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ON THE INTERSECTION MULTIPLICITY OF IMAGES UNDER AN ETALE MORPHISM

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We present a formula for the intersection multiplicity of the images of two subvarieties under an etale morphism between smooth varieties over a field k. It is a generalization of Fulton's Example 8.2.5 from [3], where a strong additional assumption has been imposed. In a special case where the base field k is algebraically closed and a proper component of the intersection is a closed point, intersection multiplicity is an invariant of etale morphisms. This corresponds with analytic geometry where intersection multiplicity is an invariant of local biholomorphisms.

1. Introduction. We present the following theorem on the intersection multiplicity of two images under an etale morphism:

MAIN THEOREM. *Let $f : X' \to X$ be a finite dominant morphism between smooth varieties over a field k of characteristic zero, V' and W' be two subvarieties of X', $V = f(V')$, $W = f(W')$, and let Z be a proper component of the intersection $V \cap W$. Denote by Z_i', $i = 1, \ldots, m$, the irreducible components of the intersection $V' \cap W'$ which lie over Z; all the Z_i' are, of course, proper components. If the morphism f is etale at the generic point ζ_i' of each subvariety Z_i', then*

$$[V' : V] \cdot [W' : W] \cdot i(Z, V \cdot W; X) = \sum_i [Z_i' : Z] \cdot i(Z_i', V' \cdot W'; X').$$

REMARK. Generally, a morphism f is *etale* if it is unramified and flat. However, if f is a dominant morphism between regular varieties, f is flat iff it is equidimensional (cf. [8]); in particular, f is flat whenever it is quasi-finite. Such a dominant quasi-finite morphism f is therefore an open mapping.

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The above theorem is a generalization of Fulton's example ([3], Example 8.2.5), where a strong additional assumption has been imposed; namely, it is required that there should exist only one irreducible component of $f^{-1}(Z)$ contained either in V' or W'. In particular, when $Z = \{P\}$ is a closed point, a unique point P' on V' or W' should lie over P. This means that not only does Fulton assume that a unique irreducible component of $V' \cap W'$ lies over Z, but moreover, that a unique subvariety of V' or W' lies over Z. His geometric apparatus of intersection theory seems to have too global character to get rid of this additional assumption, at least in a direct fashion—which is related in a sense to the fact that the two diagrams from Section 2 do not globally coincide. What enables us to achieve full generality is algebraic localization by means of the passage to quotient rings as well as the expression of intersection multiplicity directly in terms of local rings.

Besides, we wish to emphasize that Fulton's proof is essentially based on his Theorem 6.2 (*op. cit.*), and thus involves the whole heavy machinery developed in the first six fundamental chapters of his book. Instead, we make use of Samuel's formula, which is far more elementary and works exactly in the case of finite morphisms. Some structural affinity between his and our proofs consists in that both Theorem 6.2 (points a) and c) together) and Samuel's formula may be interpreted as a kind of projection formula.

A classical case of the weak (Fulton's) version of the main theorem, where the base field is algebraically closed, the component $Z = \{P\}$ is a closed point and f is a general projection, was treated by Severi [12], §11. This makes it possible to reduce intersection theory on a smooth variety to that in a projective space (also cf. [3], Example 8.2.6). Chevalley too used the weak version in his algebraic intersection theory over an algebraically closed field [2] (in Part III devoted to algebraic varieties). After proving the main theorem in Section 2, we shall show a local version in a special case where the base field k is algebraically closed and the component $Z = \{P\}$ is a closed point:

PROPOSITION. *The intersection multiplicities at points P and P' of the proper intersections of V with W and of V' with W', respectively, coincide regardless of the other intersection points lying over P.*

In other words, intersection multiplicity is an invariant of etale morphisms. This corresponds with analytic geometry where intersection multiplicity is an invariant of local biholomorphisms (cf. [2], Part II devoted to algebroid varieties). Such a localization of the problem does not need Samuel's formula, and is attained merely by the same analysis of canonical diagrams as in the proof of the main theorem, but applied to the completions of the local rings.

2. Proof of the Main Theorem. We first set up necessary notation and terminology. All schemes which occur throughout the paper are tacitly assumed to be algebraic over a fixed field. An irreducible and reduced scheme X is called a variety; the field of rational functions on X is denoted by $k(X)$. We say that a scheme X is *equidimensional* (or *of pure dimension*) if all its irreducible components are of the same dimension; then, of course, X has no embedded components. If X is of pure dimension r, we often write X^r.

Let $f : X' \to X$ be a proper morphism of schemes, W' a subvariety of X' and $W = f(W')$. One can define a number $[W' : W]$ of global character, called the *degree* of W' over W, by putting

$$[W' : W] := \begin{cases} [k(W') : k(W)] & \text{if } \dim W' = \dim W, \\ 0 & \text{if } \dim W' > \dim W, \end{cases}$$

where $[k(W') : k(W)]$ denotes the degree of the finite field extension.

Consider a morphism $f : Y^r \to X^r$ between two schemes of pure dimension r, and suppose f is quasi-finite at a point $y \in Y$. Let $B := \mathcal{O}_y$ and $A := \mathcal{O}_x$ be the local rings of the points y and $x := f(y)$ on Y and X, respectively. The completions $\widehat{B}$ and $\widehat{A}$ of the local rings B and A are unmixed (cf. [7], Chap. V, §34). By virtue of the preparation theorem, $\widehat{B}$ is a finite $\widehat{A}$-module. It follows from the last two facts that whenever the scheme X is analytically irreducible at x (i.e. if $\widehat{A}$ is a domain), no non-zero element of $\widehat{A}$ is a zero divisor in $\widehat{B}$. Then, in particular, $\widehat{A}$ is a subring of $\widehat{B}$. Under these assumptions, one can define a number $m_y f$ of local character, called the *multiplicity* of f at y, by means of Weil's formula (see e.g. [13] or [6], App. to Chap. VI; The Weil–Samuel algebraic theory of multiplicity):

$$m_y f := [\widehat{B} : \widehat{A}],$$

where $[\widehat{B} : \widehat{A}]$ denotes the dimension of the $S^{-1}\widehat{A}$-vector space $S^{-1}\widehat{B}$ with $S := \widehat{A} \setminus \{0\}$.

Let $\varphi = \{\varphi_1, \dots, \varphi_r\} \subset A := \mathcal{O}_x$ be a system of parameters at a point x on a scheme X. One can define the multiplicity $e(\varphi)$ of φ as the multiplicity of the ideal generated by the system φ:

$$e(\varphi) := e((\varphi_1, \dots, \varphi_r)).$$

The key result we shall use in the sequel is the following

Samuel's Formula (cf. [14], Chap. VIII, §10). *Let A be a local ring with maximal ideal $\mathfrak{m}$, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal of A, and A' a finite overring of A. Then A' is a semi-local ring, and $\mathfrak{q}A'$ is an open ideal in A'; let $\{\mathfrak{m}'_i\}$*

be the set of maximal ideals in A'. If no non-zero element of A is a zero divisor in A' and if the local rings $A'_{\mathfrak{m}'_i}$ have the same dimension as A, then

$$[A' : A] \cdot e(\mathfrak{q}) = \sum_i [A'/\mathfrak{m}'_i : A/\mathfrak{m}] \cdot e(\mathfrak{q}A'_{\mathfrak{m}'_i}).$$

Remarks. (1) The hypothesis that all the local rings $A'_{\mathfrak{m}'_i}$ have the same dimension as A is satisfied if the ring A is unmixed (cf. [7], Chap. V, §34), and thus whenever A is any equidimensional ring which occurs in algebraic geometry.

(2) We may interpret Samuel's formula as a special version of the projection formula from algebraic topology and algebraic geometry; namely, we must construe $\operatorname{Spec} A$ and $\operatorname{Spec} A'$ as cycles, and systems of parameters as cocycles.

(3) If the ring A is analytically irreducible, then the assumption on zero divisors for the ring extension $A \subset A'$ implies the same for the extension of the completions $\widehat{A} \subset \widehat{A'}$, and the local rings $A'_{\mathfrak{m}_i}$ always have the same dimension as A. Furthermore, we have

$$[\widehat{A'} : \widehat{A}] = [A' : A], \quad \widehat{A'} = \prod_i \widehat{A'_{\mathfrak{m}'_i}}, \quad e(\mathfrak{q}\widehat{A}) = e(\mathfrak{q})$$

and

$$e(\mathfrak{q}A') = e(\mathfrak{q}\widehat{A'}) = \sum_i e(\mathfrak{q}\widehat{A'_{\mathfrak{m}'_i}}).$$

(4) When A is an analytically irreducible local ring and $A \subset A'$ is a quasi-finite extension of local rings, we obtain a local version of Samuel's formula

$$[\widehat{A'} : \widehat{A}] \cdot e(\mathfrak{q}) = [A'/\mathfrak{m}' : A/\mathfrak{m}] \cdot e(\mathfrak{q}A').$$

In other words,

$$m_{x'} f \cdot e(\mathfrak{q}) = m_{x'} g \cdot e(\mathfrak{q}A'),$$

where $f : \operatorname{Spec}(A') \to \operatorname{Spec}(A)$, $g : \operatorname{Spec}(A'/\mathfrak{m}') \to \operatorname{Spec}(A/\mathfrak{m})$ are the induced morphisms, and x' is the closed point corresponding to the maximal ideal $\mathfrak{m}'$.

Important consequences of Samuel's formula are the general projection formula and associativity formula; these two together with the formula for product varieties form three basic theorems from which all properties of proper intersections can be derived (this was first noticed by Weil, cf. [13]). We shall need a special case of the

Associativity Formula (cf. [10], Chap. III, §4). *Consider an unmixed local Nagata ring A with maximal ideal $\mathfrak{m}$, and a system of parameters $x_1, \ldots, x_n$ ($n = \dim A$) in A. Denote by $\mathfrak{q}$ and $\mathfrak{r}$ the ideals of A generated by $x_1, \ldots, x_n$ and $x_1, \ldots, x_m$ for some $m \in \{1, \ldots, n\}$. If $\mathfrak{p}_i$ are the* *minimal primes of* $\mathfrak{r}$ (*obviously*, $\dim A_{\mathfrak{p}_i} = m$), *then*

$$e(\mathfrak{q}) = \sum_i e((\mathfrak{q} + \mathfrak{p}_i)/\mathfrak{p}_i) \cdot e(\mathfrak{r} A_{\mathfrak{p}_i}).$$

We make use of the above in the case $\mathfrak{r} = 0$. The *multiplicity* m_i of a component X_i of a scheme X is the length of the Artinian local ring $\mathcal{O}_{\xi_i}$ where ξ_i is a generic point of X_i:

$$m_i := \operatorname{length} \mathcal{O}_{\xi_i} = e(\mathcal{O}_{\xi_i});$$

of course, $m_i = 1$ whenever the scheme X is reduced. If φ is a system of parameters of a scheme X at a point x at which X is equidimensional, then the associativity formula implies that

$$e(\varphi) = \sum_i e(\varphi \mathcal{O}_x/\mathfrak{p}_i) \cdot m_i,$$

where $\mathfrak{p}_i$ are the minimal primes in the local ring $\mathcal{O}_x$ which correspond to the components X_i of X passing through x, and m_i are the multiplicities of X_i. In particular, if X is a reduced scheme, we obtain the formula

$$e(\varphi) = \sum_i e(\varphi \mathcal{O}_x/\mathfrak{p}_i).$$

Consider two equidimensional subschemes V^r and W^s of a smooth variety X^n over a field k which intersect properly along a subvariety Z (i.e. $\dim Z = r + s - n$). When one of the subschemes, say V, is a complete intersection in the vicinity of Z in X, the ideal of V in the local ring of the generic point ζ of Z on X is generated by $n - r$ elements $\varphi_1, \ldots, \varphi_{n-r}$. The canonical images of these elements in the local ring of ζ on W form a system of parameters $\overline{\varphi}$. The multiplicity $e(\overline{\varphi})$ of this system of parameters is called the *intersection multiplicity* of W and V along Z, and it is denoted by $i(Z, V \cdot W; X)$.

In order to define intersection multiplicity in general, one must apply the diagonal procedure. Denote by

$$\Delta : X \to X \times X$$

the diagonal morphism (which is a closed immersion). Each subvariety Z of X corresponds via the diagonal morphism to a unique subvariety Z^Δ of $\Delta \subset X \times X$. Since the diagonal is a complete intersection in the vicinity of Z^Δ in $X \times X$ (cf. [1], Chap. VII, §5), one can put

$$i(Z, V \cdot W; X) := i(Z^\Delta, \Delta \cdot (V \times W); X \times X).$$

Whenever V is a complete intersection, both the above definitions of intersection multiplicity coincide (Reduction Theorem; cf. [11], Chap. II, §5.7—the proof is based on the associativity formula, or cf. [3], Chap. VIII).

Now we can readily pass to the proof of the main theorem. The problem being local, we may assume that the varieties X and X' are affine. We introduce the following notation:

- ζ^{Δ} is a generic point of Z^{Δ} on $X \times X$;
- $\mathfrak{p}$ is the prime ideal of the point ζ^{Δ} in the coordinate ring $k[X \times X]$;
- $S := k[X \times X] \setminus \mathfrak{p}$;
- R and A denote the local rings of the point ζ^{Δ} on $X \times X$ and on $V \times W$ (being the localizations of the coordinate rings $k[X \times X]$ and $k[V \times W]$ with respect to the multiplicative set S);
- R' and A' denote the localizations of the coordinate rings $k[X' \times X']$ and $k[V' \times W']$ with respect to S.

Clearly, $R \subset R'$ and $A \subset A'$ are finite ring extensions. Thus R' and A' are semilocal rings; let $\mathfrak{m}'_i$ be the maximal ideals of A', $i = 1, \ldots, k$ $(k \geq m)$. Observe that there is a one-to-one correspondence between the components Z_i and those ideals $\mathfrak{m}'_i$ that lie on the diagonal $\Delta' := \Delta_{X'/k}$ of the product $X' \times X'$. We now consider the Cartesian square

$$\begin{array}{ccc} X' \times_X X' & \xrightarrow{\text{can}} & X' \times_k X' \\ {\scriptstyle\text{can}}\downarrow & & \downarrow{\scriptstyle f \times f} \\ X & \xrightarrow[\Delta_{X/k}]{} & X \times_k X \end{array}$$

(cf. [4], Chap. 0, Proposition 1.4.5), and the canonical commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\Delta_{X'/k}} & X' \times_k X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times f} \\ X & \xrightarrow[\Delta_{X/k}]{} & X \times_k X \end{array}$$

(*loc. cit.*, Proposition 1.4.8). Since the morphism f is unramified at each point ζ'_i, the diagonal morphism

$$\Delta_{X'/X} : X' \to X' \times_X X'$$

is an isomorphism of a neighbourhood U of the set $\{\zeta'_1, \ldots, \zeta'_m\}$ onto an open subset of $X' \times_X X'$ (cf. [1], Chap. VI, Proposition 3.3). Thus, identifying U both with a subset of X' and of $X' \times X'$, the above two diagrams coincide over U. This implies that the ideal $\mathfrak{Q}'$ of the diagonal $\Delta' := \Delta_{X'/k}$ in the ring R' is the extension of the ideal $\mathfrak{Q}$ of the diagonal $\Delta := \Delta_{X/k}$ in the ring R. Consequently, the ideal $\mathfrak{q}'$ of Δ' in A' is the extension of the ideal $\mathfrak{q}$ of Δ in A: $\mathfrak{q}' = \mathfrak{q}A'$, so that Samuel's formula is applicable.

Let $\mathfrak{p}_j$ be the minimal primes of the ring A. As the ring A' is flat over A, we have finite ring embeddings $A/\mathfrak{p}_j \subset A'/\mathfrak{p}_j A'$. Only at this stage do we use the assumption that k is a field of characteristic zero: then A and A' are reduced rings; moreover, the rings $A'/\mathfrak{p}_j A'$ are also reduced because A' is etale over A (cf. [5], Chap. VIII, §21). Therefore, we can easily deduce that

$$[A'/\mathfrak{p}_j A' : A/\mathfrak{p}] = [V' : V] \cdot [W' : W]$$

for all j (this is, in fact, a problem from the theory of finite separable field extensions). Hence, by Samuel's formula,

$$[V' : V] \cdot [W' : W] \cdot e((\mathfrak{q} + \mathfrak{p}_j)/\mathfrak{p}_j) = \sum_i [Z'_i : Z] \cdot e((\mathfrak{q} + \mathfrak{p}_j)A'_{\mathfrak{m}'_i}/\mathfrak{p}_j A'_{\mathfrak{m}'_i}).$$

From the associativity formula we obtain

$$\sum_j e((\mathfrak{q} + \mathfrak{p}_j)/\mathfrak{p}_j) = e(\mathfrak{q})$$

and

$$\sum_j e((\mathfrak{q} + \mathfrak{p}_j)A'_{\mathfrak{m}'_i}/\mathfrak{p}_j A'_{\mathfrak{m}'_i}) = e(\mathfrak{q}A'_{\mathfrak{m}'_i}).$$

Combining the last three equalities, we conclude that

$$[V' : V] \cdot [W' : W] \cdot e(\mathfrak{q}) = \sum_i [Z_i : Z] \cdot e(\mathfrak{q}A'_{\mathfrak{m}'_i}),$$

which completes the proof of the main theorem.

Whenever the base field k is algebraically closed and the irreducible component $Z = \{P\}$ has dimension zero, we can repeat the analysis of the canonical diagrams applied, however, to the completions of the local rings. In this manner, even without making use of Samuel's formula, we obtain the following local version of the theorem under consideration:

PROPOSITION. *Let $f : X' \to X$ be a dominant morphism between smooth varieties over an algebraically closed field k, V' and W' be two subvarieties of X' that intersect properly at a closed point P', and let $V := f(V')$, $W := f(W')$, $P := f(P')$. If the morphism f is etale at P', then*

$$i(P', V' \cdot W'; X') = i(P, V \cdot W; X).$$

In other words, intersection multiplicity is an invariant of etale morphisms.

Indeed, let A and A' be the local rings of the point P on $V \times W$ and of the point P' on $V' \times W'$. Since the base field k is algebraically closed, the schemes $X \times X$, $V \times W$, $X' \times X'$ and $V' \times W'$ are (irreducible) varieties, as well as the rings A and A' (and even their completions $\widehat{A}$ and $\widehat{A'}$ — cf. [10], Chap. VI, §1 and [7], Chap. VII, §47) are local domains with common residue field k. As before, the ideal $\mathfrak{q}'$ of the diagonal Δ' in A' is the extension of the ideal $\mathfrak{q}$ of the diagonal Δ in A: $\mathfrak{q}' = \mathfrak{q}A'$. But the ring embedding $A \subset A'$ is etale, and thus the completions $\widehat{A}$ and $\widehat{A'}$ are canonically isomorphic (cf. [1], Chap. VI, Corollary 4.5). Hence

$$i(P', V' \cdot W'; X') = e(\mathfrak{q}A') = e(\mathfrak{q}\widehat{A'}) = e(\mathfrak{q}\widehat{A}) = e(\mathfrak{q}A) = i(P, V \cdot W; X),$$

as asserted.

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