

*INVARIANT MANIFOLDS FOR ONE-DIMENSIONAL PARABOLIC  
PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER*

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**0. Introduction.** In this paper we study infinite-dimensional (semi)dynamical systems generated by semilinear one-dimensional partial differential equations (PDEs) of parabolic type

$$(0.1a) \quad u_t = u_{xx} + f(t, x, u, u_x), \quad t > 0, \quad 0 < x < L,$$

with  $f \in C^2(\mathbb{R}_+ \times [0, L] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ ,  $L > 0$ ,  $T$ -periodic in  $t$ ,  $T > 0$ , together with homogeneous Dirichlet boundary conditions

$$(0.1b) \quad u(t, 0) = u(t, L) = 0, \quad t > 0,$$

and the initial condition

$$(0.1c) \quad u(0, x) = u_0(x), \quad x \in (0, L).$$

It is well known that for PDEs of type (0.1a)+(0.1b) the number of zeros for the difference of two solutions does not increase with time (see Nickel [34], Zelenyak [43], Matano [29], [30], Angenent [5]). This fact is crucial in proving many interesting results concerning the asymptotic behavior of solutions. For instance, it was proved with the help of it that any suitably bounded solution of such an equation (independent of time) converges to a solution of the corresponding elliptic PDE (see e.g. [43] and [29]). Another consequence is that the semiflow generated by such an equation is Morse–Smale, provided the stationary solutions are hyperbolic (see Henry [22], Angenent [4], and Chen, Chen and Hale [11] for the time-periodic case). One can also mention analysis of connecting orbits (e.g. Brunovský and Fiedler [9]), investigation of Morse decompositions (Chen and Poláčik [13]), decoupling of a linear equation into the direct sum of countably many one-dimensional subbundles (Chow, Lu and Mallet-Paret [16]), and investigation of almost automorphic minimal sets for linear equations (Shen and Yi [38]).

In the present paper we analyze the structure and size of various sets of points (initial conditions for (0.1)) defined through the asymptotic behavior

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of their trajectories. The main purpose is to use the idea of the nonincreasing of the number of zeros, together with results on the existence of invariant foliations in the vicinity of a fixed point, to show that the set of those initial conditions for which the difference between the corresponding solution and its limiting solution has at least  $k$  zeros is contained in the union of finitely many Hölder manifolds-with-boundary of codimension at least  $k-1$ . This is used later (in Sections 2 and 3) to obtain results about asymptotic behavior of the solution of (0.1) with a “generic” initial condition. As the exact formulation of our results requires introducing many additional concepts, it is postponed to the end of the Introduction (Theorems 0.8–0.10).

The linear operator  $u \mapsto -u_{xx}$  together with boundary conditions (0.1b) extends uniquely to a self-adjoint densely defined sectorial (unbounded) linear operator  $A$  with compact resolvent on the Hilbert space  $L^2(0, L)$ . The domain  $\text{dom } A$  of  $A$  equals  $H_0^1(0, L) \cap H^2(0, L)$ .  $A$  generates a holomorphic  $C_0$ -semigroup on  $X := L^2(0, L)$ . Let  $X^\alpha$  denote the fractional power space for  $A$  ( $X^\alpha = \text{dom } A^\alpha$ ), with the norm  $\|\cdot\|_\alpha := \|\cdot\| + \|A^\alpha \cdot\|$ , where  $\|\cdot\|$  is the  $L^2(0, L)$ -norm. Obviously  $X^0 = L^2(0, L)$ ,  $X^1 = \text{dom } A$ .

The following convention will be useful: for a function  $G$  from  $U_1 \times U_2$  into some Banach space, where  $U_i$  is an open subset of a Banach space  $X_i$ , we denote by  $D_i G$  the (Fréchet) partial derivative of  $G$  with respect to the  $i$ th variable. For  $B \subset X^\alpha$ ,  $\text{cl } B$  stands for the closure of  $B$  in the topology of  $X^\alpha$ . For the closure of  $B$  in the topology of  $X$  we write  $\text{cl}_X B$ . By  $\mathbb{R}_+$  ( $\mathbb{Z}_+$ ) we denote the set of nonnegative reals (integers).

By Sobolev’s inequalities (see e.g. Henry’s book [21]), if  $\alpha > 3/4$  then  $X^\alpha$  embeds continuously into the Banach space  $C^1([0, L])$ .

Given  $t \in \mathbb{R}_+$  and a function  $v(\cdot) : [0, L] \rightarrow \mathbb{R}$  let  $F(t, v)(\cdot) : [0, L] \rightarrow \mathbb{R}$  be defined as  $F(t, v)(x) := f(t, v(x), v_x(x))$ . For  $\alpha > 3/4$  the substitution operator  $F$ , regarded as a mapping from  $\mathbb{R}_+ \times X^\alpha$  to  $X$ , is of class  $C^2$  (cf. Hale’s book [20]).

From now on, let  $3/4 < \alpha < 1$  be fixed. Problem (0.1a)+(0.1b)+(0.1c) can be written as an abstract semilinear parabolic differential equation

$$(0.2a) \quad \frac{du}{dt} + Au = F(t, u)$$

with an initial condition

$$(0.2b) \quad u(0) = u_0, \quad u_0 \in X^\alpha.$$

By Thm. 3.3.3 of [21] (compare also [33]), for each  $u_0 \in X^\alpha$  there is a continuous function  $u(\cdot; u_0) : [0, \tau(u_0)) \rightarrow X^\alpha$ ,  $\tau(u_0) > 0$ , with the following properties.

$$(S1) \quad u(0; u_0) = u_0.$$

$$(S2) \quad \text{The assignment } [0, \tau(u_0)) \ni t \mapsto F(t, u(t; u_0)) \in X \text{ is continuous.}$$

(S3) For each  $t \in (0, \tau(u_0))$  the derivative  $(du/dt)(t; u_0)$  exists in  $X$ .

(S4)  $u(t; u_0) \in \text{dom } A$  for each  $t \in (0, \tau(u_0))$ .

(S5) (0.2a) holds for each  $t \in (0, \tau(u_0))$ .

$u(\cdot; u_0)$  satisfying (S1)–(S5) will be called a *solution* to (0.2). A solution to (0.2) is unique (up to extending its domain of existence).

For  $u_0 \in X^\alpha$  fixed, consider the linearized (variational) equation along the solution  $u(\cdot; u_0)$ ,

$$(0.3) \quad \frac{dw}{dt} + Aw = D_2F(t, u(t; u_0))w.$$

According to Thm. 3.4.4 of [21], the solution  $w(t; u_0; w_0)$  of (0.3) with  $w(0) = w_0 \in X^\alpha$  is defined on  $[0, \tau(u_0))$ , and

$$w(t; u_0; w_0) = D_2u(t; u_0)w_0 \quad \text{for } t \in (0, \tau(u_0)).$$

The Banach space  $X^\alpha$  is partially ordered by the relation

$$u_1 \leq u_2 \quad \text{if } u_1(x) \leq u_2(x) \text{ for all } x \in [0, L].$$

The (*nonnegative*) *cone*  $(X^\alpha)_+$  in  $X^\alpha$  is defined as  $(X^\alpha)_+ := \{\tilde{u} \in X^\alpha : \tilde{u} \geq 0\}$ . It is not hard to prove that the interior  $(X^\alpha)_+^\circ$  of the cone  $(X^\alpha)_+$  equals

$$(X^\alpha)_+^\circ = \{\tilde{u} \in X^\alpha : \tilde{u}(x) > 0 \text{ for } x \in (0, L), \tilde{u}'(0) > 0, \tilde{u}'(L) < 0\}.$$

We write

$$\begin{aligned} u_1 < u_2 & \quad \text{if } u_1 \leq u_2 \text{ and } u_1 \neq u_2, \\ u_1 \ll u_2 & \quad \text{if } u_2 - u_1 \in (X^\alpha)_+^\circ. \end{aligned}$$

Given  $u_1 \leq u_2$ , the *closed order-interval* is defined as  $[u_1, u_2] := \{\tilde{u} : u_1 \leq \tilde{u} \leq u_2\}$ , and for  $u_1 \ll u_2$ , the *open order-interval* is defined as  $[[u_1, u_2]] := \{\tilde{u} : u_1 \ll \tilde{u} \ll u_2\}$ .

By giving the set  $X^\alpha$  the topology generated by the open order-intervals (called the *order topology*) we define a topological space  $(X^\alpha)_{\text{ord}}$ . It is a normable vector space. The collection

$$\{[[-(1/k)e_1, (1/k)e_1]]\}_{k \in \mathbb{Z}_+},$$

where  $e_1(x) := \sin(\pi x/L)$ , forms a neighborhood base at 0 for  $(X^\alpha)_{\text{ord}}$ . According to Hirsch's Cor. 1.12(b) in [24], the norm

$$\|\tilde{u}\|_{\text{ord}} := \max_{x \in [0, L]} |\tilde{u}(x)| + |\tilde{u}'(0)| + |\tilde{u}'(L)|$$

defines the order topology on  $X^\alpha$ . It is straightforward that on  $X^\alpha$  the order topology is essentially weaker than the original one. For more on order topology the reader is referred to Chapter I of Amann's paper [2] or to Section 19 of Deimling's book [17]. The terms *open*, *closed* etc. will refer

to the original topology, whereas speaking of the order topology we will use *order-open*, *order-closed* etc.

The Strong Maximum Principle for second order parabolic PDEs yields the following property: If  $u_1, u_2 \in X^\alpha$  and  $u_1 < u_2$ , then

$$u(t; u_1) \ll u(t; u_2) \quad \text{for all } t \in (0, \min(\tau(u_1), \tau(u_2)))$$

(compare e.g. Thm. 4.1 of Hirsch [24]). The analogous property holds for solutions of the linearized equation (0.3): For  $u_0 \in X^\alpha$  fixed, if  $w_1, w_2 \in X^\alpha$  and  $w_1 < w_2$ , then

$$w(t; u_0; w_1) \ll w(t; u_0; w_2) \quad \text{for all } t \in (0, \tau(u_0)).$$

$f$  is said to have Property (P) if

(P1)  $\limsup_{|u| \rightarrow \infty} f(t, x, u, p)/u \leq 0$  uniformly in  $t \in [0, T]$ ,  $x \in [0, L]$ ,  $p \in \mathbb{R}$ .

(P2) There is a continuous function  $c: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $|f(t, x, u, p)| \leq c(l)(1 + p^2)$  for any  $l \geq 0$  and each  $t \in [0, T]$ ,  $x \in [0, L]$ ,  $u \in [-l, l]$  and  $p \in \mathbb{R}$ .

(If  $f$  does not depend on  $u_x$  then (P2) is vacuous.)

By standard estimates (see Amann [1] and Chen, Chen and Hale [11]), from Property (P) it follows that each solution to (0.1a)+(0.1b) can be extended to the whole of  $[0, \infty)$ .

We collect what we have said so far in

**THEOREM 0.1.** *Assume that  $f \in C^2(\mathbb{R}_+ \times [0, L] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$  is  $T$ -periodic in  $t$  and has Property (P). Then*

(a) For each  $u_0 \in X^\alpha$  there exists a unique solution  $u(\cdot; u_0) : [0, \infty) \rightarrow X^\alpha$  to (0.2).

(b) The dependence  $u_0 \mapsto u(t; u_0)$  is  $C^2$  (for  $t$  fixed), and for  $t > 0$  and  $w_0 \in X^\alpha$ ,  $D_2 u(t; u_0)w_0$  equals the (unique) solution  $w(t; u_0; w_0)$  to (0.3).

(c) Given  $u_1, u_2 \in X^\alpha$ ,  $u_1 < u_2$ , one has  $u(t; u_1) \ll u(t; u_2)$  for all  $t > 0$ . For  $u_0 \in X^\alpha$  fixed, if  $w_1, w_2 \in X^\alpha$  and  $w_1 < w_2$ , then  $w(t; u_0; w_1) \ll w(t; u_0; w_2)$  for  $t > 0$ .

Define a  $C^2$  mapping  $S: X^\alpha \rightarrow X^\alpha$  as

$$Su_0 := u(T; u_0).$$

As  $f$  is  $T$ -periodic in  $t$ , so is  $F$ . This implies that if  $u(\cdot)$  is a solution of (0.2a), then so is  $u(\cdot + T)$ . Consequently,  $S^{n+1}u_0 = u((n+1)T; u_0) = u(T; u(nT; u_0))$ .  $S$  is called the *period map* for (0.2).

By Theorem 0.1(c),

$$Su_1 \ll Su_2 \quad \text{whenever } u_1 < u_2.$$

and, for each  $u_0 \in X^\alpha$ ,

$$DS(u_0)w_1 \ll DS(u_0)w_2 \quad \text{whenever } w_1 < w_2.$$

We refer to these properties by saying that  $S$  is a  $C^2$  *strongly monotone* mapping (see Mierczyński [32]).

$S$  is called *order-compact* if  $\text{cl} S[u_1, u_2]$  is compact for any closed order-interval  $[u_1, u_2]$ .

We let  $\mathcal{E}$  denote the set of fixed points of  $S$ . For  $u_0 \in X^\alpha$ , by the *forward semitrajectory* of  $u_0$  we understand the set  $\{S^n u_0 : n \in \mathbb{Z}_+\}$ . A *backward semitrajectory* of  $u_0$  is a set  $\{u_0 \cdot (-n) : n \in \mathbb{Z}_+\}$  such that  $u_0 \cdot 0 = u_0$  and  $S(u_0 \cdot (-n)) = u_0 \cdot (-n+1)$  for  $n = 1, 2, \dots$ . It is clear that whenever  $S$  is injective (as is the case for our period map), if  $u_0$  has a backward semitrajectory then it is unique. The *full trajectory* of  $u_0$  is the union of its forward and backward semitrajectories. A set  $B \subset X^\alpha$  is *forward invariant* if  $SB \subset B$ , and *invariant* if  $SB = B$ . For  $B$  forward invariant we set  $S^{-\infty}B := \bigcup_{n=0}^{\infty} S^{-n}B$ . Then  $S^{-\infty}B$  is clearly invariant.

The forward semitrajectory of  $u_0 \in X^\alpha$  is said to be *convergent* to  $v_0 \in \mathcal{E}$  (written  $\omega(u_0) = v_0$ ) if

$$\lim_{n \rightarrow \infty} \|S^n u_0 - v_0\|_\alpha = 0.$$

In the language of differential equations, this is equivalent to saying that the solution  $u(\cdot; v_0)$  is  $T$ -periodic and

$$(0.4) \quad \lim_{n \rightarrow \infty} \|u(nT + \theta; u_0) - u(\theta; v_0)\|_\alpha = 0 \quad \text{uniformly in } \theta \in [0, T].$$

Given  $v_0 \in \mathcal{E}$ , let  $K(v_0)$  denote the set of those  $\tilde{u} \in X^\alpha$  for which  $\omega(\tilde{u}) = v_0$ .

By a *global attractor* for  $S$  we understand a compact invariant set ATT such that for each bounded  $B \subset X^\alpha$  the Hausdorff distance between  $S^n B$  and ATT converges to 0 as  $n \rightarrow \infty$ . A global attractor is a maximal compact invariant set for  $S$ . For more on global attractors see Hale's book [20].

**THEOREM 0.2.** *Assume that  $f \in C^2(\mathbb{R}_+ \times [0, L] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$  is  $T$ -periodic in  $t$  and has Property (P). Then the period map  $S$  for (0.2) has the following properties:*

- (i)  $S$  is a  $C^2$  *strongly monotone injective mapping*.
- (ii) *Each forward semitrajectory is convergent*.
- (iii)  $S$  is *order-compact*.
- (iv)  $S$  has a *global attractor* ATT.
- (v)  $S$  is *continuous from  $(X^\alpha)_{\text{ord}}$  to  $X^\alpha$* .
- (vi) *Let  $\{u_0 \cdot (-n) : n \in \mathbb{Z}_+\}$  be a bounded backward semitrajectory. Then there exists  $\alpha(u_0) \in \mathcal{E}$  such that  $\lim_{n \rightarrow \infty} u_0 \cdot (-n) = \alpha(u_0)$ .*

PROOF.  $S$  is a  $C^2$  strongly monotone mapping by Theorem 0.1. The injectivity of  $S$  is a consequence of backward uniqueness for (0.1). Part (ii) was proved by Brunovský, Poláčik and Sandstede [10] (for the case of  $f$  independent of  $x$  and  $u_x$  see Chen and Matano [12]). Part (iii) is essentially Prop. 21.2 in Hess' book [23]. Part (iv) is proved in Section 6 of Chen, Chen and Hale [11], and (vi) is Prop. 4.1(b) of Chen and Poláčik [13].

It remains to prove (v). Choose  $u_0 \in X^\alpha$  and a sequence  $\{u_k\}_{k=1}^\infty \subset X^\alpha$  such that  $\lim_{k \rightarrow \infty} \|u_k - u_0\|_{\text{ord}} = 0$ . The set  $\{u_k\}$  is bounded in the  $\|\cdot\|_{\text{ord}}$ -norm, that is, contained in some closed order-interval. By (iii), from  $\{Su_k\}_{k=1}^\infty$  we can extract a subsequence converging (in the  $\|\cdot\|_\alpha$ -norm, hence in the  $\|\cdot\|_{\text{ord}}$ -norm) to some  $u^1 \in X^\alpha$ . Now, Prop. 1.10 of Hirsch [24] states that  $S$  is continuous from  $(X^\alpha)_{\text{ord}}$  into  $(X^\alpha)_{\text{ord}}$ . Thus  $\lim_{k \rightarrow \infty} \|Su_k - Su_0\|_{\text{ord}} = 0$ . Consequently,  $u^1 = Su_0$ . ■

A consequence of the existence of a global attractor is the fact that the set  $\mathcal{E}$  of fixed points is compact.

Consider a linear nonautonomous one-dimensional parabolic PDE

$$(0.6a) \quad \xi_t = \xi_{xx} + a(t, x)\xi_x + b(t, x)\xi, \quad t > 0, \quad 0 < x < L,$$

with  $a, b: \mathbb{R}_+ \times [0, L] \rightarrow \mathbb{R}$  continuous, together with homogeneous Dirichlet boundary conditions

$$(0.6b) \quad \xi(t, 0) = \xi(t, L) = 0, \quad t > 0.$$

For a continuous real-valued function  $\xi$  on  $[0, L]$  for which  $\xi(0) = \xi(L) = 0$  define the *Matano number*  $z(\xi)$  as the supremum of all  $k$  such that there are  $0 < x_1 < \dots < x_k < L$  with  $\xi(x_i)\xi(x_{i+1}) < 0$ .

THEOREM 0.3. *Let  $\xi(t, x)$  be a solution to (0.6a)+(0.6b) defined on  $[0, \infty)$ . Then*

(a) *For each  $t > 0$  one has  $z(\xi(t, \cdot)) < \infty$ .*

(b)  *$z(\xi(t_1, \cdot)) \geq z(\xi(t_2, \cdot))$  for  $0 < t_1 < t_2$ .*

(c) *If for  $t_0 > 0$  there is  $x_0 \in [0, L]$  such that  $\xi(t_0, x_0) = \xi_x(t_0, x_0) = 0$  then for any  $t_1 < t_0 < t_2$  one has  $z(\xi(t_1, \cdot)) > z(\xi(t_2, \cdot))$ .*

PROOF. See Thm. C of Angenent [5]. ■

From now on the standing assumption is:

$S$  is the period map for (0.2), with  $f \in C^2(\mathbb{R}_+ \times [0, L] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ ,  $T$ -periodic in  $t$  and having Property (P).

The next result was essentially proved in Chen, Chen and Hale [11], Thm. 3.1.

THEOREM 0.4. *Let  $v_0 \in \mathcal{E}$ . Then the spectrum  $\sigma(DS(v_0))$  of the compact linear operator  $DS(v_0)$  equals  $\{0\} \cup \{\lambda_1(v_0), \lambda_2(v_0), \dots\}$ , where*

- (i) Each  $\lambda_k(v_0)$  is an algebraically simple real eigenvalue.
- (ii)  $\lambda_1(v_0) > \dots > \lambda_k(v_0) > \lambda_{k+1}(v_0) > \dots > 0$ .
- (iii)  $\lambda_k(v_0) \rightarrow 0$  as  $k \rightarrow \infty$ .
- (iv) An eigenfunction  $\Phi_k(v_0)$  pertaining to  $\lambda_k(v_0)$  has only simple zeros and  $z(\Phi_k(v_0)) = k - 1$ .
- (v) Defining  $Z_k(v_0) := \text{cl}(\text{span}\{\Phi_l(v_0) : l \geq k + 1\})$ , one has

$$\bigcap_{k=0}^{\infty} Z_k(v_0) = \{0\}.$$

For  $v_0 \in \mathcal{E}$  we write  $Y_k(v_0) := \text{span}\{\Phi_1(v_0), \dots, \Phi_k(v_0)\}$ . One has  $X^\alpha = Y_k(v_0) \oplus Z_k(v_0)$ ,  $Z_0(v_0) = X^\alpha$ , and  $\text{codim } Z_k = k$ . A fixed point  $v_0$  is referred to as *hyperbolic* if  $\lambda_k(v_0) \neq 1$  for  $k = 1, 2, \dots$

For the proof of the next result, see Thm. 3.1 of Chen, Chen and Hale [11].

**THEOREM 0.5.** *For each  $u_0 \in X^\alpha$  the following conditions are equivalent:*

- (i) *There is  $k \in \mathbb{Z}_+$  such that  $z(S^n u_0 - \omega(u_0)) = k$  for  $n$  sufficiently large.*
- (ii)  $\lim_{n \rightarrow \infty} \frac{S^n u_0 - \omega(u_0)}{\|S^n u_0 - \omega(u_0)\|_\alpha} = \pm \Phi_{k+1}(\omega(u_0))$ ,
- (iii)  $\lim_{n \rightarrow \infty} \|S^n u_0 - \omega(u_0)\|_\alpha^{1/n} = \lambda_{k+1}(\omega(u_0))$ .

We say the forward semitrajectory of  $u_0 \in X^\alpha$  is *eventually strongly monotone* (written  $u_0 \in \mathcal{M}$ ) if either  $S^n u_0 \ll S^{n+1} u_0$  for sufficiently large  $n$ , or  $S^n u_0 \gg S^{n+1} u_0$  for sufficiently large  $n$ . We write  $u_0 \in \mathcal{N}$  whenever there exists a linearly ordered set  $J \subset \mathcal{E}$  homeomorphic (in fact,  $C^1$ -diffeomorphic) to the interval  $[0, 1] \subset \mathbb{R}$  and containing  $\omega(u_0)$  in its relative interior (see Mierczyński [32]).

**THEOREM 0.6.** *The union  $\mathcal{M} \cup \mathcal{N}$  is order-open and dense in  $X^\alpha$ .*

**Proof.** For order-openness, see Lemma 4.8 and Thm. 5.7 of Takáč [42] (compare also Mierczyński [32]). For denseness, see Thm. 4.1 of [32]. ■

We say  $u_0 \in X^\alpha$  is *order  $\omega$ -stable* (written  $u_0 \in \mathcal{S}$ ) if there are sequences  $\{\underline{u}_k\}$ ,  $\{\bar{u}_k\}$  such that  $\dots < \underline{u}_k < \underline{u}_{k+1} < \dots < u_0 < \dots < \bar{u}_{k+1} < \bar{u}_k < \dots$ ,  $\lim_{k \rightarrow \infty} \|\underline{u}_k - u_0\|_\alpha = \lim_{k \rightarrow \infty} \|\bar{u}_k - u_0\|_\alpha = 0$  and  $\lim_{k \rightarrow \infty} \|\omega(\underline{u}_k) - \omega(u_0)\|_\alpha = \lim_{k \rightarrow \infty} \|\omega(\bar{u}_k) - \omega(u_0)\|_\alpha = 0$ . Points from  $\mathcal{U} := X^\alpha \setminus \mathcal{S}$  are referred to as  *$\omega$ -unstable*.

**THEOREM 0.7.** *For any nondegenerate Gaussian measure  $\mu$  on  $X^\alpha$ ,  $\mu(\mathcal{U}) = 0$ .*

**Proof.** See Thm. 0.1 of Takáč [42]. ■

For more on Gaussian measures the reader is referred to Aronszajn [7] or Phelps [36], and in the context of strongly monotone dynamical systems, to Hirsch [24].

For  $k \in \mathbb{Z}_+$ , define  $\mathcal{C}(k)$  to be the set of those  $\tilde{u} \in X^\alpha$  for which  $z(S^n \tilde{u} - \omega(\tilde{u})) \geq k$  for all  $n \in \mathbb{Z}_+$ . Define  $\mathcal{B}(k)$  to be the complement of  $\mathcal{C}(k)$  in  $X^\alpha$ . By Theorem 0.3,  $\mathcal{B}(k) = \{\tilde{u} \in X^\alpha : \text{there is } n_0 \in \mathbb{Z}_+ \text{ such that } z(S^n \tilde{u} - \omega(\tilde{u})) < k \text{ for } n \geq n_0\}$ .

Now we are in a position to formulate the main results of this paper.

**THEOREM 0.8.** *For each  $k \geq 1$ ,  $\mathcal{C}(k)$  is contained in the union of finitely many Hölder submanifolds-with-boundary of codimension not smaller than  $k - 1$ . Furthermore, there exists  $M' \in \mathbb{Z}_+$  such that for each  $k > M'$  those submanifolds are  $C^1$  and of codimension  $k - 1$  or  $k$ .*

**Proof.** See Theorems 1.14 and 1.16. ■

For manifolds and manifolds-with-boundary (modeled on Banach spaces) the reader is referred e.g. to Lang's book [28].

It should be mentioned here that H. Koch [26] obtained results on global Hölder conjugacy between the flow of the (time-independent) equation  $u_t = u_{xx} + f(x, u)$  (with Dirichlet boundary conditions) and the direct product of a flow on a finite-dimensional inertial manifold and countably many linear one-dimensional flows; however, no characterization in terms of Matano number was given.

**THEOREM 0.9.** *The set  $\mathcal{B}(2)$  is order-open and dense in  $\mathcal{M} \cup \mathcal{N}$ , hence in  $X^\alpha$ .*

**Proof.** See Theorems 2.4–2.6. ■

We define  $\mathcal{D} := \mathcal{S} \cap \mathcal{B}(2)$ .

**THEOREM 0.10.**  *$\mu(X^\alpha \setminus \mathcal{D}) = 0$  for any nondegenerate Gaussian measure  $\mu$  on  $X^\alpha$ .*

**Proof.** See Theorem 3.4. ■

**1. Invariant manifolds.** The following auxiliary results will be of great use in the sequel.

**LEMMA 1.1.** *Let  $U$  be an open set and let  $B \subset U$  be a forward invariant set such that  $B = S^{-\infty} B \cap U$ . Then for each  $u_0 \in S^{-\infty} B$  there are  $n_0 \in \mathbb{Z}_+$  and a neighborhood  $Y \ni u_0$  such that  $S^{-\infty} B \cap Y = S^{-n_0} B \cap Y$ .*

**Proof.** Let  $n_0$  be the least nonnegative integer for which  $S^{n_0} u_0$  is in  $B$ . By continuity, there is an open neighborhood  $Y$  of  $u_0$  such that  $S^{n_0} Y \subset U$ . For each  $\tilde{u} \in S^{-\infty} B \cap Y$  we have  $S^{n_0} \tilde{u} \in S^{-\infty} B \cap S^{n_0} Y \subset S^{-\infty} B \cap U = B$ , hence  $S^{-\infty} B \cap Y \subset S^{-n_0} B \cap Y$ . The reverse inclusion is obvious. ■



LEMMA 1.2. Let  $D \subset X^\alpha$  be a  $C^1$  embedded submanifold-with-boundary of finite codimension  $k$ . Then for each  $n \in \mathbb{Z}_+$  the inverse image  $S^{-n}D$  is a  $C^1$  embedded submanifold-with-boundary of codimension  $k$ .

Proof. Compare the proof of Thm. 6.1.9(ii) in Henry’s book [21]. ■

The next theorem is a consequence of the results on invariant foliations contained in the paper [15] by Chow, Lin and Lu.

THEOREM 1.3. Assume that  $v_0 \in \mathcal{E}$  is such that  $\lambda_{k_0}(v_0) = 1$ . Then there exist a neighborhood  $U$  of  $v_0$  and

(i) A (not necessarily unique) locally invariant  $C^1$  submanifold  $W_{\text{loc}}^{\text{cs}}(v_0) \subset U$  of codimension  $k_0 - 1$  (a center-stable manifold of  $v_0$ ), tangent at  $v_0$  to  $Z_{k_0-1}(v_0)$ . (Locally invariant means that  $SW_{\text{loc}}^{\text{cs}}(v_0) \cap U \subset W_{\text{loc}}^{\text{cs}}(v_0)$  and  $S^{-1}W_{\text{loc}}^{\text{cs}}(v_0) \cap U \subset W_{\text{loc}}^{\text{cs}}(v_0)$ .)

(ii) A (not necessarily unique) locally invariant  $C^1$  submanifold  $W_{\text{loc}}^{\text{c}}(v_0) \subset U$  of dimension 1 (a center manifold of  $v_0$ ), tangent at  $v_0$  to  $\text{span } \Phi_{k_0}(v_0)$ . The center manifold  $W_{\text{loc}}^{\text{c}}(v_0)$  can be written as  $\{v_0 + \bar{u}_1 + \Psi(\bar{u}_1) : \bar{u}_1 \in U_1\}$ , where  $U_1$  is a neighborhood of zero in  $\text{span } \Phi_{k_0}(v_0)$  and  $\Psi : U_1 \rightarrow Z_{k_0}(v_0)$  is a  $C^1$  embedding with  $D\Psi(0) = 0$ .

(iii) A foliation  $\{\mathcal{F}^{\text{ss}}(\bar{u})\}$  of  $W_{\text{loc}}^{\text{cs}}(v_0)$  by pairwise disjoint submanifolds of codimension  $k_0$  (a strong stable foliation of  $W_{\text{loc}}^{\text{cs}}(v_0)$ ), indexed by points  $\bar{u}$  from  $W_{\text{loc}}^{\text{c}}(v_0)$ , with each leaf  $\mathcal{F}^{\text{ss}}(\bar{u})$  transverse to  $W_{\text{loc}}^{\text{c}}(v_0)$  at  $\bar{u}$ . For each  $\bar{u} = v_0 + \bar{u}_1 + \Psi(\bar{u}_1)$ ,  $\bar{u}_1 \in U_1$ , the leaf  $\mathcal{F}^{\text{ss}}(\bar{u})$  can be written as  $\{v_0 + \bar{u}_1 + \Psi(\bar{u}_1) + \tilde{u}_2 + \Xi(\bar{u}_1, \tilde{u}_2) : \tilde{u}_2 \in U_2\}$ , where  $U_2$  is a neighborhood of zero in  $Z_{k_0}(v_0)$ , the function  $\Xi : U_1 \times U_2 \rightarrow \text{span } \Phi_{k_0}(v_0)$  is continuous,  $\Xi(\bar{u}_1, \cdot)$  is a  $C^2$  embedding with  $D_2 D_2 \Xi$  continuous,  $\Xi(\cdot, \tilde{u}_2)$  is a  $C^1$  embedding with  $D_1 \Xi$  continuous,  $D_1 \Xi(0) = 0$  and  $D_2 \Xi(0) = 0$ . The mapping  $U_1 \times U_2 \ni (\bar{u}_1, \tilde{u}_2) \mapsto v_0 + \bar{u}_1 + \Psi(\bar{u}_1) + \tilde{u}_2 + \Xi(\bar{u}_1, \tilde{u}_2)$  is a  $C^1$  diffeomorphism onto its image. The foliation  $\{\mathcal{F}^{\text{ss}}(\bar{u})\}$  is locally forward invariant in the sense that whenever  $\tilde{u} \in \mathcal{F}^{\text{ss}}(\bar{u})$  then  $S\tilde{u} \in \mathcal{F}^{\text{ss}}(S\bar{u})$  provided that  $S\bar{u}$  is in  $U$ . Moreover, there are an equivalent norm  $\|\cdot\|_*$  and constants  $K > 0$  and  $0 < \varrho < 1$  such that the following inequalities hold:

$$(1.1) \quad \|S^n \tilde{u}^1 - S^n \tilde{u}^2\|_* \leq \varrho^n \|\tilde{u}^1 - \tilde{u}^2\|_*$$

for any  $\bar{u} \in W_{\text{loc}}^{\text{c}}(v_0)$ ,  $\tilde{u}^1, \tilde{u}^2 \in \mathcal{F}^{\text{ss}}(\bar{u})$ ,  $n \in \mathbb{Z}_+$ , as long as  $S^n \bar{u}$  is in  $U$ , and

$$(1.2) \quad \frac{\|S^n \tilde{u}^1 - S^n \tilde{u}^2\|_*}{\|S^n \bar{u} - S^n \hat{u}\|_*} \leq K \varrho^n \frac{\|\tilde{u}^1 - \tilde{u}^2\|_*}{\|\bar{u} - \hat{u}\|_*}$$

for any  $\bar{u} \in W_{\text{loc}}^{\text{c}}(v_0)$ ,  $\hat{u} \in W_{\text{loc}}^{\text{c}}(v_0)$ ,  $\bar{u} \neq \hat{u}$ ,  $\tilde{u}^1, \tilde{u}^2 \in \mathcal{F}^{\text{ss}}(\bar{u})$ ,  $n \in \mathbb{Z}_+$ , as long as  $S^n \bar{u}$  and  $S^n \hat{u}$  are in  $U$ .

The foliation  $\{\mathcal{F}^{\text{ss}}(\bar{u})\}$  is locally unique in  $W_{\text{loc}}^{\text{cs}}(v_0)$  in the sense that if  $S^n \tilde{u} \in U$  for all  $n \in \mathbb{Z}_+$ ,  $S^n \bar{u} \in W_{\text{loc}}^{\text{c}}(v_0)$  for all  $n \in \mathbb{Z}_+$ , and  $\|S^n \tilde{u} - S^n \bar{u}\|_\alpha = O(\bar{\varrho}^n)$  for some  $\bar{\varrho} < 1$ , then  $\tilde{u} \in \mathcal{F}^{\text{ss}}(\bar{u})$ .

PROOF. For a proof of (i) and (ii) see Chen and Poláčik [13]. It remains to prove (iii). For notational simplicity put  $v_0 = 0$  and write  $X_0 := Z_{k_0-1}(0)$ ,  $X_1 := \text{span } \Phi_{k_0}(0)$ ,  $X_2 := Z_{k_0}(0)$ . Since the spectral radius of  $DS(0)$  restricted to  $X_2$  is less than 1, one can find a norm  $\|\cdot\|'$  on  $X_2$  such that  $\|DS(0)|_{X_2}\|' < 1$ . Define  $\|\tilde{u}\|_* = \|\tilde{u}_1\|_\alpha + \|\tilde{u}_2\|'$ , where  $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$  with  $\tilde{u}_1 \in X_1$ ,  $\tilde{u}_2 \in X_2$ .

Consider the linearized abstract parabolic PDE on  $X_0$

$$(1.3) \quad \frac{dw}{dt} + Aw = D_2F(t, u(t; 0))w, \quad w \in X_0.$$

(1.3) gives rise to an evolution operator  $S(t, s)$  in the following way:  $S(t, s)w_0$ ,  $t \geq s$ ,  $w_0 \in X_0$ , is a solution to (1.3) satisfying the initial condition  $S(s, s)w_0 = w_0$ . Obviously,  $S(T, 0) = DS(0)|_{X_0}$ . In a less abstract language,  $S(t, s)w_0 = w(t, s; 0; w_0)$  for  $t > s$ , where  $w(\cdot, s; 0; w_0)$  is the solution of the initial-boundary value problem

$$\begin{aligned} w_t &= w_{xx} + \frac{\partial f}{\partial u_x}(t, x, u(t; 0)(x), u_x(t; 0)(x))w_x \\ &\quad + \frac{\partial f}{\partial u}(t, x, u(t; 0)(x), u_x(t; 0)(x))w, \\ w(t, 0) &= w(t, L) = 0, \quad w(s, x) = w_0(x). \end{aligned}$$

As  $t \mapsto D_2F(\cdot, u(\cdot; 0))$  is  $T$ -periodic, we have  $S(T + t, T + s) = S(t, s)$  for  $t \geq s$ .

By the theory of evolution operators ([21] or [35]) there are constants  $\omega \in \mathbb{R}$  and  $M > 0$  such that

$$\|S(t, s)w_0\|_\alpha \leq M \exp(-\omega(t - s))\|w_0\|_\alpha \quad \text{for } t \geq s, w_0 \in X^\alpha,$$

and

$$\|S(t, s)w_0\|_\alpha \leq M(t - s)^{-\alpha} \exp(-\omega(t - s))\|w_0\| \quad \text{for } t > s, w_0 \in X^\alpha.$$

Since the spectrum of  $DS(0)|_{X_0}$  is contained in  $\{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$ , it follows that for each  $\omega > 0$  we can find  $M(\omega) > 0$  such that

$$\|S(t, s)w_0\|_\alpha \leq M(\omega) \exp(-\omega(t - s))\|w_0\|_\alpha \quad \text{for } t \geq s, w_0 \in X_0,$$

and

$$(1.4) \quad \|S(t, s)w_0\|_\alpha \leq M(\omega)(t - s)^{-\alpha} \exp(-\omega(t - s))\|w_0\| \quad \text{for } t > s, w_0 \in X_0.$$

Let  $i = 0, 1, 2$ . As  $X_i$  is spanned by a set of eigenvectors for  $DS(0)$ , by injectivity it follows that the closure of  $S(T, 0)X_i$  in  $X^\alpha$  is equal to  $X_i$ . Similarly, the closure of  $S(T, 0)X_i$  in  $X$  is equal to the closure of  $X_i$  in  $X$ . Define  $X_i(t) := \text{cl}(S(t, 0)X_i)$  and  $\tilde{X}_i(t) := \text{cl}_X(S(t, 0)X_i)$  for  $t \geq 0$ . Then  $T$ -periodicity and the cocycle property ( $S(u, s) = S(u, t)S(t, s)$  for  $u \geq t \geq s$ )

entail  $X_i(nT + t) = X_i(t)$  and  $\tilde{X}_i(nT + t) = \tilde{X}_i(t)$  for  $t \in [0, T]$ ,  $n \in \mathbb{Z}_+$ . Furthermore, since  $X_1$  is one-dimensional and  $S(T, 0)$  is injective it follows that  $S(t, s)|_{X_1(s)}$  is a linear isomorphism of  $X_1(s)$  onto  $X_1(t)$  for  $t \geq s$ . This allows one to define, for  $s > t$ ,  $S(t, s)|_{X_1(s)}$  as  $(S(s, t)|_{X_1(t)})^{-1}$ . It is clear that  $X_1(t) = \tilde{X}_1(t)$  as Banach spaces. Define  $P(t)$  to be the projection of  $\tilde{X}_0(t)$  onto  $X_1(t)$  along  $\tilde{X}_2(t)$ .

Since the spectrum of  $(DS(0)|_{X_1})^{-1}$  equals  $\{1\}$ , for each  $\bar{\alpha} > 0$  we can find a constant  $\bar{M}(\bar{\alpha}) > 0$  such that

$$\|(DS(0))^{-n}w_0\|_\alpha \leq \bar{M}(\bar{\alpha}) \exp(-\bar{\alpha}(-nT))\|w_0\|_\alpha \quad \text{for } n \in \mathbb{Z}_+, w_0 \in X_1.$$

As  $S(nT + t, s)|_{X_1(s)} = DS^n(0)S(t, s)|_{X_1(s)}$ , it follows that there are positive constants  $M_1(\bar{\alpha})$  and  $M_2(\bar{\alpha})$  such that

$$(1.5a) \quad \|S(t, s)P(s)w_0\|_\alpha \leq M_1(\bar{\alpha}) \exp(-\bar{\alpha}(t - s))\|w_0\|_\alpha \quad \text{for } t \leq s, w_0 \in X_0,$$

$$(1.5b) \quad \|S(t, s)P(s)w_0\|_\alpha \leq M_2(\bar{\alpha}) \exp(-\bar{\alpha}(t - s))\|w_0\|_\alpha \quad \text{for } t \leq s, w_0 \in X_0.$$

The spectrum of  $DS(0)|_{X_2}$  is contained in  $\{\zeta \in \mathbb{C} : |\zeta| \leq \lambda_{k_0+1}(0)\}$ , hence for any  $\bar{\beta}$ ,  $\bar{\alpha} < \bar{\beta} < -\log(\lambda_{k_0+1}(0))/T$ , we can find a constant  $\bar{M}(\bar{\beta}) > 0$  such that  $\|DS^n(0)w_0\|_\alpha \leq \bar{M}(\bar{\beta}) \exp(-\bar{\beta}nT)\|w_0\|_\alpha$  for  $n \in \mathbb{Z}_+$  and  $w_0 \in X_2$ . Proceeding as above and taking account of (1.4) we show that there are constants  $M_3(\bar{\beta}) > 0$  and  $M_4(\bar{\beta}) > 0$  such that

$$(1.5c) \quad \|S(t, s)(\text{Id} - P(s))w_0\|_\alpha \leq M_3(\bar{\beta}) \exp(-\bar{\beta}(t - s))\|w_0\|_\alpha \quad \text{for } t \geq s, w_0 \in X_0,$$

$$(1.5d) \quad \|S(t, s)(\text{Id} - P(s))w_0\|_\alpha \leq M_4(\bar{\beta}) \exp(-\bar{\beta}(t - s))\|w_0\|_\alpha \quad \text{for } t > s, w_0 \in X_0.$$

In the terminology of Chow, Lin and Lu [15], the inequalities (1.5) mean that the evolution operator  $S(t, s)$  has a *pseudo-dichotomy* on the triplet  $(X_0, \text{cl}_X X_0, \text{cl}_X X_0)$ . Choose positive numbers  $\bar{\alpha}$ ,  $\bar{\beta}$ ,  $\gamma$  and  $\omega$  so that  $0 < \bar{\alpha} < \gamma < 2\gamma < \bar{\beta} < -\log(\lambda_{k_0+1}(0))/T$  and  $\gamma > \omega$ . As  $X^\alpha$  is a Hilbert space we can modify  $F$  outside some neighborhood of the compact set  $\{(t, u(t, 0)) \in X^\alpha : t \in [0, T]\} \subset [0, T] \times X^\alpha$  so that all the inequalities in Thm. 3.4 of [15] are satisfied. An application of that theorem provides the existence of the desired foliation. ■

For  $\tilde{u} \in W_{\text{loc}}^{\text{cs}}(v_0)$  we write  $\Pi\tilde{u} = \bar{u}$  if  $\tilde{u} \in \mathcal{F}^{\text{ss}}(\bar{u})$ . By the properties of the strong stable foliation,  $\Pi$  is of class  $C^1$ .

Notice that since  $W_{\text{loc}}^c(v_0)$  is tangent to  $\Phi_{k_0}(v_0)$  at  $v_0$ ,  $(\Phi_{k_0}(v_0))'(0) > 0$ ,  $z(\Phi_{k_0}(v_0))$  has only simple zeros and  $X^\alpha \subset C^1([0, L])$ , it follows that

$W_{\text{loc}}^c(v_0)$  (after possibly shrinking) can be linearly ordered by the relation

$$u_1 \prec u_2 \quad \text{if and only if} \quad u_1'(0) < u_2'(0).$$

We write  $u_1 \preceq u_2$  if  $u_1 \prec u_2$  or  $u_1 = u_2$ . For  $u_1, u_2 \in W_{\text{loc}}^c(v_0)$  with  $u_1 \prec u_2$ , write  $\langle u_1, u_2 \rangle := \{\tilde{u} \in W_{\text{loc}}^c(v_0) : u_1 \preceq \tilde{u} \preceq u_2\}$ , and for  $u_1, u_2 \in W_{\text{loc}}^c(v_0)$  with  $u_1 \prec u_2$ , write  $\langle\langle u_1, u_2 \rangle\rangle := \{\tilde{u} \in W_{\text{loc}}^c(v_0) : u_1 \prec \tilde{u} \prec u_2\}$ .

We define  $W_{\text{loc},+}^c(v_0) = \{\tilde{u} \in W_{\text{loc}}^c(v_0) : v_0 \prec \tilde{u}\}$  and  $W_{\text{loc},-}^c(v_0) = \{\tilde{u} \in W_{\text{loc}}^c(v_0) : \tilde{u} \prec v_0\}$ .

LEMMA 1.4. *Let  $v_0 \in \mathcal{E}$  be nonhyperbolic. Assume that  $W_{\text{loc}}^c(v_0)$  is so small that it is linearly ordered by  $\prec$ . Then for any  $u_1, u_2 \in W_{\text{loc}}^c(v_0)$ ,  $u_1 \prec u_2$ , such that  $Su_1, Su_2 \in W_{\text{loc}}^c(v_0)$  one has  $Su_1 \prec Su_2$ . Analogously, for any  $u_1, u_2 \in W_{\text{loc}}^c(v_0)$ ,  $u_1 \prec u_2$ , such that  $S^{-1}u_1, S^{-1}u_2$  exist and are in  $W_{\text{loc}}^c(v_0)$ , one has  $S^{-1}u_1 \prec S^{-1}u_2$ . As a consequence,  $W_{\text{loc},+}^c(v_0)$  and  $W_{\text{loc},-}^c(v_0)$  are locally invariant.*

PROOF. Let  $k_0$  be such that  $\lambda_{k_0}(v_0) = 1$ . Proceeding as in the proof of Theorem 0.5 we see that the difference  $\xi(t, x) := u(t; u_2)(x) - u(t; u_1)(x)$  satisfies a one-dimensional linear parabolic PDE of second order

$$\xi_t = \xi_{xx} + a(t, x)\xi_x + b(t, x)\xi, \quad t \in (0, 1], \quad x \in (0, L),$$

with the Dirichlet boundary conditions. In view of Theorem 0.3 it is clear that  $z(\xi(t, \cdot)) = k_0 - 1$  for all  $t \in [0, 1]$ . One has  $\xi(t, 0) = 0$  for all  $t \in [0, 1]$ ,  $\xi_x(0, 0) > 0$  and  $\xi_x(1, 0) \neq 0$ . If  $\xi_x(1, 0) < 0$  then  $\xi_x(\tau, 0) = 0$  for some  $\tau \in (0, 1)$ , which would imply (again by Theorem 0.3) that the Matano number of  $\xi$  drops at  $\tau$ , a contradiction. Consequently,  $\xi_x(1, 0) > 0$ , which means  $Su_1 \prec Su_2$ . The statement on  $S^{-1}$  follows easily. ■

From now on, when speaking of a local center manifold we shall tacitly assume that (1)  $W_{\text{loc}}^c(v_0)$  is small enough to be linearly ordered by  $\prec$ , and (2) if  $v_0$  is isolated in  $\mathcal{E} \cap (v_0 \cup W_{\text{loc},+}^c(v_0))$  [resp. in  $\mathcal{E} \cap (v_0 \cup W_{\text{loc},-}^c(v_0))$ ] then  $\mathcal{E} \cap \text{cl } W_{\text{loc},+}^c(v_0) = v_0$  [resp.  $\mathcal{E} \cap \text{cl } W_{\text{loc},-}^c(v_0) = v_0$ ].

Define  $W_{\text{loc},+}^{\text{cs}}(v_0) := \{\tilde{u} \in W_{\text{loc}}^{\text{cs}}(v_0) : H\tilde{u} \in W_{\text{loc},+}^c(v_0)\}$  and  $W_{\text{loc},-}^{\text{cs}}(v_0) := \{\tilde{u} \in W_{\text{loc}}^{\text{cs}}(v_0) : H\tilde{u} \in W_{\text{loc},-}^c(v_0)\}$ .

PROPOSITION 1.5. *Assume that  $v_0 \in \mathcal{E}$  is nonhyperbolic. Then only the following mutually exclusive cases are possible:*

- (a) For each  $\tilde{u} \in W_{\text{loc},+}^c(v_0)$ ,  $v_0 \prec \dots \prec S^{n+1}\tilde{u} \prec S^n\tilde{u} \prec \dots \prec S\tilde{u} \prec \tilde{u}$ , and  $\omega(\tilde{u}) = v_0$ .
- (b) Each  $\tilde{u} \in W_{\text{loc},+}^c(v_0)$  has backward semitrajectory  $\{\tilde{u} \cdot (-n) : n \in \mathbb{Z}_+\} \subset W_{\text{loc},+}^c(v_0)$ , with  $v_0 \prec \tilde{u} \cdot (-n) \prec \tilde{u} \cdot (-n+1) \prec \dots \prec \tilde{u} \cdot (-1) \prec \tilde{u}$  and  $\alpha(\tilde{u}) = v_0$ .
- (c)  $v_0$  is not isolated in  $\mathcal{E} \cap (v_0 \cup W_{\text{loc},+}^c(v_0))$ .

PROOF. Assume (c) does not hold. As  $W_{\text{loc}}^c(v_0)$  is linearly ordered by  $\prec$  and homeomorphic to the real interval  $(0, 1)$ , its relative topology is generated by the sets  $\langle\langle u_1, u_2 \rangle\rangle$  with  $u_1 \prec u_2$ . By the local invariance of  $W_{\text{loc}}^c(v_0)$ , there is  $u_0 \in W_{\text{loc}}^c(v_0)$  such that  $S\langle\langle v_0, u_0 \rangle\rangle \subset W_{\text{loc},+}^c(v_0)$ . The injectivity of  $S$  implies that  $S|\langle v_0, u_0 \rangle$  is a diffeomorphism from  $\langle v_0, u_0 \rangle$  onto  $\langle v_0, Su_0 \rangle$ , preserving the  $\prec$  relation. Now we proceed as in the proof of Thm. 2.1 of Mierczyński [32]. ■

In case (a) or (c) we say that  $v_0$  is *stable* in  $W_{\text{loc},+}^c(v_0)$ . In case (b) we say that  $v_0$  is *unstable* in  $W_{\text{loc},+}^c(v_0)$ .

THEOREM 1.6. *Assume that  $v_0 \in \mathcal{E}$ . Let  $k_0$  be such that  $\lambda_{k_0}(v_0) \geq 1$  and  $\lambda_{k_0+1}(v_0) < 1$ . Then there exists a nested family  $W_{k_0}(v_0) \supset W_{k_0+1}(v_0) \supset \dots$  of  $C^1$  embedded invariant submanifolds with the following properties:*

(a) *For each  $k \geq k_0$  the submanifold  $W_k(v_0)$  is tangent to  $Z_k(v_0)$  at  $v_0$  (hence  $\text{codim } W_k(v_0) = k$ ).*

$$\begin{aligned} \text{(b) } W_k(v_0) &= \{ \tilde{u} \in X^\alpha : \omega(\tilde{u}) = v_0 \text{ and } \lim_{n \rightarrow \infty} \|S^n \tilde{u} - v_0\|_\alpha^{1/n} \leq \lambda_{k+1}(v_0) \} \\ &= \left\{ \tilde{u} \in X^\alpha : \omega(\tilde{u}) = v_0 \text{ and } \right. \\ &\quad \left. \lim_{n \rightarrow \infty} \frac{S^n \tilde{u} - v_0}{\|S^n \tilde{u} - v_0\|_\alpha} = \pm \Phi_l(v_0) \text{ for some } l \geq k+1 \right\} \cup v_0 \\ &= \{ \tilde{u} \in X^\alpha : \omega(\tilde{u}) = v_0 \text{ and } z(S^n \tilde{u} - v_0) \geq k \text{ for each } n \in \mathbb{Z}_+ \}. \end{aligned}$$

(c)  $\bigcap_{k=k_0}^\infty W_k(v_0) = \{v_0\}$ .

(d) *If  $\lambda_{k_0}(v_0) = 1$  then there is a neighborhood  $U$  of  $v_0$  such that  $W_{k_0}(v_0) \cap U = \mathcal{F}^{\text{ss}}(v_0)$ , where  $\mathcal{F}^{\text{ss}}(v_0)$  is the leaf of the strong stable foliation for any local center-stable manifold  $W_{\text{loc}}^{\text{cs}}(v_0)$ .*

PROOF. For notational simplicity, assume  $v_0 = 0$ . As in the proof of Theorem 1.3, renorm  $X^\alpha$  (with a new norm  $\|\cdot\|_*$ ) so that  $\|DS(0)|Z_{k_0}(0)\|_* < 1$ .

Fix  $k \geq k_0$ . Write  $\sigma(DS(0))$  as the disjoint union  $\sigma_1(DS(0)) \cup \sigma_2(DS(0))$ , where

$$\begin{aligned} \sigma_1(DS(0)) &:= \{ \zeta \in \sigma(DS(0)) : |\zeta| < \varrho \}, \\ \sigma_2(DS(0)) &:= \{ \zeta \in \sigma(DS(0)) : |\zeta| > \varrho \}, \end{aligned}$$

$\varrho \in (\lambda_{k+1}(0), \lambda_k(0))$ . Evidently,  $\sigma_2(DS(0)) = \{\lambda_1(0), \dots, \lambda_k(0)\}$  and  $\sigma_1(DS(0)) = \{\lambda_{k+1}(0), \dots\} \cup \{0\}$ . Write  $S\tilde{u} = DS(0)\tilde{u} + R(\tilde{u})$ . As  $X^\alpha$  is a Hilbert space, we can modify  $S$  outside a neighborhood  $U$  of 0 so as to be able to apply Thm. 5.1 of Hirsch, Pugh and Shub [25] yielding the existence of a forward invariant, locally invariant  $C^1$  submanifold  $W_{k,\text{loc}}(0) \subset U$  with the following properties:

(A)  $W_{k,\text{loc}}(0)$  is tangent at 0 to  $Z_k(0)$ , and

(B)  $W_{k,\text{loc}}(0) = \{\tilde{u} \in U : \|S^n \tilde{u}\|_*/\varrho^n \rightarrow 0 \text{ as } n \rightarrow \infty\} = \{\tilde{u} \in U : \|S^n \tilde{u}\|_*/\varrho^n \text{ stays bounded as } n \rightarrow \infty\}$ .

By the characterization in (B) the submanifold  $W_{k,\text{loc}}(0)$  is locally unique. Because  $\varrho < 1$  and the norms  $\|\cdot\|_\alpha$  and  $\|\cdot\|_*$  are equivalent, one can also write  $W_{k,\text{loc}}(0) = \{\tilde{u} \in U \cap K(0) : \|S^n \tilde{u}\|_\alpha/\varrho^n \rightarrow 0 \text{ as } n \rightarrow \infty\} = \{\tilde{u} \in U \cap K(0) : \|S^n \tilde{u}\|_\alpha/\varrho^n \text{ stays bounded as } n \rightarrow \infty\}$ .

Due to Theorem 0.5 we have

$$\begin{aligned} (1.6) \quad & W_{k,\text{loc}}(0) \\ &= \{\tilde{u} \in U \cap K(0) : z(S^n \tilde{u}) \geq k \text{ for } n \text{ sufficiently large}\} \cup \{0\} \\ &= \left\{ \tilde{u} \in U \cap K(0) : \lim_{n \rightarrow \infty} \frac{S^n \tilde{u}}{\|S^n \tilde{u}\|_\alpha} = \pm \Phi_l(0) \text{ for some } l \geq k + 1 \right\} \cup \{0\}, \end{aligned}$$

whereas

$$\begin{aligned} (1.7) \quad & K(0) \cap U \setminus W_{k,\text{loc}}(0) \\ &= \{\tilde{u} \in U \cap K(0) : z(S^n \tilde{u}) < k \text{ for } n \text{ sufficiently large}\} \\ &= \left\{ \tilde{u} \in U \cap K(0) : \lim_{n \rightarrow \infty} \frac{S^n \tilde{u}}{\|S^n \tilde{u}\|_\alpha} = \pm \Phi_l(0) \text{ for some } l = 1, \dots, k \right\}. \end{aligned}$$

As the Matano number is nonincreasing, points in  $W_{k,\text{loc}}(0)$  can also be characterized as those  $\tilde{u} \in U$  for which  $\omega(\tilde{u}) = 0$  and  $z(S^n \tilde{u}) \geq k$  for all  $n \in \mathbb{Z}_+$ .

We now need to show that  $W_k(0) := S^{-\infty}W_{k,\text{loc}}(0)$  is an embedded  $C^1$  submanifold of codimension  $k$ . By (1.6) and (1.7) we conclude that  $W_{k,\text{loc}}(0) = W_k(0) \cap U$ . Now, Lemma 1.1 implies that for each  $u_0 \in W_k(0)$  there are a neighborhood  $Y$  of  $u_0$  and a nonnegative integer  $n_0$  such that  $W_k(0) \cap Y = S^{-n_0}W_{k,\text{loc}}(0) \cap Y$ . By Lemma 1.2,  $S^{-n_0}W_{k,\text{loc}}(0)$  is an embedded  $C^1$  submanifold of codimension  $k$ . The proof of (b) is complete.

In order to prove (c) it is enough to notice that by (b),  $\bigcap_{k=k_0}^\infty W_k(0) = \{\tilde{u} \in K(0) : z(S^n \tilde{u}) = \infty\}$ . An application of Theorem 0.3(a) as in the proof of Theorem 0.5 gives that this is possible only for  $\tilde{u} = 0$ .

To prove (d), let  $W_{\text{loc}}^{\text{cs}}(0) \subset U$  be a local center-stable manifold. Since  $W_{k_0}(0) \cap U$  is forward invariant, by Thm. C.4 of Chen, Chen and Hale [11] we have  $W_{k_0}(0) \cap U \subset W_{\text{loc}}^{\text{cs}}(0)$ . The fact that  $W_{k_0}(0) \cap U = \mathcal{F}^{\text{ss}}(0)$  follows by the characterization of  $W_{k_0}(0)$  in (B) and the uniqueness of the strong stable foliation. ■

REMARKS. 1. For autonomous parabolic PDEs of second order the analogous result was proved (for a hyperbolic  $v_0$ ) by Brunovský and Fiedler [8]. For an analogue for the Navier–Stokes equations, see Foias and Saut [18].

2. For autonomous parabolic PDEs of second order, not necessarily one-dimensional, the existence of a global one-codimensional  $C^1$  embedded submanifold  $W_1(v_0)$  (in our notation) for a fixed point  $v_0$  was proved by Poláčik [37] under the assumption that the spectral radius of the linearization at  $V_0$  is less than 1. This assumption was removed by Mierczyński [31]. For a time-periodic case, see Prop. 1.3 of Mierczyński [32]. Compare also a related concept of d-hypersurface in Takáč [42].

**THEOREM 1.7.** *Let  $v_0 \in \mathcal{E}$  with  $\lambda_{k_0}(v_0) = 1$ . Assume that  $v_0$  is stable in  $W_{loc,+}^c(v_0)$ . Then there exists a closed neighborhood  $\bar{V}$  of  $v_0$  with the following properties:*

- (i) *The portion  $W_{loc,+}^{cs}(v_0)$  of the local center-stable manifold  $W_{loc}^{cs}(v_0)$  of  $v_0$  is forward invariant.*
- (ii)  *$W_{loc,+}^{cs}(v_0) = \{\tilde{u} \in \bar{V} : \omega(\tilde{u}) \in \bar{V}, \omega(\tilde{u}) \succeq v_0$  and  $(S^n \tilde{u} - v_0)'(0) > 0$  for large  $n \in \mathbb{Z}_+\} \setminus W_{k_0}(v_0)$*   
 $= \{\tilde{u} \in \bar{V} \cap \mathcal{C}(k_0 - 1) : \omega(\tilde{u}) \in \bar{V}, \omega(\tilde{u}) \succeq v_0$   
*and  $(S^n \tilde{u} - v_0)'(0) > 0$  for large  $n \in \mathbb{Z}_+\} \setminus W_{k_0}(v_0)$ .*
- (iii)  *$W_{loc,+}^{cs}(v_0)$  is unique (up to the choice of the neighborhood  $\bar{V}$ ).*

**PROOF.** Choose a local center-stable manifold  $W^{cs}$  of  $v_0$  and a local center manifold  $W^c$  of  $v_0$  contained in  $W^{cs}$ . By possibly taking  $W^{cs}$  smaller we can assume that  $\lambda_{k_0-1}(\tilde{v}) > 1$  for all  $\tilde{v} \in \mathcal{E} \cap U$ , where  $U$  is the neighborhood of  $v_0$  as in Theorem 1.3. As  $v_0$  is stable in  $W_+^c$  we can find  $u_2 \in W_+^c$  such that  $Su_2 \preceq u_2$ , and consequently  $S\langle v_0, u_2 \rangle \subset \langle v_0, u_2 \rangle$ . Let  $\bar{V}$  be a closed neighborhood of  $v_0$  containing  $\bigcup\{\mathcal{F}^{ss}(\bar{u}) : \bar{u} \in \langle v_0, u_2 \rangle\}$ , and replace  $W^{cs}$  by  $W^{cs} \cap \bar{V}$ . The forward invariance of  $W^{cs}$  follows by the local forward invariance of the strong stable foliation  $\{\mathcal{F}^{ss}(\bar{u})\}$ .

We now proceed to the proof of (ii). Let  $\tilde{u} \in W_+^{cs}$ . By Theorem 1.6(d),  $W_{k_0}(v_0) \cap \bar{V} = \Pi^{-1}v_0$ , so  $\tilde{u} \notin W_{k_0}(v_0)$ . By (i),  $S^n \tilde{u} \in \bar{V}$  for all  $n \in \mathbb{Z}_+$ , hence  $\omega(\tilde{u}) \in \bar{V}$ . As a consequence of (1.1),  $\|II S^n \tilde{u} - S^n \tilde{u}\|_\alpha \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $S^n \tilde{u} \rightarrow \omega(\tilde{u})$  we have  $II S^n \tilde{u} = S^n(II \tilde{u}) \rightarrow \omega(\tilde{u})$ . Therefore  $\omega(\tilde{u})$  belongs to the closure of  $W_+^c$ , hence  $v_0 \preceq \omega(\tilde{u})$ . By the definition of  $W_+^{cs}$  we have  $v_0 \prec II \tilde{u}$ . Assume  $v_0 \prec \omega(\tilde{u})$ . This means that  $(\omega(\tilde{u}) - v_0)'(0) > 0$ . Consider the equality

$$\frac{S^n \tilde{u} - v_0}{\|S^n \tilde{u} - v_0\|_\alpha} = \frac{\omega(\tilde{u}) - v_0}{\|\omega(\tilde{u}) - v_0\|_\alpha} \cdot \frac{\|\omega(\tilde{u}) - v_0\|_\alpha}{\|S^n \tilde{u} - v_0\|_\alpha} + \frac{S^n \tilde{u} - \omega(\tilde{u})}{\|S^n \tilde{u} - v_0\|_\alpha}.$$

As the last term on the right tends to 0 and  $\|\omega(\tilde{u}) - v_0\|_\alpha / \|S^n \tilde{u} - v_0\|_\alpha$  tends to 1, it follows that  $(S^n \tilde{u} - v_0) / \|S^n \tilde{u} - v_0\|_\alpha$  tends to  $(\omega(\tilde{u}) - v_0) / \|\omega(\tilde{u}) - v_0\|_\alpha$  as  $n \rightarrow \infty$ . Since  $X^\alpha$  embeds continuously in  $C^1([0, L])$ ,  $(S^n \tilde{u} - v_0)'(0) > 0$  for  $n$  sufficiently large. Now, let  $\omega(\tilde{u}) = v_0$ . Consider the equality

$$\frac{S^n \tilde{u} - v_0}{\|S^n \tilde{u} - v_0\|_\alpha} = \frac{\|II S^n \tilde{u} - v_0\|_\alpha}{\|S^n \tilde{u} - v_0\|_\alpha} \left( \frac{II S^n \tilde{u} - v_0}{\|II S^n \tilde{u} - v_0\|_\alpha} + \frac{S^n \tilde{u} - II S^n \tilde{u}}{\|S^n \tilde{u} - II S^n \tilde{u}\|_\alpha} \right).$$

In view of (1.2), the second term in parentheses converges to zero. As concerns the first term, it converges to the direction of the normalized tangent vector of  $W_{\text{loc}}^c$  at  $v_0$ , that is, to  $\Phi_{k_0}(v_0)$ . By the inequalities

$$1 - \frac{\|S^n \tilde{u} - \Pi S^n \tilde{u}\|_\alpha}{\|S^n \tilde{u} - v_0\|_\alpha} \leq \frac{\|\Pi S^n \tilde{u} - v_0\|_\alpha}{\|S^n \tilde{u} - v_0\|_\alpha} \leq 1 + \frac{\|S^n \tilde{u} - \Pi S^n \tilde{u}\|_\alpha}{\|S^n \tilde{u} - v_0\|_\alpha}$$

and (1.2) it follows that the middle term converges to 1 as  $n \rightarrow \infty$ . Hence  $(S^n \tilde{u} - v_0)/\|S^n \tilde{u} - v_0\|_\alpha$  tends to  $\Phi_{k_0}(v_0)$ . Since  $X^\alpha$  embeds continuously in  $C^1([0, L])$ , we have  $(S^n \tilde{u} - v_0)'(0) > 0$  for  $n$  sufficiently large.

We have proved that  $W_+^{\text{cs}} \subset \{\tilde{u} \in \bar{V} : \omega(\tilde{u}) \in \bar{V}, v_0 \preceq \omega(\tilde{u}) \text{ and } (S^n \tilde{u} - v_0)'(0) > 0 \text{ for large } n \in \mathbb{Z}_+\} \setminus W_{k_0}(v_0)$ . Now, let  $\tilde{u}$  belong to the latter set. Suppose by way of contradiction that  $z(S^{n_0} \tilde{u} - \omega(\tilde{u})) < k_0 - 1$  for some  $n_0 \in \mathbb{Z}_+$ . Since the Matano number is not increasing we have  $z(S^n \tilde{u} - \omega(\tilde{u})) < k_0 - 1$  for all  $n \geq n_0$ . By Theorem 0.5,  $\|S^n \tilde{u} - \omega(\tilde{u})\|_\alpha^{1/n} \rightarrow \lambda_l(\omega(\tilde{u}))$  for some  $l \leq k_0 - 1$ . But as  $\|S^n \tilde{u} - \omega(\tilde{u})\|_\alpha \rightarrow 0$  we find that  $\|S^n \tilde{u} - \omega(\tilde{u})\|_\alpha^{1/n}$  has limit  $\leq 1$ , which contradicts  $\lambda_l(\omega(\tilde{u})) > 1$ . By Thm. C.4 of Chen, Chen and Hale [11],  $S^n \tilde{u} \in W^{\text{cs}}$  for  $n$  sufficiently large (here we take again  $\bar{V}$  smaller if necessary). As  $W^{\text{cs}}$  is locally invariant,  $\tilde{u} \in W^{\text{cs}}$ . We claim  $\Pi S^n \tilde{u} \in W_+^c$ . Indeed, if  $\Pi S^n \tilde{u} \in W_-^c$  then repeating the above reasoning we would get  $(S^n \tilde{u} - v_0)'(0) < 0$  for  $n$  large. If on the other hand  $\Pi S^n \tilde{u} = v_0$  then  $S^n \tilde{u} \in W_{k_0}(v_0)$ . We thus have  $\tilde{u} \in W_+^{\text{cs}}$ .

Part (iii) follows easily by the characterization in (ii). ■

Evidently Theorem 1.7 has its counterpart for  $W_{\text{loc},-}^{\text{cs}}(v_0)$ .

**PROPOSITION 1.8.** *Assume that  $v_0$  is a nonhyperbolic fixed point. Let  $W_{\text{loc}}^{\text{cs}}(v_0)$  be a center-stable manifold of  $v_0$  and  $U \supset W_{\text{loc}}^{\text{cs}}(v_0)$  be a neighborhood of  $v_0$  as in Theorem 1.3. If there exists  $u_0 \in U \setminus W_{k_0}(v_0)$  with  $\omega(u_0) \in U$  then  $v_0$  is stable in  $W_{\text{loc},+}^{\text{cs}}(v_0)$  or in  $W_{\text{loc},-}^{\text{cs}}(v_0)$ .*

**Proof.** Since  $\omega(u_0) \in U$ ,  $S^n u_0 \in U$  for  $n$  sufficiently large. Due to Thm. C.4 of Chen, Chen and Hale [11],  $S^n u_0 \in W_{\text{loc}}^{\text{cs}}(v_0)$  for large  $n$ . By local invariance of  $W_{\text{loc}}^{\text{cs}}(v_0)$ ,  $S^n u_0 \in W_{\text{loc}}^{\text{cs}}(v_0)$  for all  $n \in \mathbb{Z}_+$ . Consequently,  $\Pi S^n u_0 = S^n \Pi u_0$  is defined for all  $n \in \mathbb{Z}_+$ . By estimate (1.1),  $\Pi S^n u_0 \rightarrow \Pi \omega(u_0) = \omega(u_0)$ . Assume  $u_0 \in W_{\text{loc},+}^{\text{cs}}(v_0)$ , the other case being similar. This means that  $\Pi u_0 \in W_{\text{loc},+}^c(v_0)$ , hence, by Lemma 1.4,  $S^n \Pi u_0 = \Pi S^n u_0 \in W_{\text{loc},+}^c(v_0)$  for all  $n \in \mathbb{Z}_+$ . Further, as  $v_0 \prec \Pi u_0$  and  $\omega(\Pi u_0) \in U$ , we have  $v_0 \preceq \Pi \omega(u_0)$  and  $\omega(u_0) \in \text{cl } W_{\text{loc},+}^c(v_0)$ . Suppose by way of contradiction that  $v_0$  is unstable in  $W_{\text{loc},+}^{\text{cs}}(v_0)$ . Then  $v_0$  is the unique fixed point in  $\text{cl } W_{\text{loc},+}^c(v_0)$ , therefore  $\omega(u_0) = v_0$ . By Proposition 1.5,  $v_0 = \alpha(u_0)$ . We thus have a point  $u_0 \notin \mathcal{E}$  with  $\alpha(u_0) = \omega(u_0)$ , which is impossible by Prop. 5.2 of Chen and Poláčik [13]. ■



The following corollary to Theorem 1.7 and Proposition 1.8 gives a partial classification of nonhyperbolic fixed points.

**THEOREM 1.9.** *Let  $v_0 \in \mathcal{E}$  with  $\lambda_{k_0}(v_0) = 1$ . Then there exists a closed neighborhood  $\bar{V}$  of  $v_0$  such that the following mutually exclusive cases are possible:*

(i)  $\{\tilde{u} \in \bar{V} : \omega(\tilde{u}) \in \bar{V}\} = \{\tilde{u} \in \mathcal{C}(k_0 - 1) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\} = \mathcal{C}(k_0) \cap K(v_0) \cap \bar{V} = W_{k_0}(v_0) \cap \bar{V}$ ,  $v_0$  is unstable in both  $W_{\text{loc},+}^c(v_0)$  and  $W_{\text{loc},-}^c(v_0)$ , and  $v_0$  is isolated in  $\mathcal{E}$ .

(ii)  $\{\tilde{u} \in \bar{V} : \omega(\tilde{u}) \in \bar{V}\} = \{\tilde{u} \in \mathcal{C}(k_0 - 1) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\} = (W_{k_0}(v_0) \cap \bar{V}) \cup W_{\text{loc},+}^{\text{cs}}(v_0)$ ,  $W_{\text{loc},+}^{\text{cs}}(v_0)$  is unique, locally invariant and forward invariant,  $v_0$  is stable in  $W_{\text{loc},+}^c(v_0)$  and unstable in  $W_{\text{loc},-}^c(v_0)$ .

(iii)  $\{\tilde{u} \in \bar{V} : \omega(\tilde{u}) \in \bar{V}\} = \{\tilde{u} \in \mathcal{C}(k_0 - 1) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\} = (W_{k_0}(v_0) \cap \bar{V}) \cup W_{\text{loc},-}^{\text{cs}}(v_0)$ ,  $W_{\text{loc},-}^{\text{cs}}(v_0)$  is unique, locally invariant and forward invariant,  $v_0$  is stable in  $W_{\text{loc},-}^c(v_0)$  and unstable in  $W_{\text{loc},+}^c(v_0)$ .

(iv)  $\{\tilde{u} \in \bar{V} : \omega(\tilde{u}) \in \bar{V}\} = \{\tilde{u} \in \mathcal{C}(k_0 - 1) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\} = W_{\text{loc}}^{\text{cs}}(v_0)$ ,  $W_{\text{loc}}^{\text{cs}}(v_0)$  is unique, locally invariant and forward invariant,  $v_0$  is stable in both  $W_{\text{loc},+}^c(v_0)$  and  $W_{\text{loc},-}^c(v_0)$ .

In all these cases,  $\{\tilde{u} \in \bar{V} : \omega(\tilde{u}) \in \bar{V}\} = \{\tilde{u} \in \mathcal{C}(k_0 - 1) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\}$  is a locally invariant and forward invariant  $C^1$  submanifold-with-boundary of codimension at least  $k_0 - 1$ .

The next result, describing dependence of  $W_k(\tilde{v})$  on  $\tilde{v}$  in the vicinity of a nonhyperbolic fixed point, is again a consequence of the results on invariant manifolds contained in the paper [15] by Chow, Lin and Lu.

**THEOREM 1.10.** *Let  $v_0 \in \mathcal{E}$  and let  $\lambda_{k_0}(v_0) = 1$ . Then for each  $k > k_0$  there exist a closed neighborhood  $\bar{V}$  of  $v_0$  and a (not necessarily unique) locally invariant Hölder submanifold  $\mathcal{W}_{k,\text{loc}}(v_0) \subset \bar{V}$  of codimension  $k - 1$ , having the following properties.*

(i) *The manifold  $\mathcal{W}_{k,\text{loc}}(v_0)$  is foliated by pairwise disjoint  $C^1$  submanifolds  $\{\mathcal{G}_k(\bar{u})\}$  of relative codimension 1, indexed by points  $\bar{u}$  from  $W_{\text{loc}}^c(v_0)$ , with each leaf  $\mathcal{G}_k(\bar{u})$  transverse at  $\bar{u}$  to  $W_{\text{loc}}^c(v_0)$ . For each  $\bar{u} = v_0 + \bar{u}_1 + \Psi(\bar{u}_1)$ ,  $\bar{u}_1 \in U_1$  the leaf  $\mathcal{G}_k(\bar{u})$  can be written as  $\{v_0 + \bar{u}_1 + \Psi(\bar{u}_1) + \tilde{u}_3 + \Upsilon(\bar{u}_1, \tilde{u}_3) : \tilde{u}_3 \in U_3\}$ , where  $U_3$  is a neighborhood of zero in  $Z_k(v_0)$ , the function  $\Upsilon : U_1 \times U_3 \rightarrow \text{span } \Phi_{k_0}(v_0)$  is continuous,  $\Upsilon(\bar{u}_1, \cdot)$  is a  $C^1$  embedding with  $D_2\Upsilon$  continuous,  $D_2\Upsilon(0) = 0$ , and  $\Upsilon(\cdot, \tilde{u}_3)$  is Hölder continuous (with exponent  $0 < \delta < 1$ ). The foliation  $\{\mathcal{G}_k(\bar{u})\}$  is locally forward invariant, and locally unique in  $\mathcal{W}_{k,\text{loc}}(v_0)$  in the sense that if  $S^n \tilde{u} \in \bar{V}$  for all  $n \in \mathbb{Z}_+$ ,  $S^n \bar{u} \in W_{\text{loc}}^c(v_0)$  for all  $n \in \mathbb{Z}_+$ , and  $\|S^n \tilde{u} - S^n \bar{u}\|_\alpha = O(\tilde{\varrho}^n)$  for some  $\tilde{\varrho} < \lambda_{k+1}(v_0)$ , then  $\tilde{u} \in \mathcal{G}_k(\bar{u})$ .*

(ii) *For each  $\tilde{v} \in \mathcal{E} \cap \bar{V}$ ,  $\mathcal{G}_k(\tilde{v}) = W_k(\tilde{v}) \cap \bar{V}$ .*

(iii) The set  $\{\tilde{u} \in \mathcal{C}(k) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\}$  is forward invariant and contained in  $\mathcal{W}_{k, \text{loc}}(v_0)$ .

Proof. For notational simplicity put  $v_0 = 0$ . To prove (i) we utilize the same arguments as used in the proof of Theorem 1.3. We outline here only the necessary modifications. Write  $X_0 := Z_{k_0-1}(0)$ ,  $X_1 := \text{span } \Phi_{k_0}(0)$ ,  $X_3 := Z_k(0)$ ,  $X_4 := \text{span}\{\Phi_{k_0}(0), \dots, \Phi_k(0)\}$ .

For  $t \geq 0$  and  $i = 0, 3, 4$ , define  $X_i(t) := \text{cl}(S(t, 0)X_i)$  and  $\tilde{X}_i(t) := \text{cl}_X(S(t, 0)X_i)$ . As  $X_4$  is finite-dimensional, we define, for  $s > t$ ,  $S(t, s)|_{X_4(s)}$  as  $(S(s, t)|_{X_4(t)})^{-1}$ . Let  $R(t)$  be the projection of  $\tilde{X}_0(t)$  onto  $X_4(t)$  along  $\tilde{X}_3(t)$ .

The spectrum of  $(DS(0)|_{X_4})^{-1}$  is equal to  $\{(\lambda_{k_0}(0))^{-1}, \dots, (\lambda_k(0))^{-1}\}$ , hence for each  $\tilde{\alpha} > -\log(\lambda_k(0))/T$  there are positive constants  $N_1(\tilde{\alpha})$  and  $M_2(\tilde{\alpha})$  such that

$$\begin{aligned} \|S(t, s)R(s)w_0\|_\alpha &\leq N_1(\tilde{\alpha}) \exp(-\tilde{\alpha}(t-s))\|w_0\|_\alpha && \text{for } t \leq s, w_0 \in X_0, \\ \|S(t, s)R(s)w_0\|_\alpha &\leq N_2(\tilde{\alpha}) \exp(-\tilde{\alpha}(t-s))\|w_0\| && \text{for } t \leq s, w_0 \in X_0. \end{aligned}$$

The spectrum of  $DS(0)|_{X_2}$  is contained in  $\{\zeta \in \mathbb{C} : |\zeta| \leq \lambda_{k+1}(0)\}$ , so for any  $\tilde{\beta}, \tilde{\alpha} < \tilde{\beta} < -\log(\lambda_{k+1}(0))/T$ , there are positive constants  $N_3(\tilde{\beta})$  and  $N_4(\tilde{\beta}) > 0$  such that

$$\begin{aligned} \|S(t, s)(\text{Id} - R(s))w_0\|_\alpha &\leq N_3(\tilde{\beta}) \exp(-\tilde{\beta}(t-s))\|w_0\|_\alpha && \text{for } t \geq s, w_0 \in X_0, \end{aligned}$$

and

$$\begin{aligned} \|S(t, s)(\text{Id} - R(s))w_0\|_\alpha &\leq N_4(\tilde{\beta})(t-s)^{-\alpha} \exp(-\tilde{\beta}(t-s))\|w_0\| && \text{for } t > s, w_0 \in X_0. \end{aligned}$$

Choose positive numbers  $\tilde{\alpha}, \tilde{\beta}$  and  $\tilde{\gamma}$  so that  $\log(\lambda_k(0))/T < \tilde{\alpha} < \tilde{\gamma} < \tilde{\beta} < -\log(\lambda_{k+1}(0))/T$ . An application of Thm. 3.3 of Chow, Lin and Lu [15] provides the existence of a  $(k - k_0)$ -dimensional locally invariant  $C^1$  manifold  $W'$ , written as  $W' = \{\bar{u}_4 + \bar{\Psi}(\bar{u}_4) : \bar{u}_4 \in U_4\}$ , where  $U_4$  is a neighborhood of zero in  $X_4$  and  $\bar{\Psi} : U_4 \rightarrow X_3$  is a  $C^1$  embedding with  $D\bar{\Psi}(0) = 0$ . Further, proceeding along the lines of Thm. 4.4 of [15] we obtain the existence of a locally forward invariant, locally unique foliation  $\{\bar{\mathcal{G}}_k(\bar{u})\}$  of  $W_{\text{loc}}^{\text{cs}}(0)$ , indexed by points  $\bar{u} \in W'$ . For each  $\bar{u} = \bar{u}_4 + \bar{\Psi}(\bar{u}_4)$ ,  $\bar{u}_4 \in U_4$ , the leaf  $\bar{\mathcal{G}}_k(\bar{u})$  can be written as  $\{\bar{u}_4 + \bar{\Psi}(\bar{u}_4) + \tilde{u}_3 + \bar{\Upsilon}(\bar{u}_4, \tilde{u}_3) : \tilde{u}_3 \in U_3\}$ , where  $U_3$  is a neighborhood of zero in  $X_3$ , the function  $\bar{\Upsilon} : U_4 \times U_3 \rightarrow X_3$  is continuous,  $\bar{\Upsilon}(\bar{u}_4, \cdot)$  is a  $C^1$  embedding with  $D_2\bar{\Upsilon}$  continuous,  $D_2\bar{\Upsilon}(0) = 0$ , and  $\bar{\Upsilon}(\cdot, \tilde{u}_3)$  is Hölder continuous (with exponent  $0 < \delta < 1$ ). Now we need only find a center manifold  $W_{\text{loc}}^c(0)$  contained in  $W'$  and define  $\mathcal{G}_k(\bar{u})$ ,  $\bar{u} \in W_{\text{loc}}^c(0)$ , by  $\mathcal{G}_k(\bar{u}) := \bar{\mathcal{G}}_k(\bar{u})$ .

Part (ii) follows by the characterization of  $\mathcal{G}_k(\tilde{v})$  in (i) and the characterization of  $W_k(\tilde{v})$  in Theorem 1.6(b). Part (iii) follows from the local invariance of  $\mathcal{W}_{k,\text{loc}}(v_0)$  and Theorem 0.5. ■

As  $\mathcal{W}_{k,\text{loc}}(v_0)$  is contained in some local center-stable manifold  $W_{\text{loc}}^{\text{cs}}(v_0)$ , we can define  $\mathcal{W}_{k,\text{loc},+}(v_0) := \{\tilde{u} \in \mathcal{W}_{k,\text{loc}}(v_0) : \Pi\tilde{u} \in W_{\text{loc},+}^{\text{c}}(v_0)\}$  and  $\mathcal{W}_{k,\text{loc},-}(v_0) := \{\tilde{u} \in \mathcal{W}_{k,\text{loc}}(v_0) : \Pi\tilde{u} \in W_{\text{loc},-}^{\text{c}}(v_0)\}$ .

**THEOREM 1.11.** *Assume that  $v_0 \in \mathcal{E}$ ,  $\lambda_{k_0}(v_0) = 1$  and  $k > k_0$ . Further, let  $v_0$  be stable in  $W_{\text{loc},+}^{\text{c}}(v_0)$ . Then the Hölder submanifold-with-boundary  $\mathcal{W}_{k,\text{loc},+}(v_0)$  is forward invariant.*

**PROOF.** As  $v_0$  is stable in  $W_{\text{loc},+}^{\text{c}}(v_0)$ , we find  $u_2 \in W_{\text{loc},+}^{\text{c}}(v_0)$  such that  $S\langle v_0, u_2 \rangle \subset \langle v_0, u_2 \rangle$ . Now, as in the proof of Theorem 1.7, we let  $\bar{V}$  be a closed neighborhood of  $v_0$  containing  $\bigcup\{\mathcal{F}^{\text{ss}}(\bar{u}) : \bar{u} \in \langle v_0, u_2 \rangle\}$ , and replace  $W_{\text{loc}}^{\text{cs}}(v_0)$  by  $W_{\text{loc}}^{\text{cs}}(v_0) \cap \bar{V}$ . The forward invariance of  $\mathcal{W}_{k,\text{loc}}(v_0)$  follows from the local forward uniqueness of the foliation  $\{\mathcal{G}_k(\bar{u})\}$  and its characterization in Theorem 1.10(i). ■

**THEOREM 1.12.** *Let  $v_0 \in \mathcal{E}$ ,  $\lambda_{k_0}(v_0) = 1$  and  $k > k_0$ . Then there exists a closed neighborhood  $\bar{V}$  of  $v_0$  such that the following mutually exclusive cases are possible:*

(i)  $\{\tilde{u} \in \mathcal{C}(k) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\} = W_k(v_0) \cap \bar{V}$ ,  $v_0$  is unstable in both  $W_{\text{loc},+}^{\text{c}}(v_0)$  and  $W_{\text{loc},-}^{\text{c}}(v_0)$ , and  $v_0$  is isolated in  $\mathcal{E}$ .

(ii)  $\{\tilde{u} \in \mathcal{C}(k) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\} \subset (W_k(v_0) \cap \bar{V}) \cup \mathcal{W}_{k,\text{loc},+}(v_0)$ ,  $\mathcal{W}_{k,\text{loc},+}(v_0)$  is unique, locally invariant and forward invariant,  $v_0$  is stable in  $W_{\text{loc},+}^{\text{c}}(v_0)$  and unstable in  $W_{\text{loc},-}^{\text{c}}(v_0)$ .

(iii)  $\{\tilde{u} \in \mathcal{C}(k) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\} \subset (W_k(v_0) \cap \bar{V}) \cup \mathcal{W}_{k,\text{loc},-}(v_0)$ ,  $\mathcal{W}_{k,\text{loc},-}(v_0)$  is unique, locally invariant and forward invariant,  $v_0$  is stable in  $W_{\text{loc},-}^{\text{c}}(v_0)$  and unstable in  $W_{\text{loc},+}^{\text{c}}(v_0)$ .

(iv)  $\{\tilde{u} \in \mathcal{C}(k) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\} \subset \mathcal{W}_{k,\text{loc}}(v_0)$ ,  $\mathcal{W}_{k,\text{loc}}(v_0)$  is unique, locally invariant and forward invariant,  $v_0$  is stable in both  $W_{\text{loc},-}^{\text{c}}(v_0)$  and  $W_{\text{loc},+}^{\text{c}}(v_0)$ .

In all these cases,  $\{\tilde{u} \in \mathcal{C}(k) \cap \bar{V} : \omega(\tilde{u}) \in \bar{V}\}$  is contained in a locally invariant and forward invariant Hölder submanifold-with-boundary of codimension at least  $k - 1$ .

**PROOF.** This is a consequence of Theorems 1.9–1.11. ■

A natural question arises whether and under what conditions the submanifold  $\mathcal{W}_{k,\text{loc}}(v_0)$  can be proved to be more smooth. The next result gives an answer.

**THEOREM 1.13.** *There exists a nonnegative integer  $M$  such that for each nonhyperbolic fixed point  $v_0$  and each  $k > M$  the locally invariant submanifold  $\mathcal{W}_{k,\text{loc}}(v_0)$  is of class  $C^1$ .*

**Proof.** We proceed as in the proof of Theorem 1.10. According to Thm. 3.4 of Chow, Lin and Lu [15] we are able to prove that  $\mathcal{W}_{k,\text{loc}}(v_0)$  is of class  $C^1$  provided that we can find  $\tilde{\alpha}$ ,  $\tilde{\beta}$  and  $\tilde{\gamma}$  so that  $\log(\lambda_k(0))/T < \tilde{\alpha} < \tilde{\gamma} < 2\tilde{\gamma} < \tilde{\beta} < -\log(\lambda_{k+1}(0))/T$  and  $\tilde{\gamma} > \omega$ , where, as in the proof of Theorem 1.3(iii),  $\omega$  is such that  $\|S(t,s)w_0\|_\alpha \leq M \exp(-\omega(t-s))\|w_0\|_\alpha$  and  $\|S(t,s)w_0\|_\alpha \leq M(t-s)^{-\alpha} \exp(-\omega(t-s))\|w_0\|$  for  $t > s$  and  $w_0 \in X^\alpha$ . As the set  $\mathcal{E}$  of fixed points is compact, by the results of Section 4 of the paper [16] by Chow, Lu and Mallet-Paret, for each  $v_0 \in \mathcal{E}$  we have  $\lambda_k(v_0) = \exp(k^2 + O(1))T$ , where  $O(1)$  is uniform in  $v_0$ . From this it readily follows that for sufficiently large  $k$ , the positive numbers  $\tilde{\alpha}$ ,  $\tilde{\beta}$  and  $\tilde{\gamma}$  satisfying the desired inequalities can be found. ■

We now have all the ingredients needed to prove our principal result.

**THEOREM 1.14.** *For each  $k \geq 1$ ,  $\mathcal{C}(k)$  is contained in the union of finitely many Hölder submanifolds-with-boundary of codimension not smaller than  $k - 1$ .*

**Proof.** Choose  $k$ . Put  $\mathcal{E}_1(k) := \{\tilde{v} \in \mathcal{E} : \lambda_k(\tilde{v}) = 1 \text{ and } \tilde{v} \text{ is not isolated in } \mathcal{E}\}$ , and  $\mathcal{E}_2(k) := \{\tilde{v} \in \mathcal{E} : \lambda_k(\tilde{v}) < 1 \text{ and } \tilde{v} \text{ is not isolated in } \mathcal{E}\}$ ,  $\mathcal{E}_3(k) := \{\tilde{v} \in \mathcal{E} : \lambda_k(\tilde{v}) \leq 1 \text{ and } \tilde{v} \text{ is isolated in } \mathcal{E}\}$ , and  $\mathcal{E}_4(k) := \{\tilde{v} \in \mathcal{E} : \lambda_k(\tilde{v}) > 1\}$ . We represent  $\mathcal{C}(k)$  as the disjoint union  $\bigcup_{i=1}^4 \mathcal{C}_i(k)$ , where  $\mathcal{C}_i(k) := \{\tilde{u} \in \mathcal{C}(k) : \omega(\tilde{u}) \in \mathcal{E}_i(k)\}$ .

*Case 1:  $\mathcal{C}_1(k)$ .* The set  $\mathcal{E}_1(k)$  is closed, hence compact, so we can find a finite collection  $\{v_1, \dots, v_m\} \subset \mathcal{E}_1(k)$  such that  $\mathcal{E}_1(k)$  is covered by  $\bigcup_{j=1}^m V_j$ , where  $V_j$  is the interior of a closed neighborhood  $\bar{V}_j$  of  $v_j$  as in Theorem 1.9. Clearly  $\mathcal{C}_1(k) \subset \bigcup_{j=1}^m \{\tilde{u} \in \mathcal{C}(k-1) : \omega(\tilde{u}) \in \bar{V}_j\}$ . Fix  $j$ . As  $v_j$  is not isolated in  $\mathcal{E}$ , Prop. 4.2(c) of Chen and Poláčik [13] implies that  $v_j$  is not isolated in at least one of the sets  $\mathcal{E} \cap \text{cl } W_{\text{loc},+}^c(v_j)$ ,  $\mathcal{E} \cap \text{cl } W_{\text{loc},-}^c(v_j)$ , assume for definiteness that in the former. By Theorem 1.9,  $\{\tilde{u} \in \mathcal{C}(k-1) \cap \bar{V}_j : \omega(\tilde{u}) \in \bar{V}_j\}$  is equal to  $W$ , where  $W$  is either  $(W_{k_0}(v_j) \cap \bar{V}) \cup W_{\text{loc},+}^{\text{cs}}(v_j)$  or  $W_{\text{loc}}^{\text{cs}}(v_j)$ . It is straightforward that  $\{\tilde{u} \in \mathcal{C}(k-1) : \omega(\tilde{u}) \in V_j\} = S^{-\infty}W$ . By local invariance of  $W$ ,  $S^{-\infty}W \cap V_j = W$ . An application of Lemma 1.1 with  $U := V_j$  and  $B := W$  gives that for each  $u_0 \in S^{-\infty}W$  there are  $n_0$  and a neighborhood  $Y$  of  $u_0$  such that  $S^{-\infty}W \cap Y = S^{-n_0}W \cap Y$ . By Lemma 1.2 the latter set is a  $C^1$  submanifold-with-boundary of codimension  $k - 1$ .

*Case 2:  $\mathcal{C}_2(k)$ .* It is easy to check that  $\text{cl } \mathcal{E}_2(k) \subset \mathcal{E}_1(k) \cup \mathcal{E}_2(k)$ , hence the set  $\mathcal{H} := \text{cl } \mathcal{E}_2(k) \setminus \bigcup_{j=1}^m V_j$  is compact. Let  $\{v'_1, \dots, v'_{m'}\} \subset \mathcal{H}$  and  $\{V'_1, \dots, V'_{m'}\}$  be such that for each  $j = 1, \dots, m'$ ,  $V'_j$  is the interior of a

closed neighborhood  $\bar{V}'_j$  of  $v'_j$  as in Theorem 1.12, and  $\mathcal{H} \subset \bigcup_{j=1}^{m'} V'_j$ . Fix  $j$ ,  $1 \leq j \leq m'$ . As in Case 1, assume  $v'_j$  is not isolated in  $\mathcal{E} \cap \text{cl} W_{\text{loc},+}^{\text{cs}}(v'_j)$ . By Theorem 1.12,  $\{\tilde{u} \in \mathcal{C}(k) \cap \bar{V}'_j : \omega(\tilde{u}) \in \bar{V}'_j\} \subset W'$ , where  $W'$  is either  $(W_{k_0}(v'_j) \cap \bar{V}'_j) \cup \mathcal{W}_{k,\text{loc},+}(v'_j)$  or  $\mathcal{W}_{k,\text{loc}}(v'_j)$ . It is straightforward that  $\{\tilde{u} \in \mathcal{C}(k) : \omega(\tilde{u}) \in \bar{V}'_j\} = S^{-\infty} \{\tilde{u} \in \mathcal{C}(k) \cap \bar{V}'_j : \omega(\tilde{u}) \in \bar{V}'_j\} \subset S^{-\infty} W'$ . By local invariance,  $S^{-\infty} W' \cap \bar{V}'_j = W'$ . An application of Lemma 1.1 provides, for  $u_0 \in S^{-\infty} W'$ , the existence of a neighborhood  $Y$  of  $u_0$  and a nonnegative integer  $n_0$  such that  $S^{-\infty} W' \cap Y = S^{-n_0} W' \cap Y$ . Now we need only prove that the latter set is a Hölder submanifold-with-boundary of codimension  $k - 1$ . We formulate it as a separate lemma.

LEMMA 1.15.  *$S^{-n_0} W'$  is a Hölder submanifold-with-boundary of codimension  $k - 1$ .*

Proof. Let  $k_0$  be such that  $\lambda_{k_0}(v'_j) = 1$  (recall that, since  $v'_j$  is not isolated in  $\mathcal{E}$ , it is a nonhyperbolic fixed point). As  $\lambda_k(v'_j) < 1$  we have  $k_0 < k$ . By Lemma 1.2,  $S^{-n_0} \mathcal{G}_k(\Pi u_0)$  is a  $C^1$  manifold embedded in the  $C^1$  manifold-with-boundary  $S^{-n_0} (W_k(v'_j) \cup W_{\text{loc},+}^{\text{cs}}(v'_j))$ . The relative codimension of  $S^{-n_0} \mathcal{G}_k(\Pi u_0)$  in  $S^{-n_0} W_{\text{loc}}^{\text{cs}}(v'_j)$  is  $k - k_0 + 1$ . Choose a  $C^1$  manifold  $D \subset S^{-n_0} (W_k(v'_j) \cup W_{\text{loc},+}^{\text{cs}}(v'_j))$  of dimension  $k - k_0 + 1$ , transverse in  $S^{-n_0} W_{\text{loc}}^{\text{cs}}(v'_j)$  to  $S^{-n_0} \mathcal{G}_k(\Pi u_0)$  at  $u_0$ . Since  $S^{n_0}$  is an injective  $C^1$  mapping, the image  $E := S^{n_0} D$  is a  $C^1$  manifold of dimension  $k - k_0 + 1$ , transverse in  $W_{\text{loc}}^{\text{cs}}(v'_j)$  to  $\mathcal{G}_k(\Pi u_0)$  at  $S^{n_0} u_0$ , and  $S^{n_0}|_D$  is a  $C^1$  diffeomorphism onto  $E$ . As transversality is an open property,  $E$  is transverse in  $S^{-n_0} W_{\text{loc}}^{\text{cs}}(v'_j)$  to  $\mathcal{G}_k(\Pi \tilde{u})$  for  $\tilde{u}$  in some neighborhood  $N$  of  $S^{n_0} u_0$ . It intersects, then, any  $\mathcal{G}_k(\Pi \tilde{u})$ ,  $\tilde{u} \in N$ , at a unique point  $p(\Pi \tilde{u})$ . Since, for each  $\tilde{u} \in \Pi N$ ,  $p(\tilde{u})$  can be considered the fixed point of an appropriate contraction, it now suffices to notice that if a contraction depends on a parameter (here  $\tilde{u} \in \Pi N$ ) in a Hölder continuous way then its unique fixed point varies in a Hölder continuous way with the same exponent (compare e.g. the proof of Thm. 2.2 of Chow and Hale [14]). We have thus proved that the mapping  $p$  is Hölder continuous, hence  $p(\Pi N)$  is a Hölder continuous image of a real interval  $([0, 1] \subset \mathbb{R}, \text{ say})$ .  $C^1$  injectivity of  $S^{n_0}$  implies that  $I := S^{-n_0}(p(\Pi N))$  is a Hölder continuous image of  $[0, 1]$ , too. To complete the proof it suffices to use the Hölder-parameterized version of the contracting mapping principle again, this time along the lines of Lemma 1.2. ■

Case 3:  $\mathcal{C}_3(k)$ . We see that  $\text{cl} \mathcal{C}_3(k) \subset \mathcal{C}_1(k) \cap \mathcal{C}_2(k) \cap \mathcal{C}_3(k)$ . Now, the set  $\mathcal{I} := \text{cl} \mathcal{E}_3(k) \setminus (\bigcup_{j=1}^m V_j \cup \bigcup_{j=1}^{m'} V'_j)$  is compact and consists only of isolated points in  $\mathcal{E}$ , hence it is finite. Applying Theorem 1.6 we conclude that  $\{\tilde{u} \in \mathcal{E}(k) : \omega(\tilde{u}) \in \mathcal{I}\}$  is the union of the finitely many  $C^1$  manifolds  $W_k(\tilde{v})$ ,  $\tilde{v} \in \mathcal{I}$ , of codimension  $k$ .

*Case 4:  $\mathcal{C}_4(k)$ .* Now,  $\text{cl } \mathcal{E}_4(k)$  is easily seen to be contained in  $\mathcal{E}_1(k) \cup \mathcal{E}_4(k)$ . The set  $\mathcal{J} := \mathcal{E}_4(k) \setminus \bigcup_{j=1}^m V_j = \text{cl } \mathcal{E}_4(k) \setminus \bigcup_{j=1}^m V_j$  is compact. For each  $\tilde{v} \in \mathcal{J}$ , as  $\lambda_k(\tilde{v}) > 1$  we have  $\mathcal{C}_4(k) \cap K(\tilde{v}) = \mathcal{C}_4(l(\tilde{v})) \cap K(\tilde{v})$ , where  $l(\tilde{v})$  is such that  $\lambda_{l(\tilde{v})}(\tilde{v}) \leq 1$  and  $\lambda_{l(\tilde{v})-1}(\tilde{v}) > 1$ . By the continuity of the spectrum of  $DS(\tilde{v})$  it follows that the assignment  $\mathcal{E} \ni \tilde{v} \mapsto l(\tilde{v})$  is upper semicontinuous. Since  $\mathcal{E}$  is compact, there is  $l' > k$  such that  $l(\tilde{v}) \leq l'$  for each  $\tilde{v} \in \mathcal{E}$ . As a consequence,  $\{\tilde{u} \in \mathcal{C}_4(k) : \omega(\tilde{u}) \in \mathcal{J}\} \subset \bigcup_{j=k+1}^{l'} \{\tilde{u} : \omega(\tilde{u}) \in \mathcal{E}_1(l') \cup \mathcal{E}_1(l') \cup \mathcal{E}_1(l')\}$ . Case 4 thus reduces to Cases 1, 2 and 3. ■

Taking account of Theorem 1.13 in the proof of the above theorem, we obtain

**THEOREM 1.16.** *There exists a positive integer  $M'$  such that for each  $k > M'$  the set  $\mathcal{C}(k)$  is contained in the union of finitely many  $C^1$  submanifolds of codimension  $k - 1$  or  $k$ .*

**Proof.** For  $M'$  we take a positive integer that is larger than the  $M$  of Theorem 1.13 and such that  $\lambda_{M'}(\tilde{v}) < 1$  for all  $\tilde{v} \in \mathcal{E}$ . Repeating the reasoning in the proof of the previous theorem, we see that  $\mathcal{C}(k)$  is the union of  $\{\tilde{u} \in \mathcal{C}(k) : \omega(\tilde{u}) \text{ is isolated in } \mathcal{E}\}$  and  $\{\tilde{u} \in \mathcal{C}(k) : \omega(\tilde{u}) \text{ is not isolated in } \mathcal{E}\}$ . The latter set is contained in the union of finitely many sets  $S^{-\infty} \mathcal{W}_{k,\text{loc}}(v'_j)$ , which by Theorem 1.14 and Lemma 1.1 are  $C^1$  submanifolds-with-boundary of codimension  $k - 1$ . The remainder of  $\mathcal{C}(k)$  is contained in the union of finitely many  $C^1$   $k$ -codimensional manifolds  $W_k(v''_j)$ . ■

**2. Order-openness and denseness of  $\mathcal{B}(2)$ .** We begin by stating some preliminary results.

**LEMMA 2.1.** (a) *For  $v_0 \in \mathcal{E} \cap \mathcal{S}$  the spectral radius  $\lambda_1(v_0)$  is  $\leq 1$ .*

(b) *If, moreover,  $v_0$  is nonhyperbolic, then  $v_0$  is stable in both  $W_{\text{loc},+}^c(v_0)$  and  $W_{\text{loc},-}^c(v_0)$ .*

**Proof.** This follows from Thm. 2.1 and Prop. 2.5 of Mierczyński [32]. ■

**LEMMA 2.2.**  $\mathcal{M} = \mathcal{B}(1)$ .

**Proof.** By Theorem 0.5, the equality

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{S^n \tilde{u} - \omega(\tilde{u})}{\|S^n \tilde{u} - \omega(\tilde{u})\|_\alpha} = \pm \Phi_1(\omega(\tilde{u}))$$

is equivalent to  $z(S^n \tilde{u} - \omega(\tilde{u})) = 0$  for large  $n$ , that is,  $\tilde{u} \in \mathcal{B}(1)$ . From Prop. 1.3(v) of Mierczyński [32] we derive that (2.1) is equivalent to  $\tilde{u} \in \mathcal{M}$ . ■

The next result is perhaps well known, but as I was unable to find a complete proof of it, I present it here.

**LEMMA 2.3.** *The assignment  $\mathcal{N} \ni u_0 \mapsto \omega(u_0) \in \mathcal{E}$  is continuous.*

Proof. By Thm. 5.7 of Takáč [42],  $\mathcal{N} \subset \mathcal{S}$ . Fix  $u_0 \in \mathcal{N}$  and two sequences,  $\{\underline{u}_k\}$ ,  $\{\bar{u}_k\}$  such that  $\dots \ll \underline{u}_k \ll \underline{u}_{k+1} \ll \dots \ll u_0 \ll \dots \ll \bar{u}_{k+1} \ll \bar{u}_k \ll \dots$ ,  $\lim_{k \rightarrow \infty} \|\underline{u}_k - u_0\|_\alpha = \lim_{k \rightarrow \infty} \|\bar{u}_k - u_0\|_\alpha = 0$  and  $\lim_{k \rightarrow \infty} \|\omega(\underline{u}_k) - \omega(u_0)\|_\alpha = \lim_{k \rightarrow \infty} \|\omega(\bar{u}_k) - \omega(u_0)\|_\alpha = 0$  (recall the definition of  $\mathcal{S}$ ). The family  $\{[\underline{u}_k, \bar{u}_k]\}$  forms a neighborhood base for the order topology at  $u_0$ .

Let  $\{u_l\}_{l=1}^\infty \subset \mathcal{N}$  be a sequence with  $\lim_{l \rightarrow \infty} \|u_l - u_0\|_\alpha = 0$ . As the order topology is weaker than the original one, we have  $\lim_{l \rightarrow \infty} \|u_l - u_0\|_{\text{ord}} = 0$ . Therefore to each  $l \geq 1$  one can assign  $k(l) \geq 1$  such that  $u_l \in [\underline{u}_{k(l)}, \bar{u}_{k(l)}]$  and  $k(l) \rightarrow \infty$  as  $l \rightarrow \infty$ . By strong monotonicity,  $\omega(u_l) \in [\omega(\underline{u}_{k(l)}), \omega(\bar{u}_{k(l)})] \subset [[\omega(\underline{u}_{k(l)-1}), \omega(\bar{u}_{k(l)-1})]]$ , from which we conclude that  $\lim_{l \rightarrow \infty} \|\omega(u_l) - \omega(u_0)\|_{\text{ord}} = 0$ . By the definition of  $\mathcal{N}$ , there is a linearly ordered compact set  $J \subset \mathcal{E}$  homeomorphic to the real interval  $[0, 1]$  and containing  $\omega(u_0)$  in its relative interior. We can assume that  $\inf J = \omega(\underline{u}_1)$  and  $\sup J = \omega(\bar{u}_1)$ . One has

$$\omega(\omega(u_l)) = \omega(u_l) \quad \text{and} \quad \omega(u_l) \in [\omega(\underline{u}_{k(l)}), \omega(\bar{u}_{k(l)})].$$

Thm. 1.3 of Takáč [41] yields  $\lim_{l \rightarrow \infty} \|\omega(u_l) - \omega(u_0)\|_\alpha = 0$ . ■

THEOREM 2.4.  $\mathcal{B}(2)$  is open in  $\mathcal{M} \cup \mathcal{N}$ , hence in  $X^\alpha$ .

Proof. Choose  $u_0 \in \mathcal{B}(2)$ . If  $u_0 \in \mathcal{B}(1)$  then by Lemma 2.2,  $u_0$  is in  $\mathcal{M}$ . According to Prop. 1.6 of Mierczyński [32],  $\mathcal{M}$  is open, so there is a neighborhood of  $u_0$  contained in  $\mathcal{M}$ .

Assume  $u_0 \in \mathcal{B}(2) \cap \mathcal{N}$ . The sets  $\mathcal{M}$  and  $\mathcal{N}$  are disjoint, so by Lemma 2.2 it follows that there is  $n_0$  such that  $z(S^n u_0 - \omega(u_0)) = 1$  for all  $n \geq n_0$ . Theorem 0.5 gives

$$\lim_{n \rightarrow \infty} \frac{S^n u_0 - \omega(u_0)}{\|S^n u_0 - \omega(u_0)\|_\alpha} = \pm \Phi_2(\omega(u_0)).$$

As  $X^\alpha$  embeds continuously in  $C^1([0, L])$  and  $\Phi_2(\omega(u_0))$  has simple zeros,  $S^n u_0 - \omega(u_0)$  has, for large  $n$  ( $n \geq n_0$ , say), simple zeros, too.

By Lemma 2.3, there is a neighborhood  $U \subset \mathcal{N}$  (recall that  $\mathcal{N}$  is open) of  $S^{n_0} u_0$  such that  $S^{n_0} \tilde{u} - \omega(\tilde{u})$  has simple zeros and  $z(S^{n_0} \tilde{u} - \omega(\tilde{u})) = 1$  for each  $\tilde{u} \in U$ . Note that

$$z(S^n \tilde{u} - \omega(\tilde{u})) = 1 \quad \text{for all } \tilde{u} \in U \text{ and } n \in \mathbb{Z}_+.$$

Indeed, if for some  $u_1 \in U$  and  $n_1 \geq n_0$  the equality  $z(S^{n_1} u_1 - \omega(u_1)) = 0$  holds, then by Lemma 2.2,  $u_1 \in \mathcal{M}$ , which contradicts  $\mathcal{M} \cap \mathcal{N} = \emptyset$ . Thus we have proved that  $\mathcal{B}(2) \cap \mathcal{N}$  is open in  $\mathcal{N}$ . ■

THEOREM 2.5.  $\mathcal{B}(2)$  is order-open in  $X^\alpha$ .

PROOF. For  $u_0 \in \mathcal{B}(2)$ ,  $Su_0$  is in  $\mathcal{B}(2)$ , too. By Theorem 2.4 there is a neighborhood  $U$  of  $Su_0$  contained in  $\mathcal{B}$ . According to Theorem 0.2(v), the set  $S^{-1}U$  (evidently contained in  $\mathcal{B}(2)$ ) is an order-neighborhood of  $u_0$ . ■

THEOREM 2.6.  $\mathcal{B}(2)$  is dense in  $\mathcal{M} \cup \mathcal{N}$ , hence in  $X^\alpha$ .

PROOF. Choose  $u_0 \in (\mathcal{N} \cup \mathcal{M}) \setminus \mathcal{B}(2) = \mathcal{N} \setminus \mathcal{B}(2)$ . Set  $v_0 := \omega(u_0)$ . One has

$$z(S^n u_0 - v_0) \geq 2 \quad \text{for each } n \in \mathbb{Z}_+.$$

According to Theorem 1.6(b),  $u_0$  belongs to the two-codimensional  $C^1$  embedded invariant submanifold  $W_2(v_0)$ . As  $W_2(v_0)$  is  $C^1$  embedded in the one-codimensional submanifold  $W_1(v_0)$ , the relative codimension of  $W_2(v_0)$  in  $W_1(v_0)$  is one. Consequently,  $W_1(v_0) \setminus W_2(v_0)$  is dense in  $W_1(v_0)$ . To finish the proof one only needs to notice that, by Theorem 1.6(b),  $W_1(v_0) \setminus W_2(v_0) \subset \mathcal{B}(2)$ . ■

**3.  $\mathcal{D}$  has full Gaussian measure.** Recall that  $\mathcal{D} = \mathcal{S} \cap \mathcal{B}(2)$ . First, we state a lemma on the structure of Gaussian measures.

LEMMA 3.1. Assume that  $\mu$  is a nondegenerate Gaussian measure on  $X^\alpha$ . Let  $Z$  be a one-codimensional subspace of  $X^\alpha$ . Then there exists a unit vector  $f_1 \notin Z$  such that

- (i)  $Z \oplus \text{span}\{f_1\} = X^\alpha$ .
- (ii)  $\mu$  is the product of two nondegenerate Gaussian measures,  $\mu_1$  on  $\text{span}\{f_1\}$  and  $\mu_2$  on  $(\text{span}\{f_1\})^\perp$ .

PROOF. By Skorokhod [39],  $\mu$  can be represented as a countable product of nondegenerate Gaussian measures on one-dimensional subspaces  $\text{span}\{f_i\}$ , where  $\{f_i\}_{i=1}^\infty$ , the normalized eigenvectors of the covariance operator of  $\mu$ , form a complete orthonormal system in the Hilbert space  $X^\alpha$ . Since  $\text{codim } Z = 1$ , there is at least one  $f_i$  ( $f_1$ , say) not in  $Z$ . ■

PROPOSITION 3.2. Let  $M \subset X^\alpha$  be a  $C^1$  submanifold of finite nonzero codimension. Then for any nondegenerate Gaussian measure  $\mu$  on  $X^\alpha$ ,  $\mu(M) = 0$ .

PROOF. Assume that the codimension of  $M$  is one. Choose  $u_0 \in M$ , and let  $Z$  be the tangent space to  $M$  at  $u_0$ . Let  $f_1$  be a vector transverse to  $Z$  as in Lemma 3.1. Take a neighborhood  $N$  of  $u_0$  so that at each  $\tilde{u} \in M \cap N$  the line  $\tilde{u} + \text{span}\{f_1\}$  intersects  $M \cap N$  transversely at precisely one point  $\tilde{u}$ . Denote by  $P$  the orthogonal projection along  $f_1$  on  $(\text{span}\{f_1\})^\perp$ . Let  $\phi$  be the characteristic function of  $M \cap N$ . By Lemma 3.1, applying the Fubini theorem one has

$$\mu(M \cap N) = \int_N \phi d\mu = \int_{PN} \left( \int \phi d\mu_1 \right) d\mu_2$$



where the integral in parentheses, taken over some subset of  $\text{span}\{f_1\}$ , equals 0, as the one-dimensional Gaussian measure  $\mu_1$  is absolutely continuous with respect to Lebesgue measure. As  $X^\alpha$  is separable, the submanifold  $M$  can be covered by countably many such neighborhoods, which gives, via  $\sigma$ -additivity of  $\mu$ , that  $\mu(M) = 0$ . In the general case one needs to notice that locally a  $C^1$  manifold of finite codimension can be embedded in a manifold of codimension one, and make use of the completeness of the Gaussian measure. ■

PROPOSITION 3.3. *Let  $v_0 \in \mathcal{E} \cap \mathcal{S}$  be nonhyperbolic. Then*

$$\mu(S^{-\infty}\mathcal{W}_{2,\text{loc}}(v_0)) = 0$$

for any nondegenerate Gaussian measure  $\mu$ .

PROOF. By Lemma 2.1(b),  $v_0$  is stable in both  $W_{\text{loc},+}^c(v_0)$  and  $W_{\text{loc},-}^c(v_0)$ . Theorem 1.11 implies that the Hölder submanifold  $\mathcal{W}_{2,\text{loc}}(v_0)$  is locally invariant and forward invariant. Choose  $u_0 \in S^{-\infty}\mathcal{W}_{2,\text{loc}}(v_0)$ . An application of Lemma 1.1 gives the existence of a neighborhood  $Y$  of  $u_0$  and a nonnegative integer  $n_0$  such that  $S^{-\infty}\mathcal{W}_{2,\text{loc}}(v_0) \cap Y = S^{-n_0}\mathcal{W}_{2,\text{loc}}(v_0) \cap Y$ . For  $\tilde{u} \in Y$  denote by  $\mathcal{L}(\tilde{u})$  the set  $S^{-n_0}\mathcal{F}^{\text{ss}}(HS^{n_0}u_0)$ . By Lemma 1.2,  $\mathcal{L}(\tilde{u})$  is a  $C^1$  submanifold of codimension one. Let  $Z$  be the tangent space of  $\mathcal{L}(u_0)$  at  $u_0$ , and  $Z^\perp$  be its orthogonal complement in  $X^\alpha$ . According to Lemma 3.1, there exists  $f_1 \notin Z$  such that

$$\mu = \mu_1 \otimes \mu_2,$$

where  $\mu_1$  and  $\mu_2$  are nondegenerate Gaussian measures on

$$X_1 = \text{span}\{f_1\} \quad \text{and} \quad X_2 := (\text{span}\{f_1\})^\perp = \text{cl}(\text{span}\{f_2, f_3, \dots\})$$

respectively. Define  $\pi_i$ ,  $i = 1, 2$ , to be the orthogonal projection onto  $X_i$ . The vector  $f_1$  is transverse to  $Z$  at  $u_0$  and tangent spaces of  $\mathcal{L}(\tilde{u})$  depend continuously on  $\tilde{u}$ , so there is a neighborhood  $N \subset Y$  of  $u_0$  such that  $f_1$  is transverse to  $\mathcal{L}(\tilde{u})$  at each  $\tilde{u} \in N$ . Define the  $C^1$  mapping  $h : N \rightarrow W_{\text{loc}}^c(v_0)$  as  $h := \Pi \circ S^{n_0}|_N$ . As  $f_1$  is transverse to  $\mathcal{L}(u_0)$ , by  $C^1$  injectivity it follows that the vector  $DS^{n_0}(u_0)f_1$  is not tangent to the leaf of the strong stable foliation  $\{\mathcal{F}^{\text{ss}}(\bar{u})\}$  passing through  $S^{n_0}u_0$ . Consequently,  $Dh(u_0)f_1 \neq 0$ . The  $C^1$  manifold  $W_{\text{loc}}^c(v_0)$  has dimension one, so it is  $C^1$  diffeomorphic (via a diffeomorphism  $g$ ) to an interval in  $X_1$ . Define  $H : N \rightarrow X_1 \oplus X_2$  as  $H = (g \circ h, \pi_2)$ . The mapping  $H$  is clearly of class  $C^1$ . Furthermore, it is easy to check that its derivative  $DH(u_0)$  is a linear isomorphism. As a consequence of the inverse function theorem, there is a neighborhood  $N_1$  of  $u_0$  such that  $H|_{N_1}$  is a  $C^1$  diffeomorphism onto its image. Now, choose a compact interval  $[a, b] \subset X_1$  (here  $X_1$  is identified with  $\mathbb{R}$ ) satisfying  $g \circ h(u_0) \in (a, b)$  and a closed ball  $C \subset X_2$  centered at  $\pi_2 u_0$  so small that  $\tilde{N} := H^{-1}([a, b] \times C) \subset N_1$ .

For each  $y \in C$  set  $I_y := (y + X_1) \cap \tilde{N}$  and define the  $C^1$  diffeomorphism  $G[y] : [a, b] \rightarrow \pi_1 I_y$  as  $G[y] := \pi_1 \circ (g \circ h|_{I_y})^{-1}$ . Since  $[a, b]$  is compact and the derivative  $(G[y])_\xi(\xi)$  depends continuously on  $(\xi, y) \in [a, b] \times C$ , the assignment  $C \ni y \mapsto G[y] \in C^1([a, b], X^\alpha)$  is continuous.

Consider the change of variables

$$\tilde{N} \ni \tilde{u} = (\pi_1 \tilde{u}, \pi_2 \tilde{u}) = (x, y) \mapsto (\xi, y) \in [a, b] \times C,$$

where  $\xi(x, y) = g \circ h(\tilde{u})$ .

Take any Borel function  $\phi : \tilde{N} \rightarrow \mathbb{R}$ . Write  $\phi(x, y) = \tilde{\phi}(\xi, y)$ . Consider the integral

$$\int_{\tilde{N}} \phi(x, y) d\mu(x, y) = \int_{\tilde{N}} \phi(x, y) d\mu_1(x) \otimes d\mu_2(y).$$

By the Fubini theorem,

$$\int_{\tilde{N}} \phi(x, y) d\mu_1(x) \otimes d\mu_2(y) = \int_C \left( \int_{\pi_1 I_y} \phi(x, y) d\mu_1(x) \right) d\mu_2(y).$$

But for each  $y \in C$  one has

$$\int_{\pi_1 I_y} \phi(x, y) d\mu_1(x) = \int_a^b \tilde{\phi}(\xi, y) (G[y])_\xi(\xi) d\mu_1(\xi),$$

hence

$$\int_C \left( \int_{\pi_1 I_y} \phi(x, y) d\mu_1(x) \right) d\mu_2(y) = \int_C \left( \int_a^b \tilde{\phi}(\xi, y) (G[y])_\xi(\xi) d\mu_1(\xi) \right) d\mu_2(y).$$

Interchanging the order of integration we get

$$\begin{aligned} \int_C \left( \int_a^b \tilde{\phi}(\xi, y) (G[y])_\xi(\xi) d\mu_1(\xi) \right) d\mu_2(y) \\ = \int_a^b \left( \int_C \tilde{\phi}(\xi, y) (G[y])_\xi(\xi) d\mu_2(y) \right) d\mu_1(\xi). \end{aligned}$$

We have thus obtained, for any Borel function  $\phi$  on  $\tilde{N}$ ,

$$(3.4) \quad \int_{\tilde{N}} \phi(x, y) d\mu_1(x) \otimes d\mu_2(y) = \int_a^b \left( \int_C \tilde{\phi}(\xi, y) (G[y])_\xi(\xi) d\mu_2(y) \right) d\mu_1(\xi).$$

Now, specialize  $\phi(x, y)$  to be the characteristic function of the set  $\tilde{N} \cap S^{-\infty} \mathcal{W}_{2, \text{loc}}(v_0) = \tilde{N} \cap S^{-n_0} \mathcal{W}_{2, \text{loc}}(v_0)$ . We claim that for each fixed  $\xi \in [a, b]$  the integral

$$\int_C \tilde{\phi}(\xi, y) (g[y])_\xi(\xi) d\mu_2(y)$$

equals zero. Indeed, the integration is performed over the one-codimensional  $C^1$  manifold  $\mathcal{L}(H^{-1}(\xi, \pi_2 u_0))$  intersecting  $S^{-n_0} \mathcal{W}_{2,\text{loc}}(v_0)$  in the two-codimensional  $C^1$  submanifold  $S^{-n_0} \mathcal{G}_2(\Pi u_0) \cap \tilde{N}$ . The relative codimension of  $S^{-n_0} \mathcal{G}_2(\Pi u_0) \cap \tilde{N}$  in  $\mathcal{L}(H^{-1}(\xi, \pi_2 u_0))$  is one. By Proposition 3.2 it follows that  $\nu(S^{-n_0} \mathcal{G}_2(\Pi u_0) \cap \tilde{N}) = 0$  for any measure  $\nu$  equivalent to the Gaussian measure on  $\mathcal{L}(H^{-1}(\xi, \pi_2 u_0))$ . Further, since  $(G[y])_\xi(\xi) > 0$  for each  $y \in C$ , the measure with respect to which we integrate is equivalent to the Gaussian measure  $\mu_2$  on  $X_2$ . Consequently, taking (3.4) into account we get

$$\int_{\tilde{N}} \phi(x, y) d\mu(x, y) = 0.$$

Now it suffices to notice that by separability of  $X^\alpha$  the set  $S^{-\infty} \mathcal{W}_{2,\text{loc}}(v_0)$  can be covered by countably many such neighborhoods  $\tilde{N}$ . The  $\sigma$ -additivity of  $\mu$  gives the desired result. ■

**THEOREM 3.4.** *For each nondegenerate Gaussian measure  $\mu$  on  $X^\alpha$ ,  $\mu(X^\alpha \setminus \mathcal{D}) = 0$ .*

**PROOF.** The set  $X^\alpha \setminus \mathcal{D}$  is the disjoint union of  $\mathcal{U}$  and  $\mathcal{S} \cap \mathcal{C}(2)$ . Cor. 5.6 of Takáč [42] states that  $\mu(\mathcal{U}) = 0$ . It suffices therefore to show that  $\mu(\mathcal{C}(2)) = 0$ . Set  $\mathcal{E}'(2) := \{\tilde{v} \in \mathcal{E} : \lambda_1(\tilde{v}) < 1\}$ ,  $\mathcal{E}''(2) := \{\tilde{v} \in \mathcal{E} : \lambda_1(\tilde{v}) = 1\}$ . As a consequence of Lemma 2.1(a),  $\mathcal{C}(2)$  equals the disjoint union  $\mathcal{C}'(2) \cup \mathcal{C}''(2)$ , where  $\mathcal{C}'(2) := \{\tilde{u} \in \mathcal{C}(2) : \omega(\tilde{u}) \in \mathcal{E}'(2)\}$  and  $\mathcal{C}''(2) := \{\tilde{u} \in \mathcal{C}(2) : \omega(\tilde{u}) \in \mathcal{E}''(2)\}$ . As elements of  $\mathcal{E}'(2)$  are hyperbolic fixed points, they are isolated in  $\mathcal{E}$ . By the separability of  $X^\alpha$  there are at most countably many of them. Since  $\mathcal{C}'(2) = \bigcup \{W_2(\tilde{v}) : \tilde{v} \in \mathcal{E}'(2)\}$  and  $W_2(\tilde{v})$  are two-codimensional  $C^1$  manifolds, by Proposition 3.2 and  $\sigma$ -additivity of  $\mu$  we have  $\mu(\mathcal{C}'(2)) = 0$ .

We proceed to investigate the set  $\mathcal{C}''(2)$ . Let  $\{v'_1, \dots, v'_{m'}\} \subset \mathcal{E}''(2)$  and  $\{V'_1, \dots, V'_{m'}\}$  be such that for each  $j = 1, \dots, m'$ ,  $V'_j$  is the interior of a closed neighborhood  $\bar{V}'_j$  of  $v'_j$  as in Theorem 1.12 (for  $k = 2$ ), and  $\mathcal{E}''(2) \subset \bigcup_{j=1}^{m'} V'_j$ . Fix  $j$ . As, by Lemma 2.1(b),  $v'_j$  is stable in both  $W_{\text{loc},+}^c(v'_j)$  and  $W_{\text{loc},-}^c(v'_j)$ , Theorem 1.12 implies that  $\{\tilde{u} \in \mathcal{C}(2) \cap \bar{V}'_j : \omega(\tilde{u}) \in \bar{V}'_j\} \subset \mathcal{W}_{2,\text{loc}}(v'_j)$ . Hence  $\mathcal{C}''(2) \subset \bigcup_{j=1}^{m'} \mathcal{W}_{2,\text{loc}}(v'_j)$ . By Proposition 3.3,  $\mu(S^{-\infty} \mathcal{W}_{2,\text{loc}}(v'_j)) = 0$ . This completes the proof. ■

**4. Concluding remarks.** Our results are far from being as general as possible, as we do not want to be encumbered with technicalities. Here we notice that for separated (nonperiodic) boundary conditions, by Brunovský, Poláčik and Sandstede [10] any solution bounded in the  $C^1([0, L])$ -norm converges to a  $T$ -periodic solution, and all our results carry over. In the

case of a fully nonlinear equation

$$(4.1) \quad u_t = f(t, x, u, u_x, u_{xx})$$

with nonperiodic boundary conditions, where the derivative  $\partial f/\partial u_{xx}$  satisfies the uniform ellipticity condition, the theory in Henry's book [21] may not apply. One needs to use other theories guaranteeing that (4.1) gives rise to a  $C^2$  period map on some strongly ordered separable Banach space, for example those in Amann [3], or in Angenent [6]. Also, if the phase space is not Hilbert, one has to make use of the results on Gaussian measures on Banach spaces as in Kuo [27].

**5. Finite-dimensional counterparts.** Consider a system of autonomous ordinary differential equations (ODEs)

$$(5.1) \quad \dot{x} = f(x),$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^2$  vector field such that at each  $x \in \mathbb{R}^n$  the derivative of  $f$  is a *positive Jacobi matrix*, that is,  $\partial f_i/\partial x_j = 0$  for  $|i-j| > 1$  and  $\partial f_i/\partial x_j > 0$  for  $|i-j| = 1$ . As proved in Smillie [40] and Fusco and Oliva [19], every bounded forward semitrajectory of (5.1) converges to a fixed point.

There is a counterpart  $z(\cdot)$  of the Matano number (defined roughly as the number of sign changes of the coordinates of  $x \in \mathbb{R}^n$ ) which does not increase along trajectories of (5.1). Furthermore, for any  $x_0 \in \mathbb{R}^n$  with  $f(x_0) = 0$  the linearization of  $f$  at  $x_0$  has  $n$  simple real eigenvalues  $\lambda_1 > \dots > \lambda_n$ , and  $z(\Phi_i) = i$ , where  $\Phi_i$  is an eigenvector pertaining to  $\lambda_i$  (see [19]). This enables one to carry over the results of Sections 1 and 2 almost verbatim to the case of (5.1).

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