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ON QUASIREGULAR POLYNOMIAL MAPPINGS

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0. Introduction and preliminaries. The aim of the present paper is to study polynomial mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ which are quasiregular on $\mathbb{R}^n$.

0.1. Definition. A mapping $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a *polynomial mapping* iff $Q = (Q^1, \ldots, Q^n)$, where each Q^i is a polynomial in n real variables.

If $n = 2$, we shall identify $\mathbb{R}^2$ with $\mathbb{C}$ and write $Q = Q(z, \overline{z})$.

If a polynomial map depends only on z, we shall call it a *holomorphic polynomial map* and denote it by $Q(z)$.

A polynomial mapping $Q : \mathbb{R}^n \to \mathbb{R}^n$ is called *m-homogeneous* iff

$$Q(tx_1, \ldots, tx_n) = t^m(x_1, \ldots, x_n)$$

for each $t \in \mathbb{R}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Let us recall a well known algebraic fact:

0.2. Proposition. *Each polynomial mapping Q from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be written in the form*

$$Q = \sum_{k=m}^{M} Q_k + c,$$

where $m \geq 1$, $M \geq m$, $c \in \mathbb{R}^n$, Q_k is a k-homogeneous polynomial map for each k and $\|Q_M\| \cdot \|Q_m\|$ does not vanish identically on $\mathbb{R}^n$.

The number M will be called the *algebraic degree* of Q.

0.3. Definition. Let G be an open set in $\mathbb{R}^n$ and let $F : G \to \mathbb{R}^n$ be a mapping from the Sobolev space $W_p^1(G, \mathrm{loc})$, $p > 1$. We shall say that F is *quasiregular* on G iff there exists K, $1 \leq K < \infty$, such that

$$\left(\sum_{i,j=1}^{n} \left| \frac{\partial F^i}{\partial x_j} \right|^2 \right)^{n/2} \leq K n^{n/2} JF, \quad JF = \det \left[\frac{\partial F^i}{\partial x_j} \right],$$

almost everywhere on G.

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This condition has an equivalent form: There exists K, $1 \le K < \infty$, such that
$$\|DF\|^n \le KJF$$
almost everywhere on G, where $\|DF\|$ denotes the operator norm of the derivative DF.

The least such K is called the (*outer*) *dilatation* of F.

If $n = 2$, then quasiregularity can be equivalently defined as follows. Let $F : G \to \mathbb{C}$, $G = \operatorname{int} G \subset \mathbb{C}$, $F \in W_p^1(G, \mathrm{loc})$, $p > 1$. Then F is *quasiregular* iff
$$\|\mu_F\|_{L^\infty(G)} < 1, \quad \text{where } \mu_F = \left(\frac{\partial F}{\partial \overline{z}}\right) \Big/ \left(\frac{\partial F}{\partial z}\right).$$
In this case μ_F is called the *Beltrami differential* of F.

By the theorem of Reshetnyak [Re] each quasiregular mapping is open and discrete.

0.5. Definition. A quasiregular map F is *quasiconformal* iff it is injective.

Reshetnyak [Re] proved that such an F is a homeomorphism of G onto $F(G)$. A theorem of Stoilov [S] implies that if $n = 2$ then each quasiregular mapping is a composition of a quasiconformal map and a holomorphic function.

0.6. Definition. Let F be a quasiregular mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$. Then F is called of *polynomial type* iff $\lim_{x\to\infty} F(x) = \infty$. If F is a quasiregular m of polynomial type then $F(\mathbb{R}^n) = \mathbb{R}^n$ and
$$q = \max_{y\in\mathbb{R}^n} \#F^{-1}(y) < \infty.$$
The number q is called *degree* of F.

Note that if $n = 2$ then each quasiregular map of polynomial type can be expressed as $F(z, \overline{z}) = W(h(z, \overline{z}))$, where $W(z)$ is a holomorphic polynomial map on $\mathbb{C}$ and $h(z, \overline{z})$ is a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ such that $h(0) = 0$, $h(1) = 1$, $h(\infty) = \infty$.

If $n \ge 3$ then each sufficiently smooth quasiregular mapping is a local homeomorphism (see Church [C] and Rickman [Ri], p. 12).

By Zorich's theorem ([Ri], [Re]) each local homeomorphism of $\mathbb{R}^n$ into $\mathbb{R}^n$ which is quasiregular must be a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

In particular the following theorem holds:

0.7. Theorem. *If $n \ge 3$ then every quasiregular polynomial mapping is quasiconformal.*

We shall begin our study of quasiregular polynomial mappings with some examples.

1. Examples

1.1. Example.

$$F_m(x) = x\|x\|^{2m} \qquad \Big(\|x\| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}\Big).$$

This simple example can be generalized as follows: $F_W(x) = xW(\|x\|^2)$, where $W(t)$ is a polynomial of one real variable such that

$$W(t)^{n-1}(W(t) + 2tW'(t)) > 0 \quad \text{for } t > 0$$
$$\text{and } 0 < K_1 < |1 + 2W'(t) \cdot t/W(t))| < K_2 < \infty \quad \text{for } t > 0.$$

The mappings F_W together with translations and linear mappings with positive determinant generate a large semigroup S of quasiconformal polynomial mappings. (S is a semigroup with respect to composition of functions.)

However, there exist quasiconformal polynomial mappings not belonging to S, e.g.

$$Q(x) = x(\|x\|^2 + c) + e_1x_1, \quad c > 0, \ e_1 = (1, 0, \ldots, 0).$$

1.2. Remark. Mappings which are inverse to mappings from Example 1.1 are of polynomial type, but not polynomial mappings.

Such mappings can be interesting from the point of view of dynamical systems. Let $Q(z, \overline{z}) = e^{i\vartheta}z|z|^2$, where $\vartheta/(2\pi)$ is irrational. We have $Q^{-1}(w, \overline{w}) = e^{-i\vartheta}w/|w|^{2/3}$. The unit circle is a "strange attractor" for the dynamical system $(Q^{-1})^{\circ n}$. The basin of this attractor is $\overline{\mathbb{R}} \setminus \{0, \infty\}$.

The mappings from the semigroup S from Example 1.1 can also have a rich dynamics.

Let $Q(x) = (e_1 + A(x)\|x\|^2)\|e_1 + A(x)\|x\|^2\|^2$, $e_1 = (1, 0, \ldots, 0)$, $A(x_1, x_2, x_3, \ldots) = (-x_1, -x_2, x_3, \ldots, x_n)$.

Then Q has a superattracting 2-cycle $\{e_1, 0\}$, the fixed point $(t_0, 0, \ldots, 0)$ $(t_0 \in (0,1),\ (1 - t_0)^3 = t_0)$ and the invariant set $\{x \in \mathbb{R}^n : x = (1/2, 0, x_3, \ldots, x_n),\ \sum_{i=3}^n x_i^2 = 3/4\}$.

1.3. Example. The mapping $Q_n(z, \overline{z}) = z|z|^{2n} + (n+1)z + n\overline{z}$ is a quasiconformal polynomial diffeomorphism of $\mathbb{C}$ with

$$\mu_{Q_n} = \frac{n(z^2|z|^{2n-2} + 1)}{(n+1)(|z|^{2n} + 1)} \quad \text{and} \quad |\mu_{Q_n}| < n/(n+1).$$

We have $JQ_n(z) > (|z|^{2n} + 1)^2$. The equation $Q_n(z, \overline{z}) = a + bi$ can be written as

$$[r^{2n+1} + (2n+1)r]\cos\varphi = a, \quad z = re^{i\varphi},$$
$$[r^{2n+1} + r]\sin\varphi = b, \quad a^2 + b^2 \neq 0.$$

Thus

$$(*) \qquad 1 = a^2/(r^2 + (2n+1)r)^2 + b^2/(r^{2n+1} + r)^2.$$

The function on the right is strictly decreasing and $(*)$ has exactly one solution $r_0 \in [0,\infty)$. Hence Q_n is invertible.

The mapping $Q(x) = x(\|x\|^2 + c)$, $c > 0$, is a quasiconformal polynomial diffeomorphism of $\mathbb{R}^n$, since $JQ(x) = (\|x\|^2 + c)^{n-1}(\|x\|^2 + c + 2\|x\|^2) > 0$.

1.4. EXAMPLE. The mapping $z|z|^2 + a\overline{z}^2 + 2z$ is quasiconformal if $|a| < 1$.

1.5. EXAMPLE. Let $Q(z,\overline{z}) = aP(z) + b\overline{P(z)}$, where $|a| > |b|$ and $P(z)$ is a holomorphic polynomial. If the degree of P is greater than one then Q is quasiregular, but not quasiconformal. We have

$$\mu_Q = (b/a)(\overline{P'(z)}/P'(z)), \qquad JQ(x) = (|a|^2 - |b|^2)|P'(z)|^2.$$

1.6. EXAMPLE. Let $Q(z,\overline{z}) = |z|^2 + 2z^2$. Then

$$\mu_Q(z) = 1/(\overline{z}/z + 4), \quad z \neq 0, \quad |\mu_Q(z)| < 1/3.$$

We have $Q(z,\overline{z}) = (h(z))^2$ where $h(z) = (2z + \overline{z})/(2 + \overline{z}/z)^{1/2}$, $z \neq 0$, $h(0) = 0$, is a quasiconformal map from $\mathbb{C}$ onto $\mathbb{C}$.

1.7. Remark. If we multiply the function Q from Example 1.6 by z, we get the quasiconformal mapping

$$Q_1(z,\overline{z}) = z|z|^2 + 2z^3.$$

1.8. EXAMPLE. Let $Q_n(z,\overline{z}) = az^n + \overline{z}^n$, $|a| > 1$. Then $Q_n = (h)^n$, where

$$h(z) = (\overline{z}^n + az^n)/(z^{n-1}[(a + (z/\overline{z})^n)^{1/n}]^{n-1}), \qquad h(0) = 0,$$

is a quasiconformal homeomorphism of $\mathbb{C}$ onto $\mathbb{C}$.

2. General results

2.1. PROPOSITION. *Let Q be a quasiregular polynomial mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$. Write* (see Proposition 0.2)

$$Q = \sum_{k=m}^{M} Q_k + c,$$

where Q_k is a k-homogeneous polynomial map, and $\|Q_M\|\cdot\|Q_m\| \neq 0$. Then Q_M and Q_m are quasiregular.

Proof. By assumption there exists K, $1 \le K < \infty$, such that

$$(**) \qquad \left(\sum_{i,j=1}^{n}\left(\frac{\partial Q^i}{\partial x_j}\right)^2\right)^{n/2} \le Kn^{n/2}JQ \quad \text{on } \mathbb{R}^n.$$

Assume that $(**)$ is not true for Q_M. Then there exists $x_0 \in \mathbb{R}^n$ such that

$$\left(\sum_{i,j=1}^{n}\left(\frac{\partial Q_M^i}{\partial x_j}(x_0)\right)^2\right)^{n/2} > Kn^{n/2}JQ_M(x_0).$$

We now have

$$\bigg(\sum_{i,j=1}^{n}\bigg(\frac{\partial Q^i}{\partial x_j}(tx_0)\bigg)^2\bigg)^{n/2} = \bigg(t^{2M-2}\sum_{i,j=1}^{n}\bigg(\frac{\partial Q^i_M}{\partial x_j}(x_0)\bigg)^2 + \text{lower order terms}\bigg)^{n/2}$$

and

$$JQ(tx_0) = (t^{M-1})^n JQ_M(x_0) + \text{lower order terms}.$$

By taking $t\to\infty$ we get a contradiction with $(**)$ for Q.

Assume now that $(**)$ is not valid for Q_m, i.e. there exists $x_0\in\mathbb{R}^n$ such that

$$\bigg(\sum_{i,j=1}^{n}\bigg(\frac{\partial Q^i_m}{\partial x_j}(x_0)\bigg)^2\bigg)^{n/2} > Kn^{n/2}JQ_m(x_0).$$

We have

$$\bigg(\sum_{i,j=1}^{n}\bigg(\frac{\partial Q^i}{\partial x_j}(tx_0)\bigg)^2\bigg)^{n/2} = \bigg(t^{2m-2}\sum_{i,j=1}^{n}\bigg(\frac{\partial Q^i_m}{\partial x_j}(x_0)\bigg)^2 + \text{higher order terms}\bigg)^{n/2},$$

$$JQ(tx_0) = (t^{m-1})^n JQ_m(x_0) + \text{higher order terms}.$$

By taking $t\searrow 0^+$ we get a contradiction with $(**)$ again.

2.2. PROPOSITION. *If Q_m is an m-homogeneous quasiregular polynomial mapping then $Q_m(x)\neq 0$ for $x\neq 0$.*

P r o o f. By Reshetnyak's theorem [Re] a quasiregular map is open and discrete.

Propositions 2.1 and 2.2 yield the following

2.3. THEOREM. *Each quasiregular polynomial mapping is of polynomial type.*

P r o o f. Let, as above, $Q=\sum_{k=m}^{M}Q_k + c$. Put

$$R_M = \inf_{\{\|x\|=1\}}\|Q_M\| \quad\text{and}\quad \varrho_k = \sup_{\{\|x\|=1\}}\|Q_k\|,\quad k<M.$$

By Proposition 2.2, $R_M>0$. We have

$$(***)\qquad \|Q(x)\| \ge \|x\|^M R_M - \sum_{k=m}^{M-1}\|x\|^k\varrho_k - \|c\|.$$

Thus $\|Q(x)\|\to\infty$ as $\|x\|\to\infty$.

The inequality $(***)$ also yields the following

2.4. PROPOSITION. *If the algebraic degree of a quasiregular polynomial mapping* Q *is greater than one*, *then there exists* $R > 0$ *such that* $\|Q(x)\| > \|x\|$ *for* $\|x\| > R$ *and* $\lim_{x\to\infty} \|Q(x)\|/\|x\| = \infty$.

Proposition 2.4 is important from the point of view of dynamical systems. It shows that infinity is a superattracting fixed point for $Q^{\circ n}$. It also permits defining the "filled in" Julia set of Q as the complement of the immediate basin of attraction of ∞ (cf. [BSTV]).

Remark 1.2 shows that this is not true for an arbitrary quasiregular map of polynomial type (see also Remark 2.11 below).

2.5. THEOREM. *Every quasiregular* (*quasiconformal*) *polynomial mapping* $Q : \mathbb{R}^n \to \mathbb{R}^n$ *extends to a continuous mapping* (*homeomorphism*) *of the projective space* $\mathbb{RP}^n$ *onto itself.*

Proof. This follows from Proposition 2.2. Let $\sum_{k=m}^{M} Q_k + c$, Q_k and c as above. The mapping

$$\widetilde{Q}(t,x) = \Big(t^M, \sum_{k=m}^{M} Q_k(x)t^{M-k} + ct^M\Big)$$

is an M-homogeneous mapping of $\mathbb{R}^{n+1}$ onto itself which maps $\mathbb{R}^{n+1}\setminus\{0\}$ onto itself. Thus $\widetilde{Q}$ induces a mapping $\widetilde{\widetilde{Q}} : \mathbb{RP}^n \to \mathbb{RP}^n$, which is the required extension of Q.

Let us state some further properties of homogeneous polynomial quasiregular mappings.

2.6. PROPOSITION. *If* Q_m *is an* m*-homogeneous polynomial quasiregular mapping then* $Jf \ne 0$ *on* $\mathbb{R}^n \setminus \{0\}$.

Proof. Assume that $JQ_m(x_0) = 0$, $x_0 \ne 0$. The quasiregularity of Q_m implies that $(\partial Q_m^i/\partial x_j)(x_0) = 0$, $i, j = 1, \ldots, n$.

By the Euler formula,

$$mQ_m(x_0) = \sum_{i=1}^{n} x_0^i \frac{\partial Q_m^i}{\partial x_j}(x_0) = 0$$

and we get a contradiction with Proposition 2.2.

We also have

2.7. PROPOSITION. *An* m*-homogeneous polynomial mapping* Q *is quasiregular iff* $JQ > 0$ *on* $\mathbb{R}^n \setminus \{0\}$. *Its dilatation* K *is bounded from above by* $\sup_{\|x\|=1} \|DQ(x)\|^n / \inf_{\|x\|=1} JQ(x)$.

We shall now study the critical points of quasiregular polynomial mappings.

2.8. THEOREM. *Let U be an open neighborhood of x_0 in $\mathbb{R}^n$ and let $F : U \to \mathbb{R}^n$ be a quasiregular mapping of class C^∞ on U. Then there exists a neighborhood $V \subset U$ of x_0 such that $JF \neq 0$ on $V \setminus \{x_0\}$. Thus the critical points of F are isolated.*

Proof. We can assume that $x_0 = 0$ and $F(x_0) = 0$. If $JF(0) \neq 0$ then there is nothing to prove. If $JF(0) = 0$ then $(\partial F^i/\partial x_j)(0) = 0$ by quasiregularity. Theorems 7.2 and 7.3 from Reshetnyak's book imply that there exist $\alpha_1, \alpha_2 > 0$ and $c_1, c_2 > 0$ such that $c\|x\|^{\alpha_1} \le \|F(x)\| \le c_2\|x\|^{\alpha_2}$ for x from some ball $B(0,r) \subset U$. This implies that there exists $m \ge 2$ such that the mth derivative of F at zero is nonvanishing, i.e. we can find $x_0 \in B(0,r)$ such that $D^m F(0, x_0, \ldots, x_0) \neq 0$. Assume that m is the smallest number with the above property. Thus by the Taylor formula

$$F(x) = D^m F(0, x, \ldots, x) + O(\|x\|^{m+1}),$$
$$\frac{\partial F}{\partial x_i} = \frac{\partial}{\partial x_i}(D^m F(0, x, \ldots, x)) + O(\|x\|^m), \qquad i = 1, \ldots, n,\ x \in B(0,r).$$

We can now repeat the second part of the proof of Proposition 2.1 and show that $D^m F(0, x, \ldots, x)$ is an m-homogeneous quasiregular polynomial. Thus

$$\begin{aligned} J(D^m F(0, x \ldots, x))(x) &\ge \|x\|^{n(m-1)} \inf_{\|y\|=1} JD^m F(y) \\ &= \|x\|^{n(m-1)} c, \qquad c > 0. \end{aligned}$$

We have $JF(x) \ge c\|x\|^{n(m-1)} + O(\|x\|^{(m-1)n+1})$. Hence $JF(x) > 0$ if $\|x\|$ is sufficiently small and $x \neq 0$.

Theorem 2.8 implies

2.9. THEOREM. *A quasiregular polynomial mapping Q can have only a finite number of critical points.*

Proof. Let as above $Q = \sum_{k=m}^{M} Q_k + c$. We can assume that $M \ge 2$. By Proposition 2.1, Q_M is quasiregular. We have

$$JQ(x) \ge \|x\|^{n(M-1)} \inf_{\|x\|=1} JQ_M(x) + O(\|x\|^{(M-1)n-1}).$$

Thus there exists $R > 0$ such that $JQ(x) > 0$ if $\|x\| > R$. The set $\{x \in \mathbb{R}^n : JQ(x) = 0\}$ is a discrete set contained in the closed ball. Hence it must be finite.

Recently there is a great interest in the study of polynomial automorphisms of $\mathbb{R}^n$ (i.e. polynomial mappings whose inverse is also a polynomial map).

2.10. THEOREM. *If the inverse of a quasiconformal polynomial mapping Q is a polynomial mapping then Q is linear.*

Proof. The Jacobian JQ must be a constant function. This is easy to see because $JQ \cdot (JQ^{-1} \circ Q) = 1$ on $\mathbb{R}$. By assumption JQ and $JQ^{-1} \circ Q$ are polynomials on $\mathbb{R}^n$ and therefore both are constant.

The quasiconformality of Q and Q^{-1} implies that all derivatives of Q and Q^{-1} must be bounded on $\mathbb{R}^n$. Thus Q is linear.

2.11. Remark. The mappings inverse to quasiregular mappings form an interesting class of maps. They are quasiconformal maps of polynomial type which are real-analytic on $\mathbb{R}^n \setminus \{x_1, \ldots, x_S\}$, where $\{x_1, \ldots, x_S\}$ is the set of critical values of the mapping (see Theorem 2.8).

Proposition 2.4 implies that ∞ is a superrepelling fixed point for such maps.

3. Quasiregular polynomial mappings in dimension two. Examples 1.5, 1.6 and 1.8 show that in dimension two there exist quasiregular polynomial mappings which are not quasiconformal.

Theorem 2.3 yields immediately

3.1. Theorem. *Each quasiregular polynomial $Q(z, \overline{z})$ can be expressed as a composition $w(h(z, \overline{z}))$, where $w(z)$ is a holomorphic polynomial map and $h(z, \overline{z})$ is a quasiconformal map $\mathbb{C}$ onto $\mathbb{C}$, $h(0) = 0$, $h(1) = 1$, $h(\infty) = \infty$.*

3.2. Remark. Examples 1.6 and 1.8 show that h need not be a polynomial mapping.

3.3. Open Problem. Characterize the quasiconformal mappings h for which there exists a polynomial $w(z)$ such that $w \circ h$ is a polynomial mapping.

The Beltrami differential of such a mapping must be a rational function $R(z, \overline{z}) = Q_1(z, \overline{z})/Q_2(z, \overline{z})$, $|R| < k < 1$ on $\mathbb{C}$. The problem is the following: When $R(z, \overline{z}) = \widetilde{Q}_1(z, \overline{z})/\widetilde{Q}_2(z, \overline{z})$ and

$$\partial \widetilde{Q}_1/\partial z = \partial \widetilde{Q}_2/\partial \overline{z} \ ? \tag{i}$$

3.4. Example. If $R(z, \overline{z}) = z/(2\overline{z} + 6z)$ then (i) is not valid, but R can be expressed as $z^2/(2|z|^2 + 6z^2)$ and (i) holds. (R is the Beltrami differential of $z|z|^2 + 2z^3$ (Remark 1.7).)

On the other hand, $R(z) = -\frac{1}{2}z/\overline{z}$ is not a Beltrami differential of any quasiregular polynomial mappings since it is the Beltrami differential of $z/|z|^{2/3}$.

3.5. Theorem. *Let $Q(z, \overline{z}) = \sum_{k=m}^{M} Q_k(z, \overline{z}) + c$, Q_k is k-homogeneous, $|Q_M| \cdot |Q_m|$ does not vanish identically on $\mathbb{C}$. Assume that $Q(z, \overline{z})$ is a quasiregular polynomial mapping. Then*:

(1) *Q is quasiconformal iff Q_M is quasiconformal.*

(2) *If Q_M is quasiconformal then so is Q_m.*

Proof. We shall use the following theorem:

(ii) *Let D be a simply connected bounded domain in $\mathbb{C}$ with a smooth boundary and $h : \overline{D} \to \mathbb{C}$ be a C^1-smooth mapping. Assume that $h \neq 0$ on ∂D and $Jh(z) \neq 0$ if $h(z) = 0$. Then*

$$\frac{1}{2\pi} \int_{\partial D} d(\arg h) = P - N,$$

where P is the number of points $z \in D$ for which $h(z) = 0$ and $Jh(z) > 0$, and N is the number of points $z \in D$ for which $h(z) = 0$ and $Jh(z) < 0$.

(This theorem is classical and comes from Goursat's book [G], Vol. 1, pp. 388–389.)

Let $w \neq 0$. Assume also that w is not a critical value of Q. (By Theorem 2.8 there are only a finite number of such values.) For sufficiently large R we have $|Q_M - w| > |Q - Q_M|$ on $C(0, R) = \{z \in \mathbb{C} : |z| = R\}$. Thus

$$\frac{1}{2\pi} \int_{C(0,R)} d(\arg Q_M - w) = \frac{1}{2\pi} \int_{C(0,R)} d(\arg Q - w) = P - N.$$

Now $N = 0$ since $JQ_M > 0$ on $\mathbb{C} \setminus \{0\}$ and $JQ \geq 0$ on $\mathbb{C}$. Hence Q_M and Q have the same degree on $B(0, R)$, which implies (1).

(2) can be proved in the same way. We need only consider the small circles $C(0, r)$ with $r \to 0$. Let r_0 be such that $|Q_m| > |Q - Q_m - c|$ on $C(0, r)$ if $r < r_0$. Let

$$\alpha = \inf_{z \in C(0, r_0/2)} (|Q_m(z, \overline{z})| - |Q(z, \overline{z}) - Q_m(z, \overline{z}) - c|) > 0.$$

If $w \in B(0, \alpha)$ and w is not a critical value of $Q - c$ then $|Q_m - w| > |Q - Q_m - c|$, and we can use theorem (ii) for $D = B(0, r_0/2)$ and find that Q_m and $Q - c$ have the same degree on $B(0, r_0/2) \cap (Q - c)^{-1}(B(0, \alpha))$. Thus if Q is quasiconformal then so is Q_m. Hence (1)$\Rightarrow$(2).

Note that if Q is quasiconformal then Q_M and Q_m must be of an odd algebraic degree.

3.6. Remark (quasiconformal polynomial mappings of algebraic degree two). Such a mapping must also have degree two. It can be written in the form $Q(z, \overline{z}) = a(h(z))^2 + b$, where h maps quasiconformally $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$, $h(\infty) = \infty$, because each quadratic holomorphic polynomial is conjugate via an affine map to $z^2 + c$ for some $c \in \mathbb{C}$.

Thus $Q(z, \overline{z})$ is conjugate via a translation to $Q_1(z, \overline{z}) = a(h_1(z))^2 + b$, where $h_1(0) = 0$, $h_1(\infty) = \infty$.

Therefore $Q_1(z,\overline{z})$ can be written as

$$Q_1(z,\overline{z}) = \alpha z^2 + \beta\overline{z}^2 + \gamma|z|^2 + \delta.$$

Further $Q_1(z,\overline{z})$ is linearly conjugate to

$$Q_2(z,\overline{z}) = z^2 + \beta_1\overline{z}^2 + \gamma_1|z|^2 + \delta_1.$$

As a final result we obtain

3.7. Proposition. *Each quasiregular polynomial mapping of algebraic degree* 2 *is conjugate via a holomorphic affine map to a mapping of the form*

$$Q(z,\overline{z}) = z^2 + a|z|^2 + b\overline{z}^2 + c.$$

The mapping Q is also quasiregular. The quasiregularity is equivalent to the condition

$$1 + |b|^2 > |a|\,|b - a/\overline{a}| \quad \text{if } a \neq 0 \quad \text{or } a = 0 \text{ and } |b| < 1.$$

4. Bibliographical notes. Surprisingly, we have not been able to find any paper in which quasiconformal and quasiregular polynomial mappings were studied.

The quasiregular mappings of polynomial type were studied in [HK] and semigroups of quasiregular mappings on $\mathbb{C}$ in [H].

The dynamics of quasiregular maps $F_{\alpha,c} = |z|^{2\alpha-2}z^2 + c$ was described in [BSTV].

General information on quasiregular mappings can be found in [LV] (for $n = 2$), [Re] and [Ri].

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