DEGENERATIONS FOR MODULES OVER REPRESENTATION-FINITE SELFINJECTIVE ALGEBRAS

BY

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1. Introduction and main result. Let \( A \) be a finite-dimensional associative \( K \)-algebra with identity over an algebraically closed field \( K \). If \( 1 = a_1, \ldots, a_n \) is a basis of \( A \) over \( K \), we have the constant structures \( a_{ijk} \) defined by \( a_i a_j = \sum a_{ijk} a_k \). The affine variety \( \text{mod}_A(d) \) of \( d \)-dimensional unital left \( A \)-modules consists of \( n \)-tuples \( m = (m_1, \ldots, m_n) \) of \((d \times d)\)-matrices with coefficients in \( K \) such that \( m_1 \) is the identity matrix and \( m_i m_j = \sum a_{ijk} m_k \) holds for all indices \( i \) and \( j \). The general linear group \( \text{Gl}_d(K) \) acts on \( \text{mod}_A(d) \) by conjugation, and the orbits correspond to the isomorphism classes of \( d \)-dimensional modules (see [6]). We shall agree to identify a \( d \)-dimensional \( A \)-module \( M \) with the point of \( \text{mod}_A(d) \) corresponding to it. We denote by \( O(M) \) the \( \text{Gl}_d(K) \)-orbit of a module \( M \) in \( \text{mod}_A(d) \). Then one says that a module \( N \) in \( \text{mod}_A(d) \) is a degeneration of a module \( M \) in \( \text{mod}_A(d) \) if \( N \) belongs to the Zariski closure \( \overline{O(M)} \) of \( O(M) \) in \( \text{mod}_A(d) \), and we denote this fact by \( M \leq_{\text{deg}} N \). Thus \( \leq_{\text{deg}} \) is a partial order on the set of isomorphism classes of \( A \)-modules of a given dimension. It is not clear how to characterize \( \leq_{\text{deg}} \) in terms of representation theory.

There has been work by S. Abeasis and A. del Fra [1], K. Bongartz [4], [3], Ch. Riedtmann [9], and A. Skowroński and the author [11]–[14] connecting \( \leq_{\text{deg}} \) with other partial orders \( \leq_{\text{ext}} \) and \( \leq \) on the isomorphism classes in \( \text{mod}_A(d) \). They are defined in terms of representation theory as follows:

- \( M \leq_{\text{ext}} N \Leftrightarrow \) there are modules \( M_i, U_i, V_i \) and short exact sequences \( 0 \rightarrow U_i \rightarrow M_i \rightarrow V_i \rightarrow 0 \) in \( \text{mod}_A(d) \) such that \( M = M_1, M_{i+1} = U_i \oplus V_i, 1 \leq i \leq s \), and \( N = M_{s+1} \) for some natural number \( s \).
- \( M \leq N \Leftrightarrow [M, X] \leq [N, X] \) holds for all modules \( X \).

Here and later on we abbreviate \( \dim_K \text{Hom}_A(X, Y) \) by \([X, Y]\). Then for modules \( M \) and \( N \) in \( \text{mod}_A(d) \) the following implications hold:

\[
M \leq_{\text{ext}} N \Rightarrow M \leq_{\text{deg}} N \Rightarrow M \leq N
\]

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Unfortunately, the reverse implications are not true in general, and it would be interesting to find out when they are. The author proved in [14] that the orders \( \leq_{\text{deg}} \) and \( \leq \) coincide for all modules over all representation-finite algebras. Moreover, in [3] K. Bongartz proved that these orders also coincide for all modules over tame concealed algebras. The orders \( \leq_{\text{deg}} \) and \( \leq_{\text{ext}} \) do not coincide even for very simple representation-finite algebras (see [9]). The author proved in [14] and [13] that \( \leq_{\text{deg}} \) and \( \leq_{\text{ext}} \) are equivalent for all modules over an algebra \( A \) with \( \text{Ext}_A^1(X,X) = 0 \) for any indecomposable \( A \)-module \( X \), and for all modules over tame concealed algebras.

In the representation theory of algebras an important role is played by selfinjective algebras, that is, algebras \( A \) such that \( AA \) is injective. We are concerned with the question of when the partial orders \( \leq_{\text{deg}} \) and \( \leq_{\text{ext}} \) coincide for modules over representation-finite selfinjective algebras. The main aim of this paper is to prove the following theorem, which gives a complete answer to this question.

**Theorem.** Let \( A \) be a connected representation-finite selfinjective algebra. Then the following conditions are equivalent:

(i) There exist \( A \)-modules \( M, N \) such that \( M \leq_{\text{deg}} N \) and \( M \nless_{\text{ext}} N \).

(ii) There exist \( A \)-modules \( M, N \) such that \( M <_{\text{deg}} N \) and \( N \) is indecomposable.

(iii) The stable Auslander–Reiten quiver \( \Gamma_A^s \) of \( A \) is isomorphic to \( \mathbb{Z} \mathbb{D}_{3m}/(\tau^{2m-1}) \) for some \( m \geq 2 \).

For basic background on the topics considered here we refer to [4], [6], [10], and for the representation theory of representation-finite selfinjective algebras to [5], [7], [8]. The results presented in this paper form a part of the author’s doctoral dissertation written under the supervision of Professor A. Skowroński. The author gratefully acknowledges support from the Polish Scientific Grant KBN No. 2 PO3A 020 08.

**2. Proof of the main result**

2.1. Recall that \( A \) denotes a fixed finite-dimensional associative \( K \)-algebra with identity over an algebraically closed field \( K \). We denote by \( \text{mod} \ A \) the category of finite-dimensional left \( A \)-modules. By an \( A \)-module mean an object from \( \text{mod} \ A \). Further, we denote by \( \Gamma_A \) the Auslander–Reiten quiver of \( A \) and by \( \tau = \tau_A \) and \( \tau^- = \tau^-_A \) the Auslander–Reiten translations \( D \text{Tr} \) and \( \text{Tr} D \), respectively. We shall agree to identify the vertices of \( \Gamma_A \) with the corresponding indecomposable modules. By \( \Gamma_A^s \) we denote the stable translation quiver obtained from \( \Gamma_A \) by removing all projective-injective vertices and arrows attached to them. For a noninjective indecom-
posable $A$-module $U$ we denote by $\Sigma(U)$ the Auslander–Reiten sequence

$$\Sigma(U) : \quad 0 \to U \to E(U) \to \tau^{-}U \to 0,$$

and define $\pi U$ to be the unique indecomposable projective-injective direct summand of $E(U)$ if such a summand exists, or 0 otherwise.

2.2. Let $A$ be a connected representation-finite selfinjective algebra. Then $I^n_A \simeq Z\Delta/H$, where $\Delta$ is a Dynkin diagram of type $\mathbb{A}_n$ with $n \geq 1$, $\mathbb{D}_n$ with $n \geq 4$, or $\mathbb{E}_n$ with $n \in \{6, 7, 8\}$, and $H$ is an infinite cyclic group of automorphisms of $Z\Delta$ with finitely many orbits. Following [7] the vertices of $Z\Delta$ are denoted by $(p, q)$, where $p \in \mathbb{Z}$ and $q \in \Delta$, and the translation $\tau$ on $Z\Delta$ is given by $\tau(p, q) = (p-1, q)$. For a vertex $(p, q)$ of $Z\Delta$ we denote by $\overline{(p, q)}$ its orbit in $I^n_A$. Following O. Bretscher, C. Läser and C. Riedtmann (see [5, (1.1)]) we define $m_\Delta$ to be the smallest integer $m$ such that the image $\pi$ in the mesh category $K(Z\Delta)$ equals 0 for all paths $v$ in $Z\Delta$ whose length is greater than or equal to $m$. Then $m_{\mathbb{A}_n} = n$, $m_{\mathbb{D}_n} = 2n - 3$, $m_{\mathbb{E}_6} = 11$, $m_{\mathbb{E}_7} = 17$ and $m_{\mathbb{E}_8} = 29$.

2.3. Lemma. Let $A$ be a representation-finite selfinjective algebra of class $\mathbb{D}_n$ or $\mathbb{E}_n$. If $I^n_A$ is not isomorphic to $Z\mathbb{D}_m/(\tau^{2m-1})$ for $m \geq 2$, then $\text{Ext}^1_A(X, X) = 0$ for all indecomposable $A$-modules $X$.

Proof. Take any indecomposable $A$-module $X$. If $X$ is projective-injective, then $\text{Ext}^1_A(X, X) = 0$. Thus, we may assume that $X \in I^n_A$. Following O. Bretscher, C. Läser and C. Riedtmann (see [5, (1.4)]), we write $I^n_A = Z\Delta/(\tau^r\Phi)$, where $r \geq 1$ and $\Phi$ is an automorphism of $Z\Delta$ which fixes at least one vertex. Moreover, we may assume that $\Phi^k = 1_{Z\Delta}$ for some $k \geq 1$, since $\Delta = \mathbb{D}_n$ or $\Delta = \mathbb{E}_n$. Hence, every path in $Z\Delta$ starting from $Y_1$ and ending in $Y_2$ with $\overline{Y}_1 = \overline{Y}_2$ has length $2rl$ for some $l \geq 1$. Take any $W$ in $Z\Delta$ such that $\overline{W} = X$. Applying the Auslander–Reiten formula and Proposition 1.5 in [7], we obtain

$$\text{Ext}^1_A(X, X) \simeq D\text{Hom}_A(X, \tau X) \simeq \bigoplus_{\overline{v} \simeq X} \text{Hom}_{K(Z\Delta)}(Y, \tau W).$$

Then $\text{Ext}^1_A(X, X) \neq 0$ implies that there exists an integer $l \geq 1$ and a path $v$ in $Z\Delta$ of length $2rl - 2$ such that its image $\overline{v}$ in the mesh category $K(Z\Delta)$ is nonzero. By the definition of $m_\Delta$, it remains to show that $2r - 2 \geq m_\Delta$. But this is done by (1.5) and (1.6) in [5], since $\tau^r\Phi \neq \tau^{2m-1}$.

2.4. Proof of the Theorem. Clearly, (ii) implies (i).

(i)⇒(iii). Assume that $I^n_A$ is not isomorphic to $Z\mathbb{D}_m/(\tau^{2m-1})$ for any $m \geq 2$. We claim that then the orders $\leq_{\text{deg}}$ and $\leq_{\text{ext}}$ are equivalent. If $A$ is a selfinjective algebra of class $\mathbb{A}_n$, then this is done by Theorem 2 in [12]. Thus, we may assume that $A$ is of class $\mathbb{D}_n$ with $n \geq 4$, or $\mathbb{E}_n$ with
Then our claim follows from Lemma 2.3 above and Theorem 2 in [14].

(iii)⇒(ii). Assume $I^* = \mathbb{Z}D_{3m}/(\tau^{2m-1})$ for some $m \geq 2$ and $D_{3m}$ of the form

\[ 1 \to 2 \to \ldots \to (3m - 2) \to (3m - 1) \to 3m \]

The quiver $\mathbb{Z}D_{3m}$ admits a mesh-complete subquiver of the form

\[ \begin{array}{c}
(0,3m) \searrow \bullet \swarrow (0,3m-1) \\
(0,3m-2) \searrow \bullet \swarrow (0,3m-3) \\
\vdots \\
(0,m+1) \searrow \bullet \swarrow (1,m) \\
(1,m-1) \searrow \bullet \swarrow (1,3m) \\
(2,3m-2) \searrow \bullet \swarrow (2,3m-3) \\
\vdots \\
(2m-2,m+2) \searrow \bullet \swarrow (2m-1,m) \\
(2m-1,m-1) \searrow \bullet \swarrow (2m-1,3m) \\
\end{array} \]

Then there are the following short exact sequences in mod $A$:

\[ \Sigma((0,k)) : 0 \to (0,k) \to (0,k+1) \oplus (1,k-1) \oplus \pi(0,k) \to (1,k) \to 0, \]

for any $m \leq k \leq 3m - 3$,

\[ \Sigma((0,3m-2)) : 0 \to (0,3m-2) \to (0,3m-1) \oplus (0,3m) \oplus (1,3m-3) \oplus \pi(0,3m-2) \to (1,3m-2) \to 0, \]

\[ \Sigma((0,3m-1)) \oplus \Sigma((0,3m)) : 0 \to (0,3m-1) \oplus (0,3m) \to (1,3m-2) \oplus (1,3m) \oplus (2,3m-3) \oplus \pi(0,3m-1) \oplus \pi(0,3m) \to (1,3m) \oplus (1,3m) \oplus (2,3m-3) \to 0, \]

\[ \Sigma((1,3m-2)) : 0 \to (1,3m-2) \to (1,3m-1) \oplus (1,3m) \oplus (2,3m-3) \oplus (2,3m-2) \to (2,3m-2) \to 0, \]

\[ \Sigma((l,3m-1-l)) : 0 \to (l,3m-1-l) \to (l,3m-l) \oplus (l+1,3m-2-l) \oplus \pi(l,3m-1-l) \to (l+1,3m-1-l) \to 0, \]

for any $2 \leq l \leq 2m - 2$.

Applying Lemma (3 + 3 + 2) from [2, (2.1)] to these sequences, we get a short exact sequence

\[ 0 \to (0,m) \to (1,m-1) \oplus (2m-1,m) \oplus \pi \to (2m-1,m+1) \to 0, \]
where
\[ \pi = \bigoplus_{k=m}^{3m} \pi(0,k) \oplus \bigoplus_{l=1}^{2m-2} \pi(l,3m-1-l). \]

Of course, \((2m-1,m) = (0,m)\). Finally, applying [9, Proposition 3.4], we infer that \((1,m-1) \oplus \pi <_{\text{deg}} (2m-1,m+1)\). This finishes the proof.

REFERENCES


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