], \ldots, [S_n]$ to $\mathcal{T} = [P_1], \ldots, [P_n]$ is given by the Cartan matrix $A := (\langle P_k, P_l\rangle)_{k,l}$. Its inverse is the matrix $D := A^{-1} = (\langle S_l, S_k\rangle)_{k,l}$.*

(b) *The change of base (for $\mathrm{K}_0\mathbb{X}$) from $\tau^-\mathcal{S} = [\tau^- S_1], \ldots, [\tau^- S_n]$ to $\mathcal{T} = [P_1], \ldots, [P_n]$ is given by the matrix $-A^t$; its inverse is the matrix $-D^t$ (A, D as in (a)).*

(c) *In the basis $\mathcal{S}$ [$\mathcal{T}$, resp.] of $\mathrm{K}_0\mathbb{X}$, the automorphism τ^- (the inverse of the Coxeter transformation) is given by the matrix $\tau_{\mathcal{S}}^- := -A \cdot A^{-t} = -A \cdot D^t$ [$\tau_{\mathcal{T}}^- := -A^{-t} \cdot A = -D^t \cdot A$, resp.] ($A, D$ as in (a)).*

Proof. (a) $\mathcal{S}$ and $\mathcal{T}$ are dual bases due to $\langle P_k, S_l\rangle = \delta_{kl}$ (cf. Lemma 2.1). Let $A = (a_{kl})$ and $D = (d_{kl})$. Applying $\langle P_k, -\rangle$ to $[P_l] = \sum_{i=1}^n a_{il}[S_l]$ yields $a_{k,l} = \langle P_k, P_l\rangle$. Applying $\langle -, S_k\rangle$ to $[S_l] = \sum_{i=1}^n d_{il}[P_l]$ yields $d_{kl} = \langle S_l, S_k\rangle$.

(b) follows by $\langle X, Y\rangle = -\langle \tau^- Y, X\rangle$.

(c) By (a) and (b), $A \cdot (-A^{-t})$ describes the change of base from $\mathcal{S}$ to $\tau^-\mathcal{S}$. The matrix $\tau_{\mathcal{T}}^-$ of the automorphism τ^- in the basis $\mathcal{T}$ is obtained by conjugating $\tau_{\mathcal{S}}^-$ with A:

$$\tau_{\mathcal{T}}^- = A^{-1} \cdot (-AD^t) \cdot A = -D^t \cdot A. \blacksquare$$

Consider the exceptional sequence $P_1, \ldots, P_n$ of indecomposable projective Σ-modules.

PROPOSITION 2.3. *The simple Σ-modules $S_n, \ldots, S_1$, in reversed order, constitute an exceptional sequence within $\mathcal{D}^b\mathrm{coh}\,\mathbb{X}$.*

P r o o f. Each simple Σ-module S_i must either belong to $\mathrm{mod}_+ \Sigma$ or $\mathrm{mod}_- \Sigma$, hence either S_i or $S_i[-1]$ lies in $\mathrm{coh}\,\mathbb{X}$. In both cases $\mathrm{Ext}^1_\Sigma(S_i, S_i) = \mathrm{Ext}^1_{\mathrm{coh}\,\mathbb{X}}(S_i, S_i) = 0$ and $\mathrm{End}_\Sigma(S_i) = \mathrm{End}_{\mathrm{coh}\,\mathbb{X}}(S_1) \cong k$, so each S_i is an exceptional object in $\mathcal{D}^\flat \mathrm{coh}\,\mathbb{X}$. On the other hand, for $l < m$, $\mathrm{Ext}^1_\Sigma(S_l, S_m) \neq 0$ would correspond to a non-zero morphism $\varphi : P_m \to P_l$, a contradiction. Finally, $\mathrm{Ext}^2_\Sigma(S_l, S_m) \neq 0$ would indicate a relation from P_m to P_l in $\mathcal{T} = P_1 \oplus \ldots \oplus P_n$, which would necessarily imply the existence of a path from P_m to P_l in $\mathcal{T}$, again contradicting $l < m$ since $P_1, \ldots, P_n$ constitute an exceptional sequence. ∎

THEOREM 2.4. *For any chosen pair* l, m, *only one of* $\mathrm{Ext}^i_\Sigma(S_l, S_m)$, $i = 0, 1, 2$, *can be non-zero. In particular*, *the* k-*dimension of these spaces is determined by the Euler characteristic* $\langle S_l, S_m\rangle$.

P r o o f. First, recall that $\mathrm{gl.dim}\,\Sigma \leq 2$ (cf. [1]).

$\mathrm{Ext}^2_\Sigma(S_l, S_m) \neq 0$ can only occur for $S_l \in \mathrm{mod}_- \Sigma$ and $S_m \in \mathrm{mod}_+ \Sigma$ and necessarily implies that $m < l$. Then $S_l[-1], S_m \in \mathrm{coh}\,\mathbb{X}$, and so $\mathrm{Ext}^1_{\mathrm{coh}\,\mathbb{X}}(S_l[-1], S_m) = \mathrm{Ext}^2_\Sigma(S_l, S_m) \neq 0$, hence by Lemma 1.3 we have $0 = \mathrm{Hom}_{\mathrm{coh}\,\mathbb{X}}(S_l[-1], S_m) = \mathrm{Ext}^1_\Sigma(S_l, S_m)$. ∎

3. Rank additivity. With the notation of the previous section, $D = (d_{m,l}) = (\langle S_l, S_m\rangle)$ can be interpreted as an *adjacency matrix* of the tilting sheaf $\mathcal{T}$: irreducible morphisms $u : P_m \to P_l$ with respect to $\mathrm{add}\,\mathcal{T}$ correspond to elements of $\mathrm{Ext}^1_\Sigma(S_l, S_m)$, while minimal relations starting in P_m and ending in P_l correspond to elements of $\mathrm{Ext}^2_\Sigma(S_l, S_m)$; and by Theorem 2.4, for each pair of simple Σ-modules S_l, S_m, at most one of these two vector spaces can be non-zero.

If therefore $\langle S_l, S_m\rangle$ is *negative* for a pair $m \neq l$, its absolute value coincides with the k-dimension of $\mathrm{irr}_\mathcal{T}(P_m, P_l)$, the vector space of all irreducible morphisms from P_m to P_l with respect to $\mathrm{add}(\mathcal{T})$ (cf. [12]); if $\langle S_l, S_m\rangle$ is *positive*, its value equals the k-dimension of the vector space of minimal relations from P_m into P_l. Finally, $\langle S_m, S_m\rangle = 1$.

It is therefore natural to introduce the *Tits quiver* corresponding to $\mathcal{T}$. The vertices of this quiver are the isomorphism classes of exceptional direct summands P_m of $\mathcal{T}$; between these we have integer-valued arrows equipped with multiplicities

$$P_m \xrightarrow{-\langle S_l, S_m\rangle = -d_{m,l}} P_l \quad \text{for } m \neq l.$$

For fixed l, we call those vertices P_m $(m \neq l)$ from which an arrow valued with $d_{m,l} \neq 0$ starts to end in P_l, or in which an arrow valued with $d_{l,m} \neq 0$ ends, starting from P_l, the *neighbours* of P_l in $\mathcal{T}$. Note that the definition of neighbour neglects the orientation of the arrows in the Tits quiver $\mathcal{T}$. So the non-zero entries $d_{m,l}$ $(m \neq l)$ of the lth *column* of D correspond to vertices

P_m from which an irreducible morphism or a minimal relation (terminating in P_l) starts, while the non-zero entries $d_{l,j} \neq 0$ ($j \neq 0$) of the lth *row* of D correspond to vertices P_j in which an irreducible morphism or a minimal relation (starting in P_l) ends.

Note that the Tits quiver determines the Tits form of $\Sigma = \mathrm{End}(\mathcal{T})$. Since we do not consider the morphisms and explicit relations $\mathcal{T}$ is equipped with, but rather the k-dimensions of the corresponding vector spaces, the Tits quiver provides less information than a quiver with relation does.

A couple of examples will be given at the end of this section. In the graphical presentation, arrows with value 0 will be omitted, and arrows with negative values will be drawn as dashed arrows. In natural examples the Tits quiver will always be endowed with an *orientation*. For additivity, however, we only consider the underlying *bigraph* $G(\mathcal{T})$, thus neglecting the orientation and only considering the neighbours adjacent to each vertex.

We distinguish two sorts of edges in $G(\mathcal{T})$ and define

$$b_{l,m} = \text{number of } \textit{solid} \text{ edges between } P_l \text{ and } P_m,$$
$$c_{l,m} = \text{number of } \textit{dashed} \text{ edges between } P_l \text{ and } P_m.$$

Definition 3.1. A linear function $\lambda : \mathrm{K}_0\mathbb{X} \to \mathbb{Z}$ is said to be *additive* on a tilting sheaf if the bigraph $G(\mathcal{T})$ underlying its Tits quiver satisfies: $\lambda(P_k) \geq 0$ for all k, and for each vertex P_l,

$$2 \cdot \lambda(P_l) = \sum_{k—l} b_{k,l} \cdot \lambda(P_l) - \sum_{k--l} c_{k,l} \cdot \lambda(P_l).$$

For the valuation of a tilting sheaf $\mathcal{T}$ given by the rank $\mathrm{rk} : \mathrm{K}_0\mathbb{X} \to \mathbb{Z}$ one obtains the following property depending only on the non-oriented $\mathbb{Z}$-valued bigraph $G(\mathcal{T})$ underlying $\mathcal{T}$:

Theorem 3.2. *The rank function is additive on each tilting sheaf.*

Proof. Note that $\mathrm{rk}(P_l) \geq 0$ for each l. Using the notation above, the row matrix $\mathrm{rk}_{\mathcal{T}} = [\mathrm{rk}(P_1), \ldots, \mathrm{rk}(P_n)]$ describes the linear function rk in the basis $[P_1], \ldots, [P_n]$. Since now D $[-D^t$, resp.$]$ encodes the change of base from $\mathcal{T} = [P_1], \ldots, [P_n]$ to $\mathcal{S} = [S_1], \ldots, [S_n]$ [or to $\tau^-\mathcal{S} = [\tau^- S_1], \ldots, [\tau^- S_n]$, resp.] (as above S_k denotes the simple Σ-module with projective cover P_k), we have

$$[\mathrm{rk}(S_1), \ldots, \mathrm{rk}(S_n)] = [\mathrm{rk}(P_1), \ldots, \mathrm{rk}(P_n)] \cdot D,$$
$$[\mathrm{rk}(\tau^- S_1), \ldots, \mathrm{rk}(\tau^- S_n)] = -[\mathrm{rk}(P_1), \ldots, \mathrm{rk}(P_n)] \cdot D^t.$$

Recall that the rank function is τ-invariant, therefore the left-hand sides of these two equations coincide, and by subtraction we obtain the formula

$$[\mathrm{rk}(P_1), \ldots, \mathrm{rk}(P_n)] \cdot (D + D^t) = [0, \ldots, 0]. \tag{3.1}$$

This formula already yields the rank additivity claimed in the theorem: Assume that $P_1, \ldots, P_n$ are ordered so that $\mathrm{Hom}(P_l, P_m) = 0$ for $l > m$, and the $P_1, \ldots, P_n$ as well as $S_n, \ldots, S_1$ each constitute an exceptional sequence within $\mathrm{coh}\,\mathbb{X}$ or $\mathrm{mod}(\Sigma)$; hence the matrices A and D of Proposition 2.2 are each upper triangular, having entries 1 on their main diagonals. The symmetric matrix $D + D^t$ has the shape

$$\begin{bmatrix} 2 & & d_{1l} & & \\ & \ddots & d_{ml} & & \\ \ldots & \ldots & 2 & \ldots & \ldots \\ & & d_{lj} & \ddots & \\ & & d_{ln} & & 2 \end{bmatrix};$$

its lth column contains, *above* the main diagonal, the data related to the predecessors of P_l, while, *below* the main diagonal, it contains the data related to the successors of P_l; hence the lth column of this matrix contains the data for all *neighbours* of P_l.

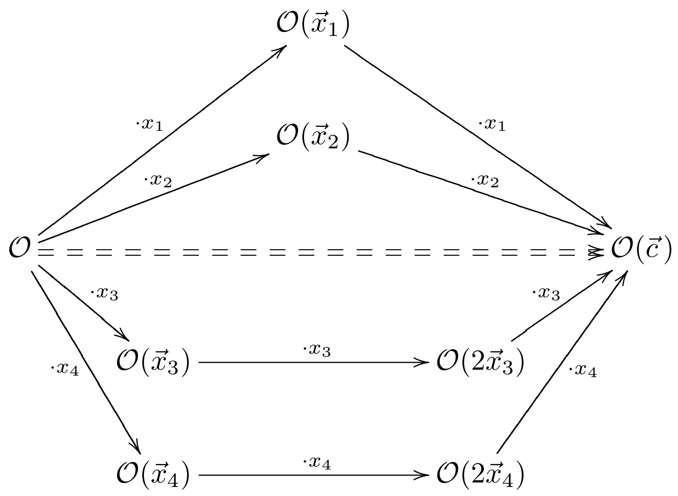
Formula (3.1) now states that the lth entry of the row vector $[\mathrm{rk}(P_1), \ldots, \mathrm{rk}(P_n)] \cdot (D + D^t)$ equals 0 (for each l); this entry is the sum of the ranks of all neighbours of P_l, weighted by the multiplicities of the corresponding arrows, plus $2 \cdot \mathrm{rk}(P_l)$. ∎

In [5] and [12], the Happel–Vossieck list of all tame-concealed algebras is given. In the language of sheaf theory over weighted projective lines, this list contains all *tilting bundles* in $\mathrm{coh}\,\mathbb{X}$ for $\mathbb{X}$ being of tame-domestic representation type, with the integers denoting the rank of indecomposable direct summands. A couple of explicit further examples will be given at the end of this section.

COROLLARY 3.3. *Let $\mathcal{T}$ be a tilting sheaf, T be its maximal sub-bundle and let $\mathcal{B}_1, \ldots, \mathcal{B}_m$ denote the branches consisting of sheaves of finite length which enlarge T to $\mathcal{T}$. Then, for each branch $\mathcal{B}_l$, there must be at least one relation terminating at its root point and starting from a direct summand of T.*

Proof. Each tilting sheaf $\mathcal{T} \in \mathrm{coh}\,\mathbb{X}$ is obtained from a partial tilting bundle T by branch enlargement with branches $\mathcal{B}_1, \ldots, \mathcal{B}_m$ (cf. [12]) chosen from exceptional tubes of sheaves of finite length. There may be relations within T; as well as within any branch $\mathcal{B}_l$ attached to T. Due to rank additivity—applied to each of the root points of each branch $\mathcal{B}_l$ (which is of rank 0, and among whose neighbours at least one of the exceptional vector bundles of T has to occur)—there must be at least one relation terminating at each root point and starting from a direct summand of T. There are no further relations than those mentioned above (cf. [10]). ∎

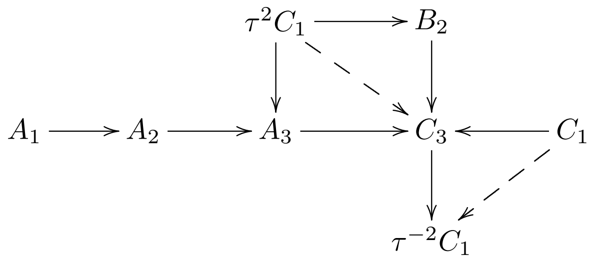
Example 3.4. First, let $\underline{\mathbf{p}} = (2,2,3,3)$ and consider the "canonical configuration"



whose endomorphism ring equals the canonical algebra $\Lambda(2,2,3,3)$. Here the dashed arrows symbolize the relations $x_3^3 = x_2^2 - \lambda_3 x_1^2$ and $x_4^3 = x_2^2 - \lambda_4 x_1^2$ for suitable $\lambda_3, \lambda_4 \in \mathbb{P}^1(k)$.

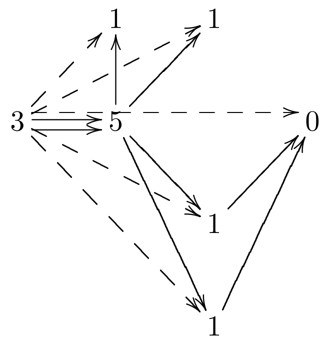
The rank function takes value 1 on each vertex and is clearly additive.

Example 3.5. Now let $\underline{\mathbf{p}} = (2,3,4)$ and consider the tilting bundle $\mathcal{T}$ on $\operatorname{coh}\mathbb{X}$ whose Tits quiver is depicted below; it is—in different language—a tame-concealed quiver of type $\widetilde{\mathbb{E}}_7$ (compare the Happel–Vossieck list in [5] and [12]):



The subindices indicate the rank. Observe that we have one commutativity relation and one zero relation. (As before, τ denotes the Auslander–Reiten translation on $\operatorname{coh}\mathbb{X}$.)

Example 3.6. Finally, let $\underline{\mathbf{p}} = (2,2,2,3)$ and $\mathcal{T}$ be the tilting sheaf with the following underlying Tits quiver (for simplicity, the exceptional summands of $\mathcal{T}$ have been replaced by their corresponding ranks):



4. Quasi-tilted algebras of canonical type. Recall from [3] that an algebra arising as the endomorphism ring of a tilting object in a *hereditary* category is called *quasi-tilted*.

DEFINITION 4.1 [10]. An algebra Σ is called *quasi-tilted of canonical type* if Σ can be realized as the endomorphism ring of a tilting object in a hereditary category $\mathcal{H}$ which is derived equivalent to a category $\operatorname{coh}\mathbb{X}$ for some weighted projective line $\mathbb{X}$.

Equivalently we could require $\mathcal{H}$ to be derived equivalent to the module category $\operatorname{mod}\Lambda$ over a canonical algebra Λ (cf. [12], [1]).

THEOREM 4.2. *The rank function is additive on the quiver of any representation-infinite quasi-tilted algebra of canonical type.*

Proof. Theorem 3.3 of [10] proves that a representation-infinite algebra Σ is quasi-tilted of canonical type if and only if Σ occurs as the endomorphism ring of a tilting object in a category $\mathcal{C}(\mathbb{X}',\mathbb{X}'') \subseteq \mathcal{D}^b\operatorname{coh}\mathbb{X}$. Here $\mathbb{X} = \mathbb{X}' \amalg \mathbb{X}''$ is an arbitrary partition of the point set of the curve $\mathbb{X}$, and

$$\mathcal{C}_0' = \coprod_{x\in\mathbb{X}'} \mathcal{U}_x, \qquad \mathcal{C}_0'' = \coprod_{x\in\mathbb{X}''} \mathcal{U}_x.$$

[$\mathcal{U}_x$ denotes the uniserial subcategory of sheaves of finite length concentrated at the point $x \in \mathbb{X}$.]

Hence the partition $\mathbb{X} = \mathbb{X}' \amalg \mathbb{X}''$ above yields a corresponding partition of tube categories

$$\operatorname{coh}_0\mathbb{X} = \mathcal{C}_0' \amalg \mathcal{C}_0''.$$

Let $\mathcal{C}_+$ denote the subcategory $\operatorname{vect}\mathbb{X}$ of vector bundles in $\operatorname{coh}\mathbb{X}$, and define

$$\mathcal{C}(\mathbb{X}',\mathbb{X}'') := \mathcal{C}_0'[-1] \vee \mathcal{C}_+ \vee \mathcal{C}_0'' \subseteq \mathcal{D}^b\operatorname{coh}\mathbb{X}$$

as the additive closure of $\mathcal{C}_0'[-1], \mathcal{C}_+, \mathcal{C}_0''$ in $\mathcal{D}^b\operatorname{coh}\mathbb{X}$. The category $\mathcal{C}(\mathbb{X}',\mathbb{X}'')$ is derived equivalent to $\operatorname{coh}\mathbb{X}$ (and for $\operatorname{coh}\mathbb{X}$, categories of this type are the only possible hereditary categories which are derived equivalent to $\operatorname{coh}\mathbb{X}$); moreover, we have (cf. [10], Proposition 3.1):

$\Sigma_0'[-1] \oplus \Sigma_+ \oplus \Sigma_0''$ (for $\Sigma_0' \in \mathcal{C}_0'$, $\Sigma_+ \in \mathcal{C}_+$, $\Sigma_0'' \in \mathcal{C}_0''$) constitutes a *tilting object* in $\mathcal{C}(\mathbb{X}', \mathbb{X}'')$ if and only if $\Sigma_+ \oplus \tau_{\mathbb{X}}^{-1} \Sigma_0' \oplus \Sigma_0''$ constitutes a *tilting sheaf* in $\operatorname{coh} \mathbb{X}$. Here $\Sigma_r = \Sigma_+ \oplus \Sigma_0''$ is a tilting object in the category ${}^{\perp}(\Sigma_0')$ (which is isomorphic to a sheaf category $\operatorname{coh} \mathbb{X}_r$ of reduced weight type, cf. [2]) and is obtained from the partial tilting bundle Σ_+ as a branch-co-extension with Σ_0'' using branches from $\mathcal{C}_0''$. The dual statement holds: $\Sigma_l = \Sigma_0'[-1] \oplus \Sigma_+$ is obtained from Σ_+ as a branch-extension with Σ_0' using branches from $\mathcal{C}_0'$; Σ_l is a tilting object in a category isomorphic to $\operatorname{coh} \mathbb{X}_l^{\mathrm{op}}$ for a curve $\operatorname{coh} \mathbb{X}_l$ of reduced weight type. Σ_+ constitutes a tilting bundle in ${}^{\perp}(\Sigma_0') \cap (\Sigma_0'')^{\perp} \cong \operatorname{coh} \mathbb{Y}$, where $\mathbb{Y}$ is obtained by weight reduction with respect to the corresponding points from $\mathbb{X}''$ (yielding $\mathbb{X}_r$) and the corresponding points from $\mathbb{X}'$ (yielding $\mathbb{X}_l$).

Using the same notations as in the considerations above, rank additivity has only to be shown for exceptional summands of Σ of rank 0: for the summands of Σ_0'' it is immediately obtained from the additivity of the tilting sheaf $\Sigma_r \in \operatorname{coh} \mathbb{X}_r$, while for those summands of $\Sigma_0'[-1]$ it follows from the additivity of Σ_l, which is a tilting object in the category $\operatorname{coh} \mathbb{X}_l^{\mathrm{op}}$. ∎

Finally, we would like to investigate rank additivity for tilting complexes in the last two examples:

EXAMPLE 4.3. The following example illustrates that the rank function need *not* be additive on *tilting complexes*:

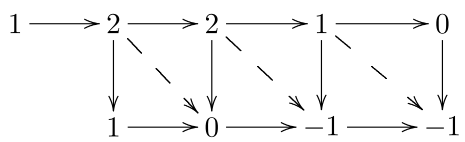
The algebra defined by

$$1 \xrightarrow{\alpha} 0 \xrightarrow{\alpha} -1 \xrightarrow{\alpha} 0 \xrightarrow{\alpha} 1 \xrightarrow{\alpha} 0 \xrightarrow{\alpha} -1 \xrightarrow{\alpha} 0 \xrightarrow{\alpha} 1,$$

subject to all relations $\alpha^3 = 0$, has a realisation as a tilting complex of type $(2, 3, 5)$, but is *not* rank additive. (Please keep in mind the relations!)

EXAMPLE 4.4. On the other hand, there exist rank additive *tilting complexes* having no realisations as *tilting sheaves*:

The tilting complex with underlying Tits quiver



of type $(2, 3, 5)$ (corresponding to $\widetilde{\mathbb{E}}_8$) yields an example.

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