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DEGENERATIONS FOR MODULES OVER REPRESENTATION-FINITE SELFINJECTIVE ALGEBRAS

BY

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1. Introduction and main result. Let A be a finite-dimensional associative K-algebra with identity over an algebraically closed field K. If $1 = a_1, \ldots, a_n$ is a basis of A over K, we have the constant structures a_{ijk} defined by $a_i a_j = \sum a_{ijk} a_k$. The affine variety $\mathrm{mod}_A(d)$ of d-dimensional unital left A-modules consists of n-tuples $m = (m_1, \ldots, m_n)$ of $(d \times d)$-matrices with coefficients in K such that m_1 is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$ holds for all indices i and j. The general linear group $\mathrm{Gl}_d(K)$ acts on $\mathrm{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules (see [6]). We shall agree to identify a d-dimensional A-module M with the point of $\mathrm{mod}_A(d)$ corresponding to it. We denote by $\mathcal{O}(M)$ the $\mathrm{Gl}_d(K)$-orbit of a module M in $\mathrm{mod}_A(d)$. Then one says that a module N in $\mathrm{mod}_A(d)$ is a *degeneration* of a module M in $\mathrm{mod}_A(d)$ if N belongs to the Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\mathrm{mod}_A(d)$, and we denote this fact by $M \le_{\mathrm{deg}} N$. Thus $\le_{\mathrm{deg}}$ is a partial order on the set of isomorphism classes of A-modules of a given dimension. It is not clear how to characterize $\le_{\mathrm{deg}}$ in terms of representation theory.

There has been work by S. Abeasis and A. del Fra [1], K. Bongartz [4], [3], Ch. Riedtmann [9], and A. Skowroński and the author [11]–[14] connecting $\le_{\mathrm{deg}}$ with other partial orders $\le_{\mathrm{ext}}$ and $\le$ on the isomorphism classes in $\mathrm{mod}_A(d)$. They are defined in terms of representation theory as follows:

- $M \le_{\mathrm{ext}} N \Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\mathrm{mod}\, A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \le i \le s$, and $N = M_{s+1}$ for some natural number s.
- $M \le N \Leftrightarrow [M, X] \le [N, X]$ holds for all modules X.

Here and later on we abbreviate $\dim_K \mathrm{Hom}_A(X, Y)$ by $[X, Y]$. Then for modules M and N in $\mathrm{mod}_A(d)$ the following implications hold:

$$M \le_{\mathrm{ext}} N \Rightarrow M \le_{\mathrm{deg}} N \Rightarrow M \le N$$

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(see [4], [9]). Unfortunately, the reverse implications are not true in general, and it would be interesting to find out when they are. The author proved in [14] that the orders $\le_{\mathrm{deg}}$ and $\le$ coincide for all modules over all representation-finite algebras. Moreover, in [3] K. Bongartz proved that these orders also coincide for all modules over tame concealed algebras. The orders $\le_{\mathrm{deg}}$ and $\le_{\mathrm{ext}}$ do not coincide even for very simple representation-finite algebras (see [9]). The author proved in [14] and [13] that $\le_{\mathrm{deg}}$ and $\le_{\mathrm{ext}}$ are equivalent for all modules over an algebra A with $\mathrm{Ext}_A^1(X,X)=0$ for any indecomposable A-module X, and for all modules over tame concealed algebras.

In the representation theory of algebras an important role is played by selfinjective algebras, that is, algebras A such that ${}_AA$ is injective. We are concerned with the question of when the partial orders $\le_{\mathrm{deg}}$ and $\le_{\mathrm{ext}}$ coincide for modules over representation-finite selfinjective algebras. The main aim of this paper is to prove the following theorem, which gives a complete answer to this question.

THEOREM. *Let A be a connected representation-finite selfinjective algebra. Then the following conditions are equivalent*:

(i) *There exist A-modules M, N such that $M \le_{\mathrm{deg}} N$ and $M \not\le_{\mathrm{ext}} N$.*

(ii) *There exist A-modules M, N such that $M <_{\mathrm{deg}} N$ and N is indecomposable.*

(iii) *The stable Auslander–Reiten quiver Γ_A^{s} of A is isomorphic to $\mathbb{Z}\mathbb{D}_{3m}/(\tau^{2m-1})$ for some $m \ge 2$.*

For basic background on the topics considered here we refer to [4], [6], [10], and for the representation theory of representation-finite selfinjective algebras to [5], [7], [8]. The results presented in this paper form a part of the author's doctoral dissertation written under the supervision of Professor A. Skowroński. The author gratefully acknowledges support from the Polish Scientific Grant KBN No. 2 PO3A 020 08.

2. Proof of the main result

2.1. Recall that A denotes a fixed finite-dimensional associative K-algebra with identity over an algebraically closed field K. We denote by $\operatorname{mod} A$ the category of finite-dimensional left A-modules. By an A-module mean an object from $\operatorname{mod} A$. Further, we denote by Γ_A the Auslander–Reiten quiver of A and by $\tau{=}\tau_A$ and $\tau^-{=}\tau_A^-$ the Auslander–Reiten translations $D\,\mathrm{Tr}$ and $\mathrm{Tr}D$, respectively. We shall agree to identify the vertices of Γ_A with the corresponding indecomposable modules. By Γ_A^{s} we denote the stable translation quiver obtained from Γ_A by removing all projective-injective vertices and arrows attached to them. For a noninjective indecomposable A-module U we denote by $\Sigma(U)$ the Auslander–Reiten sequence

$$\Sigma(U): \quad 0 \to U \to E(U) \to \tau^{-}U \to 0,$$

and define πU to be the unique indecomposable projective-injective direct summand of $E(U)$ if such a summand exists, or 0 otherwise.

2.2. Let A be a connected representation-finite selfinjective algebra. Then $\Gamma_A^{\mathrm{s}} \simeq \mathbb{Z}\Delta/\Pi$, where Δ is a Dynkin diagram of type $\mathbb{A}_n$ with $n \geq 1$, $\mathbb{D}_n$ with $n \geq 4$, or $\mathbb{E}_n$ with $n \in \{6,7,8\}$, and Π is an infinite cyclic group of automorphisms of $\mathbb{Z}\Delta$ with finitely many orbits. Following [7] the vertices of $\mathbb{Z}\Delta$ are denoted by (p,q), where $p \in \mathbb{Z}$ and $q \in \Delta$, and the translation τ on $\mathbb{Z}\Delta$ is given by $\tau(p,q) = (p-1,q)$. For a vertex (p,q) of $\mathbb{Z}\Delta$ we denote by $\overline{(p,q)}$ its orbit in Γ_A^{s}. Following O. Bretscher, C. Läser and C. Riedtmann (see [5, (1.1)]) we define m_Δ to be the smallest integer m such that the image $\overline{v}$ in the mesh category $K(\mathbb{Z}\Delta)$ equals 0 for all paths v in $\mathbb{Z}\Delta$ whose length is greater than or equal to m. Then $m_{\mathbb{A}_n} = n$, $m_{\mathbb{D}_n} = 2n-3$, $m_{\mathbb{E}_6} = 11$, $m_{\mathbb{E}_7} = 17$ and $m_{\mathbb{E}_8} = 29$.

2.3. Lemma. *Let A be a representation-finite selfinjective algebra of class $\mathbb{D}_n$ or $\mathbb{E}_n$. If Γ_A^{s} is not isomorphic to $\mathbb{Z}\mathbb{D}_{3m}/(\tau^{2m-1})$ for $m \geq 2$, then $\mathrm{Ext}_A^1(X,X) = 0$ for all indecomposable A-modules X.*

Proof. Take any indecomposable A-module X. If X is projective-injective, then $\mathrm{Ext}_A^1(X,X) = 0$. Thus, we may assume that $X \in \Gamma_A^{\mathrm{s}}$. Following O. Bretscher, C. Läser and C. Riedtmann (see [5, (1.4)]), we write $\Gamma_A^{\mathrm{s}} = \mathbb{Z}\Delta/(\tau^r\Phi)$, where $r \geq 1$ and Φ is an automorphism of $\mathbb{Z}\Delta$ which fixes at least one vertex. Moreover, we may assume that $\Phi^k = 1_{\mathbb{Z}\Delta}$ for some $k \geq 1$, since $\Delta = \mathbb{D}_n$ or $\Delta = \mathbb{E}_n$. Hence, every path in $\mathbb{Z}\Delta$ starting from Y_1 and ending in Y_2 with $\overline{Y}_1 = \overline{Y}_2$ has length $2rl$ for some $l \geq 1$. Take any W in $\mathbb{Z}\Delta$ such that $\overline{W} = X$. Applying the Auslander–Reiten formula and Proposition 1.5 in [7], we obtain

$$\mathrm{Ext}_A^1(X,X) \simeq D\underline{\mathrm{Hom}}_A(X,\tau X) \simeq \bigoplus_{\overline{Y} \simeq X} \mathrm{Hom}_{K(\mathbb{Z}\Delta)}(Y,\tau W).$$

Then $\mathrm{Ext}_A^1(X,X) \neq 0$ implies that there exists an integer $l \geq 1$ and a path v in $\mathbb{Z}\Delta$ of length $2rl-2$ such that its image $\overline{v}$ in the mesh category $K(\mathbb{Z}\Delta)$ is nonzero. By the definition of m_Δ, it remains to show that $2r-2 \geq m_\Delta$. But this is done by (1.5) and (1.6) in [5], since $\tau^r\Phi \neq \tau^{2m-1}$.

2.4. Proof of the Theorem. Clearly, (ii) implies (i).

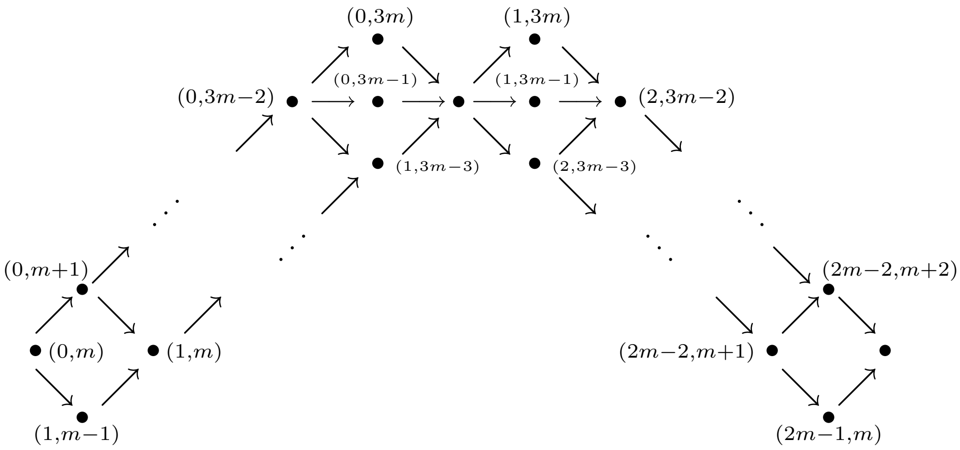
(i)⇒(iii). Assume that Γ_A^{s} is not isomorphic to $\mathbb{Z}\mathbb{D}_{3m}/(\tau^{2m-1})$ for any $m \geq 2$. We claim that then the orders $\leq_{\mathrm{deg}}$ and $\leq_{\mathrm{ext}}$ are equivalent. If A is a selfinjective algebra of class $\mathbb{A}_n$, then this is done by Theorem 2 in [12]. Thus, we may assume that A is of class $\mathbb{D}_n$ with $n \geq 4$, or $\mathbb{E}_n$ with

$n \in \{6,7,8\}$. Then our claim follows from Lemma 2.3 above and Theorem 2 in [14].

(iii)⇒(ii). Assume $\Gamma_A^{\mathrm{s}} = \mathbb{Z}\mathbb{D}_{3m}/(\tau^{2m-1})$ for some $m \ge 2$ and $\mathbb{D}_{3m}$ of the form

$$1 \to 2 \to \ldots \to (3m-2) \begin{array}{l} \nearrow (3m-1) \\ \searrow 3m \end{array}$$

The quiver $\mathbb{Z}\mathbb{D}_{3m}$ admits a mesh-complete subquiver of the form



Then there are the following short exact sequences in mod A:

$$\Sigma(\overline{(0,k)}) : 0 \to \overline{(0,k)} \to \overline{(0,k+1)} \oplus \overline{(1,k-1)} \oplus \pi\overline{(0,k)} \to \overline{(1,k)} \to 0,$$
$$\text{for any } m \le k \le 3m-3,$$
$$\Sigma(\overline{(0,3m-2)}) : 0 \to \overline{(0,3m-2)} \to \overline{(0,3m-1)} \oplus \overline{(0,3m)} \oplus \overline{(1,3m-3)} \oplus \pi\overline{(0,3m-2)} \to \overline{(1,3m-2)} \to 0,$$
$$\Sigma(\overline{(0,3m-1)}) \oplus \Sigma(\overline{(0,3m)} : 0 \to \overline{(0,3m-1)} \oplus \overline{(0,3m)} \to \overline{(1,3m-2)} \oplus \overline{(1,3m-2)} \oplus \pi\overline{(0,3m-1)} \oplus \pi\overline{(0,3m)} \to \overline{(1,3m-1)} \oplus \overline{(1,3m)} \to 0,$$
$$\Sigma(\overline{(1,3m-2)}) : 0 \to \overline{(1,3m-2)} \to \overline{(1,3m-1)} \oplus \overline{(1,3m)} \oplus \overline{(2,3m-3)} \oplus \pi\overline{(1,3m-2)} \to \overline{(2,3m-2)} \to 0,$$
$$\Sigma(\overline{(l,3m-1-l)}) : 0 \to \overline{(l,3m-1-l)} \to \overline{(l,3m-l)} \oplus \overline{(l+1,3m-2-l)} \oplus \pi\overline{(l,3m-1-l)} \to \overline{(l+1,3m-1-l)} \to 0, \text{ for any } 2 \le l \le 2m-2.$$

Applying Lemma (3 + 3 + 2) from [2, (2.1)] to these sequences, we get a short exact sequence

$$0 \to \overline{(0,m)} \to \overline{(1,m-1)} \oplus \overline{(2m-1,m)} \oplus \pi \to \overline{(2m-1,m+1)} \to 0,$$

where

$$\pi = \bigoplus_{k=m}^{3m} (\pi\overline{(0,k)}\,) \oplus \bigoplus_{l=1}^{2m-2} (\pi\overline{(l,3m-1-l)}\,).$$

Of course, $\overline{(2m-1,m)} = \overline{(0,m)}$. Finally, applying [9, Proposition 3.4], we infer that $\overline{(1,m-1)} \oplus \pi <_{\mathrm{deg}} \overline{(2m-1,m+1)}$. This finishes the proof.

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