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ON THE PLANARITY OF PEANO GENERALIZED CONTINUA: AN EXTENSION OF A THEOREM OF S. CLAYTOR

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We extend a theorem of S. Claytor in order to characterize the Peano generalized continua which are embeddable into the 2-sphere. We also give a characterization of the Peano generalized continua which admit closed embeddings in the Euclidean plane.

1. Introduction. The celebrated Kuratowski Theorem [7] states that a finite graph G is embeddable in the 2-sphere S^2 if and only if G contains no subgraph homeomorphic to the complete bipartite graph $K_{3,3}$ or to the complete graph with five vertices K_5 (see Fig. 1).

Fig. 1

Later S. Claytor [2] characterized Peano continua which are embeddable in S^2 by adding to $K_{3,3}$ and K_5 two further forbidden curves L_1 and L_2 which are non-polyhedral 1-dimensional Peano continua constructed from $K_{3,3}$ and K_5 respectively (see Fig. 2).

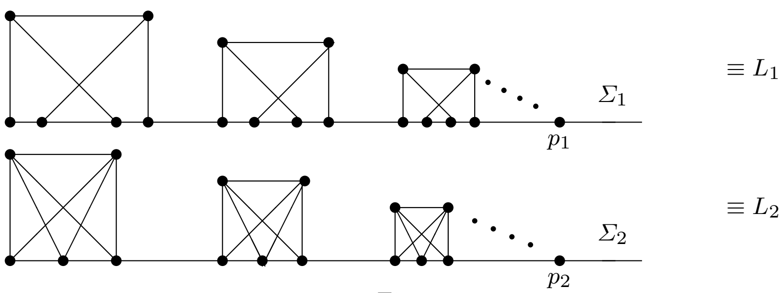

Fig. 2

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Actually, L_1 and L_2 are due to Kuratowski who already suggested a role for them in his paper [7].

The Kuratowski Theorem was extended to any locally finite graph G by Dirac and Schuster [3] by proving that G is planar if and only if all its finite subgraphs are planar. In addition, R. Halin [6] characterized the locally finite graphs which admit planar embeddings without vertex accumulation points (VAP-free embedding) by six forbidden graphs. Namely K_5, $K_{3,3}$, and the four infinite graphs in Fig. 3 below. Later C. Thomassen ([10]; Cor. 4.1) showed that for connected locally finite graphs VAP-free embeddings are the same as closed embeddings.

Also two-dimensional finite complexes embeddable in S^2 have been characterized by Mardešić and Segal in [8]. The authors use Claytor's Theorem to show that $K_5, K_{3,3}$ and the space $F \subseteq \mathbb{R}^3$ called the "spiked disk" given by $F = \{(x,y,0) : x^2+y^2 \leq 1\} \cup \{(0,0,z) : 0 \leq z \leq 1\}$ are the forbidden subspaces for the planarity of finite 2-complexes.

In this paper we show that Claytor's Theorem still holds for Peano generalized continua. See §2 for definitions. Namely we prove

Theorem 1.1. *Let X be a Peano generalized continuum. Then the following statements are equivalent:*

(1) *X is embeddable in S^2 (or equivalently in $\mathbb{R}^2$ if $X \neq S^2$).*
(2) *Any subcontinuum $K \subseteq X$ embeds in S^2.*
(3) *X contains no set homeomorphic to $K_5, K_{3,3}, L_1, L_2$.*
(4) *The Freudenthal compactification $\widehat{X}$ of X is embeddable in S^2.*

We also characterize the Peano generalized continua which admit closed embeddings in $\mathbb{R}^2$. More explicitly, a Peano generalized continuum P is said to be *properly planar* if there exists a proper (or equivalently closed) embedding of P into the Euclidean plane $\mathbb{R}^2$. See §2 for definitions.

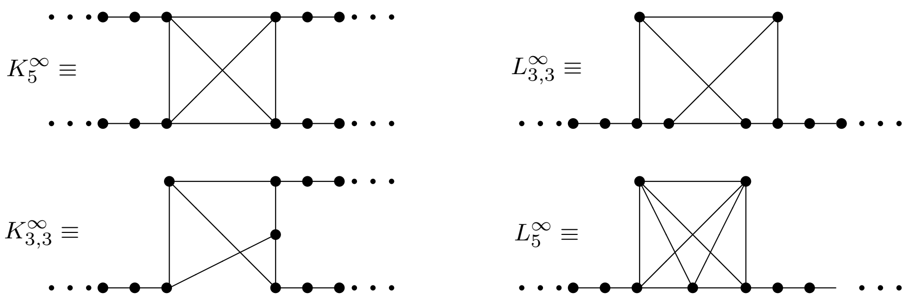

Fig. 3

Theorem 1.2. *Let $X \neq S^2$ be a Peano generalized continuum. Then the following statements are equivalent:*

(a) *X is properly planar.*

(b) *The one-point compactification X^+ is embeddable in S^2.*

(c) *X contains no closed set homeomorphic to $K_5, K_{3,3}, L_1, L_2$ or any of the four Halin graphs of Fig.* 3.

These results are extensions to generalized continua of the quoted Dirac–Schuster Theorem and Halin–Thomassen result respectively. Moreover, we derive from Theorems 1.1 and 1.2 a characterization of (properly) planar locally compact 2-polyhedra. Actually the characterization of planar compact polyhedra in [8] can now be extended to locally compact polyhedra with small changes. Namely, if one uses 1.1 instead of Claytor's Theorem in the proof of (c)⇒(a) in ([8]; §3) one gets

Corollary 1.3. *A locally compact two-dimensional polyhedron P is embeddable in S^2 if and only if P contains no set homeomorphic to $K_{3,3}, K_5$ or the spiked disk F.*

Similarly by using 1.2 one gets

Corollary 1.4. *A locally compact two-dimensional polyhedron $P \neq S^2$ is properly planar if and only if P contains no closed set homeomorphic to $K_{3,3}, K_5, K_5^\infty, L_5^\infty, K_{3,3}^\infty, L_{3,3}^\infty$ or the spiked disk F.*

Remark 1.5. Alternative proofs of Corollaries 1.3 and 1.4 can be found in [1] among other characterizations of (properly) planar polyhedra.

2. Basic definitions and proofs of Theorems 1.1 and 1.2. We recall that a *Peano continuum* X is a metrizable compact connected locally connected space. When compactness is replaced by local compactness the space X is called a *Peano generalized continuum*. Any Peano generalized continuum is arc connected by ([9]; 4.2.5). Moreover, it follows from ([4]; 4.4.F(c)) that any Peano generalized continuum is separable and hence second countable and σ-compact (([4]; 4.1.16) and ([4]; 3.8.C(b))). The local compactness together with the σ-compactness yield that X is a countable union $\bigcup_{n=1}^\infty K_n$ of compact subsets $K_n \subseteq X$ with $K_n \subseteq \operatorname{int} K_{n+1}$. Actually, we can assume without loss of generality that each K_n is connected and all the components of $X - K_n$ are unbounded. Indeed, each K_n is contained in a finite union of open connected subsets of compact closure K_n'. If K_n' is not connected we can consider a new K_n' by adding to K_n' compact and connected subspaces joining its (finite) components. If some components of $X - K_n$ are bounded then we consider a new K_n'' by adding to K_n all the bounded components in $X - K_n$. The sequence $\{K_n\}_{n\geq 1}$ with these two properties will be called a *decomposition* of X.

Given a decomposition $\{K_n\}_{n\geq 1}$ of X a *Freudenthal end* of X is a sequence $\varepsilon = (C_n)_{n\geq 1}$ of components $C_n \subseteq X - K_n$ with $C_{n+1} \subseteq C_n$. Let $\mathcal{F}(X)$

be the set of Freudenthal ends of X and let $e(X)$ denote the cardinal number of $\mathcal{F}(X)$. The set $\widehat{X} = X \cup \mathcal{F}(X)$ admits a compact topology whose basis consists of the open sets of X together with the sets $\widehat{C}_n = C_n \cup \{\varepsilon \in \mathcal{F}(X) : C_n$ appears in $\varepsilon\}$ $(n \geq 1)$. This topology (which does not depend on the sequence $\{K_n\}_{n\geq 1}$) is called the Freudenthal topology and $\widehat{X}$ is called the *Freudenthal compactification* of X. Moreover, the subspace $\mathcal{F}(X)$ turns out to be homeomorphic to a closed subset of the Cantor set (see [5] for details).

A *proper map* $f : X \to Y$ is a continuous map such that $f^{-1}(K)$ is compact for any compact subset $K \subseteq Y$. If X and Y are Peano generalized continua the proper map f is necessarily closed. Moreover, the properness of f is equivalent to the continuity of the extension $f^+ : X^+ \to Y^+$ with $f^+(\infty) = \infty$ between the corresponding one-point compactifications. In fact, f extends to a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$ which restricts to a continuous map $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$. Namely if $\varepsilon = (C_n)_{n\geq 1}$, then $\widehat{f}(\varepsilon) = f_*(\varepsilon) = (D_k)_{k\geq 1}$ where $f(C_{n_k}) \subset D_k$ for some increasing subsequence $(C_{n_k})_{k\geq 1}$ of ε. A proper map $r : [0,\infty) \to X$ is called a *ray*, and $\varepsilon \in \mathcal{F}(X)$ is said to be the end induced by r if $r_*(\infty) = \varepsilon$. In fact one can find for any end ε a ray $r : [0,\infty) \to X$ which induces ε. Moreover, two rays induce the same end if and only if they can be joined outside any compact subset $K \subseteq X$.

In order to apply Claytor's Theorem [2] in the proofs of Theorems 1.1 and 1.2 we shall use the following lemma

LEMMA 2.1. *Let X be a Peano generalized continuum. Then its Freudenthal compactification $\widehat{X}$ as well as its one-point compactification X^+ are Peano continua.*

Proof. As a consequence of the Hahn–Mazurkiewicz Theorem ([9]; 4.2.7) continuous images of Peano continua are Peano continua. Hence the map $q : \widehat{X} \to X^+$ with $q(\mathcal{F}(X)) = \infty$ shows that it will suffice to prove that $\widehat{X}$ is a Peano continuum.

If $X = \bigcup_{n=1}^{\infty} K_n$ is a decomposition of X, the compactness of K_{n+1} implies that the number of components of $X - K_n$ is finite. Hence the open sets $\widehat{C}_n$ form a countable neighbourhood basis of all $\varepsilon \in \mathcal{F}(X)$ in $\widehat{X}$. As X is second countable, it follows that $\widehat{X}$ is second countable. Therefore Urysohn's Metrization Theorem ([4]; 4.2.8) yields that $\widehat{X}$ is a metrizable space. Moreover, $\widehat{X}$ is connected since X is ([4]; 6.1.11). Finally, a ray $r : [0,\infty) \to C_n$ can be regarded as an arc $\widehat{r} : [0,\infty] \to \widehat{C}_n$ with $\widehat{r}(\infty) \in \mathcal{F}(X)$. Hence the open neighbourhoods $\widehat{C}_n \subseteq \widehat{X}$ are arc connected and so $\widehat{X}$ is locally (arc) connected. This finishes the proof. ∎

Now we are ready to prove Theorem 1.1.

Proof of 1.1. Only (3)⇒(4) needs to be checked. Assume that (4) does not hold. Then by Lemma 2.1 we can apply Claytor's Theorem to find a subspace

$A \subseteq \widehat{X}$ homeomorphic to one of the spaces in $\mathcal{S} = \{K_5, K_{3,3}, L_1, L_2\}$. Since we assume (3), one necessarily has $A \cap \mathcal{F}(X) \neq \emptyset$. We now proceed to replace A by a new A' still homeomorphic to a space in $\mathcal{S}$ with $A' \cap \mathcal{F}(X) = \emptyset$, that is, $A' \subseteq X$, which will give a contradiction.

Case 1: $A = K_{3,3}$ or K_5 and no vertex of A is in $\mathcal{F}(X)$. In order to obtain A' from A we firstly observe that for each edge $E \subseteq A$ the intersection $F = E \cap \mathcal{F}(X)$ is homeomorphic to a closed subset of the middle-third Cantor set. Hence we can cover F by finitely many disjoint open sets $W_1, \ldots, W_k$ in $\widehat{X}$ such that all $W_i' = W_i \cap X$ are connected components of the complement $X - K$ of a certain compact set $K \subseteq X$ (depending on F). Moreover, we can assume without loss of generality that $W_i \cap E' = \emptyset$ for any edge $E' \neq E$ of A. Furthermore, the intersections $F_i = F \cap W_i$ are also closed in F, and hence compact subsets of E. Let x_i and y_i denote the first and last element in F_i respectively. Here we identify E with $[0,1]$ by a linear homeomorphism. Notice that x_i and y_i are not vertices of A. Let $x_i' \leq x_i$ and $y_i' \geq y_i$ be points in F_i . Then we replace the segment $< x_i', y_i' > \subseteq E$ by an arc $C_i \subseteq W_i'$ joining x_i' to y_i'. By proceeding in this way for each edge of A we obtain a new graph $A' \subseteq \widehat{X}$ which is homeomorphic to A. We have a contradiction, and hence Case 1 is finished.

Case 2: $A = K_{3,3}$ or K_5 and some vertex of A is in $\mathcal{F}(X)$. By definition of the Freudenthal topology of $\widehat{X}$ we can find a basis of open neighbourhoods $\{U_n\}$ of v in $\widehat{X}$ such that $U_n' = U_n - \mathcal{F}(X) \subseteq X$ is arc connected. Moreover, we can assume that U_n contains no vertex other than v. Since $\mathcal{F}(X)$ is 0-dimensional, any edge incident to v meets U_n'. We now take an arc γ_1 in U_n' joining two of the edges containing v, say Γ_1 and Γ_2. If $D_1 \subseteq \Gamma_1$ is the open segment from v to $q_1 = \gamma_1 \cap \Gamma_1$ we consider the new graph $A_1 = (A - D_1) \cup \gamma_1$. Let $U_2 \subset U_1$ be a new neighbourhood of v in $\widehat{X}$ with $\gamma_1 \cap U_2 = \emptyset$. Let $\gamma_2 \subseteq U_2'$ be an arc joining $\Gamma_2 \subseteq A_1$ to another edge of A other than Γ_1, say Γ_3. Notice that $\Gamma_3 \subseteq A_1$. Let $A_2 = (A_1 - D_2) \cup \gamma_2$ where $D_2 \subseteq \Gamma_2$ is the open segment from v to $q_2 = \gamma_2 \cap \Gamma_2$. If $A = K_{3,3}$ we finish here, otherwise we still have to take a new neighbourhood $U_3 \subseteq U_2$ of v in $\widehat{X}$. We can proceed in this way for each vertex in $A \cap \mathcal{F}(X)$, and we get a new graph $A_0 \subseteq \widehat{X}$ for which $A_0 \cap \mathcal{F}(X)$ does not contain vertices of A_0 and so we are in Case 1. Notice that A_0 is homeomorphic to A if $A = K_{3,3}$ and it contains a subgraph homeomorphic to $K_{3,3}$ if $A = K_5$. See Fig. 4.

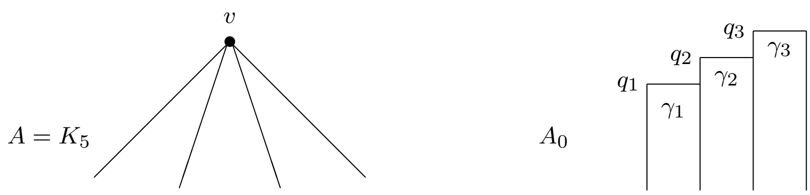

Fig. 4

CASE 3: $A = L_1$ or L_2. Let $N_i = L_i - \Sigma_i$ be the complement of the segment Σ_i in L_i. If $A \cap \mathcal{F}(X)$ contains the limit point $p_i = \Sigma_i \cap \overline{N}_i$, the connectedness of X implies the existence of an arc $\gamma \subseteq X$ from the segment Σ_i to N_i. Here we use the 0-dimensionality of $\mathcal{F}(X)$ to ensure $N_i \cap X \neq \emptyset$ and $\Sigma_i - \{p_i\} \cap X \neq \emptyset$. Then the union $A \cup \gamma$ contains a copy of K_5 if $A = L_2$ or a copy of $K_{3,3}$ if $A = L_1$. That is, we are in the previous cases.

If $p_i \notin A \cap \mathcal{F}(X)$ then we can find a neighbourhood Ω of p_i in X with $\Omega \cap \mathcal{F}(X) = \emptyset$. Moreover, $\Omega \cap A \subseteq X$ contains a homeomorphic copy of L_i and hence we reach a contradiction with assumption (3). The proof is now complete. ■

REMARK 2.2. The equivalences (1)⇔(2)⇔(3) correspond to the extension of Dirac–Schuster Theorem [3] to Peano generalized continua.

We now proceed to prove Theorem 1.2.

Proof of 1.2. We shall show (a)⇒(c)⇒(b)⇒(a). In fact, only (c)⇒(b) needs to be checked. We use the notation of the proof of 1.1.

Assume on the contrary that X^+ is not embeddable in S^2. According to 2.1 and Claytor's Theorem we can find a closed subspace $A \subseteq X^+$ homeomorphic to a continuum in the family $\{K_5, K_{3,3}, L_1, L_2\}$. Moreover, we can assume $\infty \in A$ since otherwise $A \subseteq X$ and this contradicts (c).

If $A = K_5$ or $K_{3,3}$ then it easily follows that $A - \{\infty\}$ is one of the Halin graphs in Fig. 4. In case $A = L_i$, we can also assume $p_i = \infty$ since otherwise we can always get a copy of L_i in any neighbourhood Ω of p_i with $\infty \notin \Omega$. So assume $p_i = \infty$. Since X is connected we can find an arc $\gamma \subseteq X$ joining the segment $\Sigma_i - \{p_i\}$ to $N_i = L_i - \Sigma_i$. Then $(A - \{p_i\}) \cup \gamma$ contains a copy of L_5^∞ if $A = L_2$ or $L_{3,3}^\infty$ if $A = L_1$. This contradicts (c) and the proof is finished. ■

REMARK 2.3. Notice that the one-point compactification of a Halin graph is a Kuratowski graph. Moreover, if we join two vertices of different infinite edges in K_5^∞ we get a copy of $K_{3,3}^\infty$ embedded in the new graph. When we proceed in the same way with $K_{3,3}^\infty$ we get an embedding of $L_{3,3}^\infty$. And, if we join three vertices of three different infinite edges of K_5^∞ we get an embedding of $L_{3,3}^\infty$ in the new graph.

COROLLARY 2.4. *Let P a planar Peano generalized continuum with $e(P) = k$. Then P is not properly planar if and only if it contains a Halin graph H with $e(H) \leq k$.*

P r o o f. If $H \subseteq P$ is a Halin graph with $e(H) > k$ then at least two ends of H are the same in P. Since P is planar, Remark 2.3 yields a new Halin graph H' with $e(H') < e(H)$. The result follows after at most three steps. ■

Corollary 2.5. *Let P be a one-ended Peano generalized continuum. Then P is planar if and only if P is properly planar.*

Remark 2.6. The equivalence (a)$\Leftrightarrow$(c) in 1.2 is the extension to Peano generalized continua of the characterization of Halin and Thomassen of properly planar locally finite connected graphs ([6] and ([10]; Cor 4.1)). In fact, the proof of 1.2 shows that Claytor's Theorem implies this characterization since L_i $(i = 1, 2)$ cannot be embedded in any graph H (otherwise the points of valence ≥ 3 define a set of vertices in H having p_i as a cluster point in the topology of H).

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