ON THE PLANARITY OF PEANO GENERALIZED CONTINUA: 
AN EXTENSION OF A THEOREM OF S. CLAYTOR

BY

R. AYALA, M. J. CHÁVEZ AND A. QUINTERO (SEVILLA)

We extend a theorem of S. Claytor in order to characterize the Peano generalized continua which are embeddable into the 2-sphere. We also give a characterization of the Peano generalized continua which admit closed embeddings in the Euclidean plane.

1. Introduction. The celebrated Kuratowski Theorem [7] states that a finite graph $G$ is embeddable in the 2-sphere $S^2$ if and only if $G$ contains no subgraph homeomorphic to the complete bipartite graph $K_{3,3}$ or to the complete graph with five vertices $K_5$ (see Fig. 1).

![Fig. 1](attachment:fig1.png)

Later S. Claytor [2] characterized Peano continua which are embeddable in $S^2$ by adding to $K_{3,3}$ and $K_5$ two further forbidden curves $L_1$ and $L_2$ which are non-polyhedral 1-dimensional Peano continua constructed from $K_{3,3}$ and $K_5$ respectively (see Fig. 2).

![Fig. 2](attachment:fig2.png)

1991 Mathematics Subject Classification: 54C10, 54F15.
Actually, $L_1$ and $L_2$ are due to Kuratowski who already suggested a role for them in his paper [7].

The Kuratowski Theorem was extended to any locally finite graph $G$ by Dirac and Schuster [3] by proving that $G$ is planar if and only if all its finite subgraphs are planar. In addition, R. Halin [6] characterized the locally finite graphs which admit planar embeddings without vertex accumulation points (VAP-free embedding) by six forbidden graphs. Namely $K_5$, $K_{3,3}$, and the four infinite graphs in Fig. 3 below. Later C. Thomassen ([10]; Cor. 4.1) showed that for connected locally finite graphs VAP-free embeddings are the same as closed embeddings.

Also two-dimensional finite complexes embeddable in $S^2$ have been characterized by Mardešić and Segal in [8]. The authors use Claytor’s Theorem to show that $K_5, K_{3,3}, L_1, L_2$ are the forbidden subspaces for the planarity of finite 2-complexes.

In this paper we show that Claytor’s Theorem still holds for Peano generalized continua. See §2 for definitions. Namely we prove

**Theorem 1.1.** Let $X$ be a Peano generalized continuum. Then the following statements are equivalent:

1. $X$ is embeddable in $S^2$ (or equivalently in $R^2$ if $X \neq S^2$).
2. Any subcontinuum $K \subseteq X$ embeds in $S^2$.
3. $X$ contains no set homeomorphic to $K_5, K_{3,3}, L_1, L_2$.
4. The Freudenthal compactification $\hat{X}$ of $X$ is embeddable in $S^2$.

We also characterize the Peano generalized continua which admit closed embeddings in $R^2$. More explicitly, a Peano generalized continuum $P$ is said to be *properly planar* if there exists a proper (or equivalently closed) embedding of $P$ into the Euclidean plane $R^2$. See §2 for definitions.

**Theorem 1.2.** Let $X \neq S^2$ be a Peano generalized continuum. Then the following statements are equivalent:
(a) $X$ is properly planar.
(b) The one-point compactification $X^+$ is embeddable in $S^2$.
(c) $X$ contains no closed set homeomorphic to $K_5, K_{3,3}, L_1, L_2$ or any of the four Halin graphs of Fig. 3.

These results are extensions to generalized continua of the quoted Dirac–Schuster Theorem and Halin–Thomassen result respectively. Moreover, we derive from Theorems 1.1 and 1.2 a characterization of (properly) planar locally compact 2-polyhedra. Actually the characterization of planar compact polyhedra in [8] can now be extended to locally compact polyhedra with small changes. Namely, if one uses 1.1 instead of Claytor’s Theorem in the proof of (c) $\Rightarrow$ (a) in ([8]; §3) one gets

**Corollary 1.3.** A locally compact two-dimensional polyhedron $P$ is embeddable in $S^2$ if and only if $P$ contains no set homeomorphic to $K_{3,3}, K_5$ or the spiked disk $F$.

Similarly by using 1.2 one gets

**Corollary 1.4.** A locally compact two-dimensional polyhedron $P \neq S^2$ is properly planar if and only if $P$ contains no closed set homeomorphic to $K_{3,3}, K_5, K_5^\infty, L_5^\infty, K_{3,3}^\infty, L_{3,3}^\infty$ or the spiked disk $F$.

**Remark 1.5.** Alternative proofs of Corollaries 1.3 and 1.4 can be found in [1] among other characterizations of (properly) planar polyhedra.

2. Basic definitions and proofs of Theorems 1.1 and 1.2. We recall that a Peano continuum $X$ is a metrizable compact connected locally connected space. When compactness is replaced by local compactness the space $X$ is called a Peano generalized continuum. Any Peano generalized continuum is arc connected by ([9]; 4.2.5). Moreover, it follows from ([4]; 4.4.F(c)) that any Peano generalized continuum is separable and hence second countable and $\sigma$-compact (([4]; 4.1.16) and ([4]; 3.8.C(b))). The local compactness together with the $\sigma$-compactness yield that $X$ is a countable union $\bigcup_{n=1}^{\infty} K_n$ of compact subsets $K_n \subseteq X$ with $K_n \subseteq \text{int} K_{n+1}$. Actually, we can assume without loss of generality that each $K_n$ is connected and all the components of $X - K_n$ are unbounded. Indeed, each $K_n$ is contained in a finite union of open connected subsets of compact closure $K_n'$. If $K_n'$ is not connected we can consider a new $K'_n$ by adding to $K_n'$ compact and connected subspaces joining its (finite) components. If some components of $X - K_n$ are bounded then we consider a new $K''_n$ by adding to $K_n''$ all the bounded components in $X - K_n$. The sequence $\{K_n\}_{n \geq 1}$ with these two properties will be called a decomposition of $X$.

Given a decomposition $\{K_n\}_{n \geq 1}$ of $X$ a Freudenthal end of $X$ is a sequence $\varepsilon = (C_n)_{n \geq 1}$ of components $C_n \subseteq X - K_n$ with $C_{n+1} \subseteq C_n$. Let $\mathcal{F}(X)$
be the set of Freudenthal ends of $X$ and let $e(X)$ denote the cardinal number of $\mathcal{F}(X)$. The set $\hat{X} = X \cup \mathcal{F}(X)$ admits a compact topology whose basis consists of the open sets of $X$ together with the sets $\hat{C}_n = C_n \cup \{ \varepsilon \in \mathcal{F}(X) : C_n \text{ appears in } \varepsilon \}$ ($n \geq 1$). This topology (which does not depend on the sequence $\{K_n\}_{n \geq 1}$) is called the Freudenthal topology and $\hat{X}$ is called the Freudenthal compactification of $X$. Moreover, the subspace $\mathcal{F}(X)$ turns out to be homeomorphic to a closed subset of the Cantor set (see [5] for details).

A proper map $f : X \to Y$ is a continuous map such that $f^{-1}(K)$ is compact for any compact subset $K \subseteq Y$. If $X$ and $Y$ are Peano generalized continua the proper map $f$ is necessarily closed. Moreover, the properness of $f$ is equivalent to the continuity of the extension $f^+ : X^+ \to Y^+$ with $f^+(\infty) = \infty$ between the corresponding one-point compactifications. In fact, $f$ extends to a continuous map $\hat{f} : \hat{X} \to \hat{Y}$ which restricts to a continuous map $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$. Namely if $\varepsilon = (C_n)_{n \geq 1}$, then $\hat{f}(\varepsilon) = f_*(\varepsilon) = (D_k)_{k \geq 1}$ where $f(C_n) \subseteq D_k$ for some increasing subsequence $(C_n^*)_{k \geq 1}$ of $\varepsilon$. A proper map $r : [0, \infty) \to X$ is called a ray, and $r(\infty) = \varepsilon$. In fact one can find for any end $\varepsilon$ a ray $r : [0, \infty) \to X$ which induces $\varepsilon$. Moreover, two rays induce the same end if and only if they can be joined outside any compact subset $K \subseteq X$.

In order to apply Claytor’s Theorem [2] in the proofs of Theorems 1.1 and 1.2 we shall use the following lemma.

**Lemma 2.1.** Let $X$ be a Peano generalized continuum. Then its Freudenthal compactification $\hat{X}$ as well as its one-point compactification $X^+$ are Peano continua.

**Proof.** As a consequence of the Hahn–Mazurkiewicz Theorem ([9]; 4.2.7) continuous images of Peano continua are Peano continua. Hence the map $q : \hat{X} \to X^+ = \mathcal{F}(X)$ shows that it will suffice to prove that $\hat{X}$ is a Peano continuum.

If $X = \bigcup_{n=1}^{\infty} K_n$ is a decomposition of $X$, the compactness of $K_{n+1}$ implies that the number of components of $X - K_n$ is finite. Hence the open sets $\hat{C}_n$ form a countable neighbourhood basis of all $\varepsilon \in \mathcal{F}(X)$ in $\hat{X}$. As $X$ is second countable, it follows that $\hat{X}$ is second countable. Therefore Urysohn’s Metrization Theorem ([4]; 4.2.8) yields that $\hat{X}$ is a metrizable space. Moreover, $\hat{X}$ is connected since $X$ is ([4]; 6.1.11). Finally, a ray $r : [0, \infty) \to C_n$ can be regarded as an arc $\hat{r} : [0, \infty) \to \hat{C}_n$ with $\hat{r}(\infty) \in \mathcal{F}(X)$. Hence the open neighbourhoods $\hat{C}_n \subseteq \hat{X}$ are arc connected and so $\hat{X}$ is locally (arc) connected. This finishes the proof.

Now we are ready to prove Theorem 1.1.

**Proof of 1.1.** Only (3) $\implies$ (4) needs to be checked. Assume that (4) does not hold. Then by Lemma 2.1 we can apply Claytor’s Theorem to find a subspace
$A \subseteq \hat{X}$ homeomorphic to one of the spaces in $S = \{K_5, K_{3,3}, L_1, L_2\}$. Since we assume (3), one necessarily has $A \cap \mathcal{F}(X) \neq \emptyset$. We now proceed to replace $A$ by a new $A'$ still homeomorphic to a space in $S$ with $A' \cap \mathcal{F}(X) = \emptyset$, that is, $A' \subseteq X$, which will give a contradiction.

**Case 1**: $A = K_{3,3}$ or $K_5$ and no vertex of $A$ is in $\mathcal{F}(X)$. In order to obtain $A'$ from $A$ we firstly observe that for each edge $E \subseteq A$ the intersection $F = E \cap \mathcal{F}(X)$ is homeomorphic to a closed subset of the middle-third Cantor set. Hence we can cover $F$ by finitely many disjoint open sets $W_1, \ldots, W_k$ in $\hat{X}$ such that all $W_i = W_i \cap X$ are connected components of the complement $X - K$ of a certain compact set $K \subseteq X$ (depending on $F$). Moreover, we can assume without loss of generality that $W_i \cap E' = \emptyset$ for any edge $E' \neq E$ of $A$. Furthermore, the intersections $F_i = F \cap W_i$ are also closed in $F$, and hence compact subsets of $E$. Let $x_i$ and $y_i$ denote the first and last element in $F_i$ respectively. Here we identify $E$ with $[0, 1]$ by a linear homeomorphism. Notice that $x_i$ and $y_i$ are not vertices of $A$. Let $x_i' \leq x_i$ and $y_i' \geq y_i$ be points in $F_i$. Then we replace the segment $< x_i', y_i' > \subseteq E$ by an arc $C_i \subseteq W_i$ joining $x_i'$ to $y_i'$. By proceeding in this way for each edge of $A$ we obtain a new graph $A' \subseteq \hat{X}$ which is homeomorphic to $A$. We have a contradiction, and hence Case 1 is finished.

**Case 2**: $A = K_{3,3}$ or $K_5$ and some vertex of $A$ is in $\mathcal{F}(X)$. By definition of the Freudenthal topology of $\hat{X}$ we can find a basis of open neighbourhoods $\{U_n\}$ of $v$ in $\hat{X}$ such that $U_n' = U_n - \mathcal{F}(X) \subseteq X$ is arc connected. Moreover, we can assume that $U_n$ contains no vertex other than $v$. Since $\mathcal{F}(X)$ is 0-dimensional, any edge incident to $v$ meets $U_n'$. We now take an arc $\gamma_1$ in $U_n'$ joining two of the edges containing $v$, say $\Gamma_1$ and $\Gamma_2$. If $D_1 \subseteq \Gamma_1$ is the open segment from $v$ to $q_1 = \gamma_1 \cap \Gamma_1$ we consider the new graph $A_1 = (A - D_1) \cup \gamma_1$. Let $U_2 \subseteq U_1$ be a new neighbourhood of $v$ in $\hat{X}$ with $\gamma_1 \cap U_2 = \emptyset$. Let $\gamma_2 \subseteq U_2'$ be an arc joining $\Gamma_2 \subseteq A_1$ to another edge of $A$ other than $\Gamma_1$, say $\Gamma_3$. Notice that $\Gamma_3 \subseteq A_1$. Let $A_2 = (A_1 - D_2) \cup \gamma_2$ where $D_2 \subseteq \Gamma_2$ is the open segment from $v$ to $q_2 = \gamma_2 \cap \Gamma_2$. If $A = K_{3,3}$ we finish here, otherwise we still have to take a new neighbourhood $U_3 \subseteq U_2$ of $v$ in $\hat{X}$. We can proceed in this way for each vertex in $A \cap \mathcal{F}(X)$, and we get a new graph $A_0 \subseteq \hat{X}$ for which $A_0 \cap \mathcal{F}(X)$ does not contain vertices of $A_0$ and so we are in Case 1. Notice that $A_0$ is homeomorphic to $A$ if $A = K_{3,3}$ and it contains a subgraph homeomorphic to $K_{3,3}$ if $A = K_5$. See Fig. 4.
Case 3: \( A = L_1 \) or \( L_2 \). Let \( N_i = L_i - \Sigma_i \) be the complement of the segment \( \Sigma_i \) in \( L_i \). If \( A \cap \mathcal{F}(X) \) contains the limit point \( p_i = \Sigma_i \cap \overline{N_i} \), the connectedness of \( X \) implies the existence of an arc \( \gamma \subseteq X \) from the segment \( \Sigma_i \) to \( N_i \). Here we use the 0-dimensionality of \( \mathcal{F}(X) \) to ensure \( N_i \cap X \neq \emptyset \) and \( \Sigma_i - \{p_i\} \cap X \neq \emptyset \). Then the union \( A \cup \gamma \) contains a copy of \( K_5 \) if \( A = L_2 \) or a copy of \( K_{3,3} \) if \( A = L_1 \). That is, we are in the previous cases.

If \( p_i \notin A \cap \mathcal{F}(X) \) then we can find a neighbourhood \( \Omega \) of \( p_i \) in \( X \) with \( \Omega \cap \mathcal{F}(X) = \emptyset \). Moreover, \( \Omega \cap A \subseteq X \) contains a homeomorphic copy of \( L_i \) and hence we reach a contradiction with assumption (3). The proof is now complete.

Remark 2.2. The equivalences (1) \( \iff \) (2) \( \iff \) (3) correspond to the extension of Dirac–Schuster Theorem [3] to Peano generalized continua.

We now proceed to prove Theorem 1.2.

Proof of 1.2. We shall show (a) \( \Rightarrow \) (c) \( \Rightarrow \) (b) \( \Rightarrow \) (a). In fact, only (c) \( \Rightarrow \) (b) needs to be checked. We use the notation of the proof of 1.1.

Assume on the contrary that \( X^+ \) is not embeddable in \( S^2 \). According to 2.1 and Claytor’s Theorem we can find a closed subspace \( A \subseteq X^+ \) homeomorphic to a continuum in the family \( \{K_5, K_{3,3}, L_1, L_2\} \). Moreover, we can assume \( \infty \in A \) since otherwise \( A \subseteq X \) and this contradicts (c).

If \( A = K_5 \) or \( K_{3,3} \) then it easily follows that \( A - \{\infty\} \) is one of the Halin graphs in Fig. 4. In case \( A = L_i \), we can also assume \( p_i = \infty \) since otherwise we can always get a copy of \( L_i \) in any neighbourhood \( \Omega \) of \( p_i \) with \( \infty \notin \Omega \).

So assume \( p_i = \infty \). Since \( X \) is connected we can find an arc \( \gamma \subseteq X \) joining the segment \( \Sigma_i - \{p_i\} \) to \( N_i = L_i - \Sigma_i \). Then \( (A - \{p_i\}) \cup \gamma \) contains a copy of \( L_5^\infty \) if \( A = L_2 \) or \( L_{3,3}^\infty \) if \( A = L_1 \). This contradicts (c) and the proof is finished.

Remark 2.3. Notice that the one-point compactification of a Halin graph is a Kuratowski graph. Moreover, if we join two vertices of different infinite edges in \( K_5^\infty \) we get a copy of \( K_{3,3}^\infty \) embedded in the new graph. When we proceed in the same way with \( K_{3,3}^\infty \) we get an embedding of \( L_{3,3}^\infty \). And, if we join three vertices of three different infinite edges of \( K_5^\infty \) we get an embedding of \( L_{3,3}^\infty \) in the new graph.

Corollary 2.4. Let \( P \) a planar Peano generalized continuum with \( e(P) = k \). Then \( P \) is not properly planar if and only if it contains a Halin graph \( H \) with \( e(H) \leq k \).

Proof. If \( H \subseteq P \) is a Halin graph with \( e(H) > k \) then at least two ends of \( H \) are the same in \( P \). Since \( P \) is planar, Remark 2.3 yields a new Halin graph \( H' \) with \( e(H') < e(H) \). The result follows after at most three steps.
Corollary 2.5. Let $P$ be a one-ended Peano generalized continuum. Then $P$ is planar if and only if $P$ is properly planar.

Remark 2.6. The equivalence $(a) \Leftrightarrow (c)$ in 1.2 is the extension to Peano generalized continua of the characterization of Halin and Thomassen of properly planar locally finite connected graphs ([6] and ([10]; Cor 4.1)). In fact, the proof of 1.2 shows that Claytor’s Theorem implies this characterization since $L_i$ ($i = 1, 2$) cannot be embedded in any graph $H$ (otherwise the points of valence $\geq 3$ define a set of vertices in $H$ having $p_i$ as a cluster point in the topology of $H$).

Acknowledgements. This work was partially supported by the project DGICYT PB96-1374.

The authors thank the referee for his/her helpful suggestions.

REFERENCES


R. Ayala and A. Quintero
Departamento de Geometría y Topología
Facultad de Matemáticas
Universidad de Sevilla
Apartado 1160
41080-Sevilla, Spain
E-mail: quintero@cica.es

M. J. Chávez
Departamento de Matemática Aplicada I
Escuela Universitaria de Arquitectura Técnica
Universidad de Sevilla
Avda. Reina Mercedes s/n
41012-Sevilla Spain
E-mail: mjchavez@cica.es

Received 6 May 1996; revised 15 April 1997