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LOCAL SPECTRUM AND KAPLANSKY'S THEOREM ON ALGEBRAIC OPERATORS

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Using elementary arguments we improve former results of P. Vrbová [10] concerning local spectrum. As a consequence, we obtain a new proof of Kaplansky's theorem on algebraic operators on a Banach space.

0. Introduction. Let $T \in \mathcal{B}(X)$ and $x \in X$. We define Ω_x to be the set of $\alpha \in \mathbb{C}$ for which there exists a neighbourhood V_α and u analytic on V_α with values in X such that $(\lambda - T)u(\lambda) = x$ on V_α. This set is open and contains the complement of the spectrum of T. The function u is called a *local resolvent* of T on V_α. By definition, the *local spectrum* of T at x, denoted by $\mathrm{Sp}_x(T)$, is the complement of Ω_x. So it is a compact subset of $\mathrm{Sp}(T)$. In general, $\mathrm{Sp}_x(T)$ may be empty, even for $x \neq 0$ (see Example 1 below). But for $x \neq 0$, the local spectrum of T is non-empty if T satisfies the *uniqueness property for the local resolvent*. That is, $(\lambda - T)v(\lambda) = 0$ implies $v = 0$ for any analytic function v defined on any domain D of $\mathbb{C}$ with values in X. It is easy to see that T has this property if the spectrum of T has no interior points (see [6]). So this happens if T has a finite, countable or real spectrum. It is still an open problem whether the class of commuting operators with uniqueness property for the local resolvent is stable for addition. For more details on this interesting problem the reader is referred to [2], [3] and [6]. For operators satisfying the uniqueness property for the local resolvent there is a unique local resolvent which is the analytic extension of $(\lambda - T)^{-1}x$. Using this property and Liouville's Theorem, it is easy to conclude that $\mathrm{Sp}_x(T) \neq \emptyset$ for $x \neq 0$. Also in this case the *local spectral radius* $r_x(T) = \max\{|\mu| : \mu \in \mathrm{Sp}_x(T)\}$ is equal to $\overline{\lim}_{k\to\infty} \|T^k x\|^{1/k}$. In general this last property is false.

In this note, we first introduce three simple examples to clarify the concept of the local spectrum. Secondly, we use elementary arguments to improve former results of P. Vrbová [10], which roughly say that $\partial\,\mathrm{Sp}(T) \subset$

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$\mathrm{Sp}_x(T)$ for a large set of vectors x. As a consequence, we obtain an elementary proof of a well known result of Kaplansky, that is, a locally algebraic bounded linear operator on a Banach space is in fact algebraic.

1. Results. The first example will show that even for $x \neq 0$, the local spectrum of T at x is not in general non-empty. The third example will show that Gelfand's spectral radius formula is not in general true for the local spectral radius. The second example will show that there are many operators with the property of the uniqueness of the local resolvent.

EXAMPLE 1. On the Hilbert space l^2 consider the left shift operator

$$T : (\xi_1, \xi_2, \ldots) \to (\xi_2, \xi_3, \ldots).$$

It is well known that the spectrum of T is the closed unit disc (see [8], problem 82). Denoting by (e_n) the canonical orthonormal basis we have $T^k e_k = 0$ for $k \geq 1$, so for $\lambda \neq 0$ the relation

$$e_k = (\lambda^k - T^k)\frac{e_k}{\lambda^k} = (\lambda - T)(\lambda^{k-1} + \ldots + T^{k-1})\frac{e_k}{\lambda^k} \tag{1}$$

implies that $\mathrm{Sp}_{e_k}(T) \subset \{0\}$. If we consider the right shift operator

$$S : (\xi_1, \xi_2, \ldots) \to (0, \xi_1, \xi_2, \ldots)$$

which is an isometry and satisfies $TS = I$, it is easy to verify that we have for $x \in l^2$, $x \neq 0$, $|\lambda| < 1$,

$$(\lambda - T)\Big(-\sum_{n=0}^{\infty} \lambda^n S^{n+1} x\Big) = x. \tag{2}$$

The series defines an analytic function in λ, so we have $\mathrm{Sp}_x(T)$ included in the unit circle. Consequently, $\mathrm{Sp}_{e_k}(T)$ is empty for $k \geq 1$, which implies that $\mathrm{Sp}_x(T) = \emptyset$ if x is in the linear subspace F generated by $e_1, e_2, \ldots$ There are some x not in F for which $\mathrm{Sp}_x(T)$ is non-empty, in which case $\mathrm{Sp}_x(T)$ is included in the unit circle. For instance we may choose $x = \sum_{n=1}^{\infty} (1/n) e_{2^n}$ for which $\varrho_x(T) = 1$. Consequently, by Theorem 1.2 below, $r_x(T) = 1$.

EXAMPLE 2. Let $\{w_n\}_{n=1}^{\infty}$ be an arbitrary sequence of non-zero complex numbers. On the Hilbert space l^2 consider the operator T_k defined by $T_k e_n = w_n e_{n+k}$ for $n = 1, 2, \ldots$ It is well known that T_0 has the property of uniqueness of the local resolvent. We show here that this is true for all T_k. In fact, suppose that $(\lambda - T_k)v(\lambda) = 0$ for every $\lambda \in D(\lambda_0, r)$. Then it follows from the definition of T_k that if $v(\lambda) = (v_1, v_2, \ldots)$, then

$$(0, \ldots, 0, w_1 v_1, w_2 v_2, \ldots) = (\lambda v_1, \ldots).$$

Hence,

$$\lambda v_n = 0 \quad \text{for } n = 1, \ldots, k \quad \text{and} \quad \lambda v_{n+k} = w_n v_n \quad \text{for } n = 1, 2, \ldots$$

CASE 1: If $\lambda = 0$, then $v_n = 0$ for all n.

CASE 2: If $\lambda \neq 0$, then $v_n = 0$ for $n = 1, \ldots, k$. On the other hand, we have $\lambda v_{k+1} = w_1 v_1 = 0$. So, $v_{k+1} = 0$. Repeating this process, we get $v_{n+k} = 0$ for $n = 2, 3, \ldots$ Consequently, $v = 0$.

REMARK. Using the same idea as above we can easily prove that quasi-similarity preserves the property of uniqueness of the local resolvent.

We denote by $r(T)$ the spectral radius of T which is equal to $\lim_{k\to\infty} \|T^k\|^{1/k}$. By convention, for $x \neq 0$,

$$r_x(T) = \begin{cases} \max\{|\alpha| : \alpha \in \mathrm{Sp}_x(T)\} & \text{if } \mathrm{Sp}_x(T) \neq \emptyset, \\ 0 & \text{if } \mathrm{Sp}_x(T) = \emptyset, \end{cases}$$

and $\varrho_x(T) = \limsup_{k\to\infty} \|T^k x\|^{1/k} \le r(T)$. It is well known that in general, $\|T^k x\|^{1/k}$ has no limit. From the formal identity

$$(\lambda - T)\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{T}{\lambda}\right)^k x = x,$$

we conclude that

$$r_x(T) \le \varrho_x(T) \le r(T). \tag{3}$$

The next example shows that in general $r_x(T)$ is less than $\varrho_x(T)$, even if $\mathrm{Sp}_x(T)$ is non-empty.

EXAMPLE 3. Let $X = l^2 \oplus \mathbb{C}^2$, with the norm $\|(x_1, x_2)\| = \max(\|x_1\|, \|x_2\|)$, on which we consider the left shift operator T on l^2 and M a 2×2 matrix having eigenvalues $1/2$ and $3/2$. We have $\mathrm{Sp}(A) = \mathrm{Sp}(I+T) \cup \{1/2, 3/2\} = \overline{D}(1,1)$, the closed unit disk centred at 1. So $r(A) = 2$. Let u be an eigenvector of M corresponding to $1/2$. Then we have $\varrho_{(e_1,u)}(A) = \max(\varrho_{e_1}(I+T), \varrho_u(M)) = 1$. Moreover, since $\mathrm{Sp}_{e_1}(I+T)$ is empty, we obtain

$$\mathrm{Sp}_{(e_1,u)}(A) = \mathrm{Sp}_{e_1}(I+T) \cup \mathrm{Sp}_u(M) = \mathrm{Sp}_u(M) = \{1/2\}.$$

So $r_{(e_1,u)}(A) = 1/2$.

LEMMA 1.1. *Suppose* $T \in \mathcal{B}(X)$, $x, y \in X$, $\alpha \in \mathbb{C}$. *Then*

(i) $\varrho_{\alpha x}(T) = \varrho_x(T)$,
(ii) $\varrho_{Sx}(T) \le \varrho_x(T)$ *if* $S \in \mathcal{B}(X), ST = TS$,
(iii) $\varrho_{x+y}(T) \le \max(\varrho_x(T), \varrho_y(T))$,
(iv) *if* $\varrho_x(T) > \varrho_y(T)$ *then* $\varrho_{x+y}(T) = \varrho_x(T)$.

The proof is straightforward. It follows from the conditions (i)–(iii) of Lemma 1.1 that for a real number r, the set $\{x \in X : \varrho_x(T) \le r\}$ is a linear (not necessarily closed) subspace of X, hyperinvariant under T. Apparently nobody has noticed until now the following elementary results.

THEOREM 1.2. *Let $T \in \mathcal{B}(X)$, $x \in X$, $x \neq 0$. If $r_x(T) = r(T)$ then $r_x(T) = \varrho_x(T) = r(T)$. If $r_x(T) < r(T)$ then $\varrho_x(T) < r(T)$.*

Proof. The first part is obvious by (3). Suppose that $r_x(T) < r(T)$, which means that the circle Γ with centre 0 and radius $r(T)$ is included in Ω_x. For $\alpha \in \Gamma$ there exists a disk neighbourhood V_α and a local resolvent u such that $(\lambda - T)u(\lambda) = x$ on V_α. But if $\lambda \in V_\alpha$ with $|\lambda| > r(T)$ then $\lambda - T$ is invertible so $u(\lambda)$ coincides with $R(\lambda, x) = (1/\lambda)\sum_{k=0}^{\infty}(T/\lambda)^k x$. If we now take $\alpha, \beta \in \Gamma$ with $V_\alpha \cap V_\beta \neq \emptyset$, then the two corresponding local resolvents coincide on $V_\alpha \cap V_\beta \cap \{z : |z| > r(T)\}$. By the Identity Principle they coincide on $V_\alpha \cap V_\beta$. Using these arguments we can extend analytically $R(\lambda, x)$ onto $\{z : |z| > \varrho\}$ for some $\varrho < r(T)$ so, by definition of the radius of convergence of the series, we have $\varrho_x(T) < \varrho < r(T)$.

THEOREM 1.3. *Suppose $T \in \mathcal{B}(X)$ and $r(T) > 0$. Then the set Y of $x \in X$ such that $r_x(T) < r(T)$ is a linear subspace of X which is an F_σ-set invariant under any operator S commuting with T.*

Proof. Obviously Y contains zero. Suppose $x \neq 0$. By Theorem 1.2, $r_x(T) < r(T)$ is equivalent to $\varrho_x(T) < r(T)$. By Lemma 1.1(i)–(iii), Y is a linear subspace of X invariant under any S commuting with T. Moreover, we have $Y = \bigcup_{p,q \in \mathbb{N}} Y_{p,q}$ where

$$Y_{p,q} = \left\{ x \in X : \|T^k x\| \le \left(r(T) - \frac{1}{p+1} \right)^k \text{ for } k \ge q \right\}$$

is closed, so Y is an F_σ-set.

Supposing that T has the uniqueness property for the local resolvent, Dunford proved that $\mathrm{Sp}(T) = \bigcup_{x \in X} \mathrm{Sp}_x(T)$. The argument is very simple. Let $\alpha \in \mathrm{Sp}(T) \setminus \bigcup_{x \in X} \mathrm{Sp}_x(T)$; from the definition of Ω_x this implies that $\alpha - T$ is onto, so it is not one-to-one. Consequently, there exists $u \neq 0$ such that $(\alpha - T)u = 0$, in which case $(\lambda - T)\frac{u}{\lambda - \alpha} = u$ for $\lambda \neq \alpha$, so $\mathrm{Sp}_u(T) \subset \{\alpha\}$; but $\mathrm{Sp}_u(T)$ is non-empty and this implies that $\mathrm{Sp}_u(T) = \{\alpha\}$, which gives a contradiction. If $\mathrm{Sp}_x(T)$ is empty for some $x \neq 0$, then Example 1 shows that this result cannot be true in general. Gray [7] claims to prove this result in general, but his argument is invalid.

P. Vrbová [10] improved Dunford's result when T has the uniqueness property for the local resolvent. She proved that there exists $x \in X$ such that $\mathrm{Sp}(T) = \mathrm{Sp}_x(T)$, a result which is obviously false for the general case. She also proved that the set of $x \in X$ for which $\mathrm{Sp}_x(T)$ contains the boundary of $\mathrm{Sp}(T)$ is non-meager (i.e. it is not of the first category) in X.

We now improve Vrbová's theorem using an elementary argument.

LEMMA 1.4 (Holomorphic Functional Calculus for Local Spectrum). *Let $T \in \mathcal{B}(X)$, $x \neq 0$, and f holomorphic in a neighbourhood D of* $\mathrm{Sp}(T)$. *Then*

$f(\mathrm{Sp}_x(T))$ is included in $\mathrm{Sp}_x(f(T))$. If f is injective on D then $f(\mathrm{Sp}_x(T)) = \mathrm{Sp}_x(f(T))$. Moreover, if T has the uniqueness property for the local resolvent then we have the same property for any f holomorphic.

Proof. The proof is essentially given in [5], Theorem 1.6, p. 6. The injective case is obtained by applying the first case to f^{-1} and the operator $f(T)$.

COROLLARY 1.5. *Let $T \in \mathcal{B}(X)$ be invertible. Then*

$$\mathrm{dist}(0, \mathrm{Sp}_x(T)) = 1/r_x(T^{-1}).$$

Proof. If $\mathrm{Sp}_x(T)$ is empty then $\mathrm{dist}(0, Sp_x(T)) = \infty$, so the formula is obvious. Suppose $\mathrm{Sp}_x(T)$ is non-empty. By Lemma 1.4 applied to $1/z$, which is holomorphic and injective on a neighbourhood of $\mathrm{Sp}(T)$, we have $\mathrm{Sp}_x(T^{-1}) = \{1/z : z \in \mathrm{Sp}_x(T)\}$, so we get the result.

LEMMA 1.6. *Let $T \in \mathcal{B}(X)$ be invertible. Then there exists an $x \in X$ such that* $\mathrm{dist}(0, \mathrm{Sp}_x(T)) = \mathrm{dist}(0, \mathrm{Sp}(T))$.

Proof. Without loss of generality, we may suppose that $\mathrm{dist}(0, \mathrm{Sp}(T)) = 1$. Since $\mathrm{Sp}_x(T) \subset \mathrm{Sp}(T)$ we have $\mathrm{dist}(0, \mathrm{Sp}_x(T)) \geq 1$. Suppose that $\mathrm{dist}(0, \mathrm{Sp}_x(T)) > 1$ for every $x \in X$. By Corollary 1.5, we have $r_x(T^{-1}) < 1$. Consequently, by Theorem 1.2, $\varrho_x(T^{-1}) < 1$ for every $x \in X$. Hence, for every $x \in X$ there exists $\varrho_x < 1$ and an integer N_x such that

$$\|T^{-k}x\| \leq \varrho_x^k \quad \text{for } k \geq N_x.$$

Let λ be fixed with $|\lambda| = 1$. Then we have

$$\sum_{k=N_x}^{\infty} \left\| \frac{T^{-k}}{\lambda^k} x \right\| = \sum_{k=N_x}^{\infty} \|T^{-k}x\| \leq \sum_{k=N_x}^{\infty} \varrho_x^k < \infty.$$

Setting

$$S_n x = \frac{1}{\lambda} \sum_{k=0}^{n} \frac{T^{-k}}{\lambda^k} x,$$

we have $S_n \in \mathcal{B}(X)$ and by the previous inequalities,

$$\sup_n \|S_n x\| < \infty$$

for every $x \in X$, so by the Banach–Steinhaus Theorem the operator defined by

$$Sx = \sum_{k=0}^{\infty} \left(\frac{T^{-1}}{\lambda} \right)^k x$$

is bounded. Moreover, we have $(\lambda - T^{-1})Sx = S(\lambda - T^{-1})x = x$ for every $x \in X$, so $\lambda \notin \mathrm{Sp}(T^{-1})$. Consequently, we proved that the unit circle is disjoint from $\mathrm{Sp}(T^{-1})$, so it is also disjoint from $\mathrm{Sp}(T)$, and this is a contradiction because $\mathrm{dist}(0, \mathrm{Sp}(T)) = 1$. Hence, there exists some $x \in X$ for which $\mathrm{dist}(0, \mathrm{Sp}_x(T)) = 1$.

THEOREM 1.7. *Let $T \in \mathcal{B}(X)$. There exists a subset E of X, which is a countable union of linear subspaces of X invariant under any operator commuting with T, such that $X \setminus E$ is a dense G_δ-subset of X and such that $x \notin E$ implies $\partial\,\mathrm{Sp}(T) \subset \mathrm{Sp}_x(T)$.*

Proof. Let (s_n) be an enumeration of the countable set $(Q+iQ)\setminus\mathrm{Sp}(T)$ and let $Y_n = \{x \in X : \mathrm{dist}(s_n, \mathrm{Sp}_x(T)) > \mathrm{dist}(s_n, \mathrm{Sp}(T))\}$. By Corollary 1.5 and Theorem 1.3 applied to $(T - s_n)^{-1}$ we know that Y_n is a linear subspace of X invariant under any operator commuting with T, and it is also an F_σ-subset of X. Let E be the union of all those Y_n; is an F_σ-subset of X. First we prove that $X \setminus E$ is dense in X. Suppose the contrary. Then there exists an open ball included in E, so by Baire's Category Theorem, one of the $Y_{n,p,q}$, as defined in the proof of Theorem 1.3 for the operator $(T - s_n)^{-1}$, contains interior points. So this is also true for Y_n. But Y_n is a linear subspace of X, which implies that $X = Y_n$ for some integer n. But, by Lemma 1.6 applied to $T - s_n$, we get a contradiction.

We now prove that $x \notin E$ implies $\partial\,\mathrm{Sp}(T) \subset \mathrm{Sp}_x(T)$. Suppose that $\alpha \in \partial\,\mathrm{Sp}(T)$ and $\alpha \notin \mathrm{Sp}_x(T)$. Let $\varrho > 0$ be such that the closed disc $\overline{D}(\alpha, \varrho)$ is disjoint from $\mathrm{Sp}_x(T)$. Since $\alpha \in \partial\,\mathrm{Sp}(T)$ there exists n such that $|\alpha - s_n| < \varrho/2$. Therefore if $x \notin E$ we have

$$\varrho/2 > |\alpha - s_n| \geq \mathrm{dist}(s_n, \mathrm{Sp}(T)) = \mathrm{dist}(s_n, \mathrm{Sp}_x(T)) \geq \varrho/2,$$

which is absurd. Therefore the result is proved.

REMARK. If instead of one operator $T \in \mathcal{B}(X)$ we have a sequence (T_n) of operators, then for each n there exists some subset E_n as defined by Theorem 1.7. Consequently, the union E of all those E_n has the properties of Theorem 1.7 for any T_n. Therefore, Theorem 1.7 generalizes previous results on $\varrho_x(T)$.

We say that $T \in \mathcal{B}(X)$ is *locally algebraic* if for every $x \in X$ there exists a non-zero polynomial p such that $p(T)x = 0$. A well-known result due to Kaplansky [9] states that a locally algebraic bounded linear operator on a Banach space is in fact algebraic. Several different proofs of this important result are known (see for instance [1], p. 86, and [9]). We add a new one as an application of Theorem 1.7.

COROLLARY 1.8. *If $T \in \mathcal{B}(X)$ is locally algebraic then it is algebraic.*

Proof. From Baire's Category Theorem, it is easy to prove that there exists an integer N such that for every $x \in X$ there exists a non-zero polynomial p satisfying $p(T)x = 0$ and the degree of p is less than or equal to N (see the proof of Corollary 4.2.8 in [1]). By Lemma 1.4 we have $p(\mathrm{Sp}_x(T)) \subset \mathrm{Sp}_x(p(T)) \subset \{0\}$ because $\varrho_x(p(T)) = 0$. So, for every $x \in X$, $\mathrm{Sp}_x(T)$ has at most N points. By Theorem 1.7, $\partial\,\mathrm{Sp}(T)$ has at most N points, so the same is true for $\mathrm{Sp}(T)$. Denote by $\beta_1, \dots, \beta_m$ ($m \le N$) the m distinct points of $\mathrm{Sp}(T)$. Let x be arbitrary in X and let p of minimal degree and leading coefficient one be such that $p(T)x = 0$. Denote by $\beta_1^x, \dots, \beta_k^x$ ($k \le N$) the distinct zeros of p (depending on x). We have

$$p(T)x = (T - \beta_1^x) \dots (T - \beta_k^x)x = 0,$$

and the k elements $\beta_1^x, \dots, \beta_k^x$ are in $\mathrm{Sp}(T)$, otherwise, if for instance $\beta_1^x \notin \mathrm{Sp}(T)$, then the operator $T - \beta_1^x$ is invertible and we would have $q(T)x = 0$ with $q(z) = p(z)/(z - \beta_1^x)$, which contradicts the minimality of the degree of p. So, we have

$$(T - \beta_1)^N \dots (T - \beta_m)^N x = 0 \qquad \text{for every } x \in X.$$

Hence $(T - \beta_1)^N \dots (T - \beta_m)^N = 0$.

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