COLLOQUIUM MATHEMATICUM

VOL. 75 1998 NO. 1

ON THE WITT RING OF A RELATIVE PROJECTIVE LINE

BY

M. SZYJEWSKI (KATOWICE)

1. Introduction. After Witt the classical algebraic theory of quadratic forms deals with the Witt ring of Witt classes of symmetric bilinear spaces. A *symmetric bilinear space* is a pair (V, β), where V is a finite-dimensional vector space over a field K of characteristic different from 2 and $\beta : V \to V^\wedge$ is a self-dual ($\beta = \beta^\wedge$, or, equivalently, $\beta(u)(v) = \beta(v)(u)$ for arbitrary $u, v \in V$) isomorphism of V with its dual space $V^\wedge$. Factoring out by trivial in some sense (e.g. for the problem of representability of elements of K by the quadratic form $v \mapsto q(v) = \beta(v)(v)$) hyperbolic spaces

$$\left(M \oplus M^\wedge, \begin{bmatrix} 0 & 1_M \\ 1_{M^\wedge} & 0 \end{bmatrix}\right)$$

yields the *Witt ring* $W(K)$ of the field K, consisting of classes of symmetric bilinear forms up to hyperbolic direct summands. Addition in it is induced by the direct sum and multiplication is induced by the tensor product.

This theory has numerous applications in algebraic number theory, theory of algebras, field theory, Galois theory, cohomology theory, algebraic K-theory and algebraic geometry, and conversely. Extensive bibliography may be found in [10].

There are several generalizations obtained by changing the main objects: skew-symmetric bilinear forms, hermitian forms, algebras with involution and Ranicki formations. On the other hand, there is a natural way to generalize the notion of a Witt ring: take a ring in place of a field and finitely generated projective (i.e. locally free) modules in place of vector spaces. For local rings with 2 invertible the theory is similar to the classical one. In general, a difficult theory for fields becomes more difficult for, say, hermitian forms over group rings. The Witt ring of a group ring has significant applications in geometry and topology (e.g. for the group ring $\mathbb{Z}[\pi(X)]$ of the fundamental group in surgery theory).

1991 *Mathematics Subject Classification*: Primary 11E81; Secondary 14C35.

Research supported by KBN (State Committee for Scientific Research) under Grant 2 P301 020 05.

The next step is due to Knebusch [6], [7]: consider schemes in place of (spectra of) rings and vector bundles (locally free coherent sheaves of $\mathcal{O}_X$-modules) in place of projective modules. Let $(X, \mathcal{O}_X)$ be a scheme and let $\mathcal{L}$ be a line bundle over X. From here on we write $\pm$ to indicate two possibilities: "$+$-symmetric" means simply "symmetric", while "$-$-symmetric" should be read as "skew-symmetric".

A *$\pm$-symmetric $\mathcal{L}$-valued space* (V, β) consists of a vector bundle V and an isomorphism $\beta : V \to \mathcal{H}om_{\mathcal{O}_X}(V, \mathcal{L}) = V^{\wedge} \otimes \mathcal{L}$ such that $\beta^{\wedge\mathcal{L}} = (\beta^{\wedge} \otimes 1_{\mathcal{L}}) \circ (1 \otimes \mu^{-1}) = \pm\beta$, where $\mu : \mathcal{L}^{\wedge} \otimes \mathcal{L} \to \mathcal{O}_X$ is the evaluation isomorphism.

For a subbundle $\iota : W \rightarrowtail V$ its *orthogonal complement* $W^{\perp}$ is a subbundle of V defined as $W^{\perp} = \operatorname{Ker}(i^{\wedge\mathcal{L}} \circ \beta)$.

A subbundle W of a bilinear space is said to be *totally isotropic* or *sublagrangian* iff $W \subset W^{\perp}$, and is *lagrangian* if $W = W^{\perp}$. Equivalently, a lagrangian subbundle of a bundle (V, β) is a totally isotropic subbundle of rank equal to half the rank of V.

A bilinear space (V, β) is *metabolic* iff it possesses a lagrangian subbundle, i.e. if there exists an exact sequence

$$0 \to W \xrightarrow{\iota} V \xrightarrow{\iota' \circ \beta} W^{\wedge\mathcal{L}} \to 0$$

of vector bundles, where $\iota' = \iota^{\wedge\mathcal{L}} : \mathcal{H}om_{\mathcal{O}_X}(V, \mathcal{L}) \to \mathcal{H}om_{\mathcal{O}_X}(W, \mathcal{L})$ is the restriction to W.

Two $\pm$-symmetric $\mathcal{L}$-valued bilinear spaces (V, β) and (W, γ) are *Witt equivalent* iff there exist metabolic $\pm$-symmetric $\mathcal{L}$-valued bilinear spaces (M, μ) and (N, ν) such that

$$(V, \beta) \oplus (M, \mu) \cong (W, \gamma) \oplus (N, \nu).$$

The *Witt group* $W^{\pm}(X, \mathcal{L})$ of $\pm$-symmetric $\mathcal{L}$-valued bilinear spaces consists of the classes of Witt equivalence of $\pm$- symmetric $\mathcal{L}$-valued bilinear spaces with direct sum as addition. In the case of the trivial line bundle $\mathcal{L} = \mathcal{O}_X$ and symmetric forms we write

$$W(X) = W^{+}(X, \mathcal{O}_X).$$

The tensor product induces multiplication on $W(X)$, so $W(X)$ is a ring, the *Witt ring* of the scheme X. The Witt ring is a (co)functor: for a morphism $f : X \to Y$ of schemes the inverse image functor f^* induces a ring homomorphism $f^* : W(Y) \to W(X)$.

The theory of quadratic forms over schemes is the theory of the functor W. There are two separate theories, in fact. The global theory relates in general the properties of Witt rings to the geometry of schemes, e.g. for divisorial schemes X,

- there exists a surjective ring homomorphism $W(X) \to \mathbb{Z}$ if and only if there exists a closed point $x \in X$ with formally real local ring $\mathcal{O}_{X,x}$,
- otherwise there exists an integer n such that $2^n \cdot W(X) = 0$.

([7, Chapt. III.2, Theorems 2 and 3].) Moreover, the global theory describes $W(X)$ as an abstract ring and as a $W(S)$-module for various classes of schemes X/S. For example,

- if 2 is an invertible element in a ring R, then $W(\mathbb{A}^n_R) = W(R)$ (Karoubi theorem),
- if 2 is an invertible element in a ring R, then $W(\mathbb{P}^n_R) = W(R) \oplus I$, where $2^{n+1}I^{n+1} = 0$; if in addition R is a regular ring, then $I^{n+1} = 0$ ([7, Chapt. III.7, Theorem 2]),
- if F is a field of characteristic different from 2, then $W(\mathbb{P}^n_F) = W(F)$ ([1, Satz]),
- if X is an elliptic curve over a field F of characteristic different from 2, then $W(X)$ is a $W(F)$-module with generators corresponding to elements of order ≤ 2 in $\mathrm{Pic}(X)$ ([2]),
- if X is a split projective quadric of even dimension ([12]), or a Grassmann variety $\mathrm{Gr}(2, n)$ of planes ([13]) over a field F of characteristic different from 2, then $W(X) = W(F) \oplus I$ with a nonzero ideal I.

The global theory is difficult, and a major part of research is devoted to the local theory. So we give neither a review of known results from the Minkowski–Hasse local-global principle to the newest results, nor a complete list of references. F. Fernández-Carmena paper [3] and Jaworski's recent paper [4] are examples of this branch of the theory. In the local theory various generalizations of the localization sequence

$$0 \to W(X) \to W(K(X)) \to \coprod_x W(k(x))$$

are studied. Such a sequence is exact for regular curves X, and if X is the spectrum of a Dedekind ring. Limitation of local methods consists in the fact that complexes with the Witt ring of a scheme as a member, constructed by means of localization, are exact only for schemes of dimension 1, 2 and possibly 3. Probably localization complexes fit into a spectral sequence—Sansuc and Barge constructed such a spectral sequence for affine schemes, but these results are still unpublished.

Arason proved in 1980 that for a field K with $\operatorname{char} K \neq 2$, the canonical map $W(K) \to W(\mathbb{P}^n_K)$ induced by the structure map $\mathbb{P}^n_K \to \mathrm{Spec}\ K$ is an isomorphism. The proof depends on a result of Horrocks on representing bundles as direct sums of line bundles and on properties of the bundles Ω^r of differential forms. In 1991 M. Ojanguren asked if Arason's theorem may be generalized to the case of a projective space over a ring. Now there are tools to construct an infinite sequence of regular rings R with $\dim R \equiv 6 \pmod 8$ such that the canonical map $W(R) \to W(\mathbb{P}^1_R)$ is not surjective.

The idea consists in studying a group $E^+(X)$, a subfactor of $K_0(X)$, closely related to $W(X)$ and much easier to compute. The group $E^+(X)$ together with the homomorphism $e^0 : W(X) \to E^+(X)$ and its generalizations $E^\pm(X, \mathcal{L})$ are introduced in Section 2. The general Theorem 3.1 in Section 3 describes the E-groups of a projective bundle. This description shows the way to construct an element of $E^+(\mathbb{P}^n_R)$ which is outside the image of the map $E^+(\mathrm{Spec}\ R) \to E^+(\mathbb{P}^n_R)$ and exhibits special properties of (projective modules over) R that provide the construction. Rings with the required property are coordinate rings of affine split quadrics (Section 5), and computations are possible in the framework of Swan's K-theory of quadrics (Section 4). Thus commutativity of the diagram

$$\begin{array}{ccc} W(X) & \xrightarrow{e^0} & E(X) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ W(Y) & \xrightarrow{e^0} & E(Y) \end{array}$$

for every map $f : X \to Y$ shows that it is enough to find a bilinear space $(\mathcal{M}, \beta)$ with a prescribed value of $e^0(\mathcal{M}, \beta)$ to give a negative answer to Ojanguren's question. This is done for the projective line in Section 6 by means of the theory of Ranicki formations developed by W. Pardon for rings [8] and by F. Fernández-Carmena for schemes [3]. A theorem due to Fernández-Carmena provides a construction of a symmetric bilinear form over a scheme for a given formation over a closed subscheme of codimension one.

2. E-groups and the invariant e^0. Any line bundle $\mathcal{L}$ on a scheme X defines an exact involutive contravariant functor ${}^{\wedge\mathcal{L}}$ on the category of vector bundles on X,

$$M \mapsto M^{\wedge\mathcal{L}} = M^\wedge \otimes \mathcal{L}, \quad \varphi^{\wedge\mathcal{L}} = \varphi^\wedge \otimes 1_{\mathcal{L}} \quad \text{for } \varphi : M \to N.$$

This involution induces the analogous involution on the Q-construction. Since its geometric realization interchanges paths $0 \twoheadleftarrow 0 \rightarrowtail A$ and $0 \twoheadleftarrow A \rightarrowtail A$ which form a loop corresponding to the object A, there is an induced involutive automorphism ${}^{\wedge\mathcal{L}}$ of K-groups (homotopy groups of the Q-construction) which acts on $K_0(X)$ as $[M] \mapsto -[M^{\wedge\mathcal{L}}]$. Nevertheless, we define

$$[M]^{\wedge\mathcal{L}} = [M^{\wedge\mathcal{L}}].$$

We are interested in the Tate cohomology of the two-element group $\{1, {}^{\wedge\mathcal{L}}\}$

with values in $K_0(X)$. Denote by $C(X,\mathcal{L})$ the complete resolution

$$C(X,\mathcal{L}):\quad \dots \xrightarrow{1-\wedge^{\mathcal{L}}} K_0(X) \xrightarrow{1+\wedge^{\mathcal{L}}} K_0(X) \xrightarrow{1-\wedge^{\mathcal{L}}} K_0(X) \xrightarrow{1-\wedge^{\mathcal{L}}} \dots \tag{2.1}$$

Definition 2.1.

$$E^+(X,\mathcal{L}) = \operatorname{Ker}(1-\wedge^{\mathcal{L}})/\operatorname{Im}(1+\wedge^{\mathcal{L}}),$$
$$E^-(X,\mathcal{L}) = \operatorname{Ker}(1+\wedge^{\mathcal{L}})/\operatorname{Im}(1-\wedge^{\mathcal{L}}).$$

We will refer to *E-groups* meaning the collection of $E^+(X,\mathcal{L})$ and $E^-(X,\mathcal{L})$ for all line bundles $\mathcal{L}$. Types of E-groups of a scheme X correspond to elements of the factor group $\operatorname{Pic}(X)/2\operatorname{Pic}(X)$.

Proposition 2.1. *For every line bundle $\mathcal{K}$ there are isomorphisms*

$$E^+(X,\mathcal{L}\otimes\mathcal{K}^{\otimes 2}) \cong E^+(X,\mathcal{L}),$$
$$E^-(X,\mathcal{L}\otimes\mathcal{K}^{\otimes 2}) \cong E^-(X,\mathcal{L}).$$

Proof. Tensoring with $\mathcal{K}$ induces an isomorphism of complexes $C(X,\mathcal{L}) \xrightarrow{[\mathcal{K}]\cdot} C(X,\mathcal{L}\otimes\mathcal{K}^{\otimes 2})$:

$$\begin{tikzcd} \cdots \arrow[r, "1-\alpha"] & K_0(X) \arrow[r, "1+\alpha"] & K_0(X) \arrow[r, "1-\alpha"] & \cdots \\ \cdots \arrow[r, "1-\beta"] & K_0(X) \arrow[r, "1+\beta"] \arrow[u, "{[\mathcal{K}]\cdot}"] & K_0(X) \arrow[r, "1-\beta"] \arrow[u, "{[\mathcal{K}]\cdot}"] & \cdots \end{tikzcd}$$

where $\alpha(\mathcal{P}) = \mathcal{L}\otimes\mathcal{K}^{\otimes 2}\otimes\mathcal{P}^\wedge$ and $\beta(\mathcal{P}) = \mathcal{L}\otimes\mathcal{P}^\wedge$. ∎

Definition 2.2. The forgetful functor induces a group homomorphism

$$e^0_{\mathcal{L}} : W^+(X,\mathcal{L})\oplus W^-(X,\mathcal{L}) \to E^+(X,\mathcal{L}),$$
$$e^0_{\mathcal{L}}(\mathcal{P},\beta) = [\mathcal{P}] \pmod{\operatorname{Im}(1+\wedge^{\mathcal{L}})}.$$

The inverse image functor f^* for a morphism $f: Y\to X$ of schemes induces a homomorphism $f^*: E(X,\mathcal{L}) \to E(Y,f^*\mathcal{L})$. As an example we prove the homotopy property of E-groups.

Proposition 2.2 (Homotopy property). *If $f: X\to Y$ is a flat morphism of regular noetherian separated schemes whose fibres are affine spaces, then*

$$E^+(Y,\mathcal{L}) \cong E^+(X,f^*\mathcal{L}) \quad \textit{and} \quad E^-(Y,\mathcal{L}) \cong E^-(X,f^*\mathcal{L}).$$

Proof. By the homotopy property of K-groups the map $f^*: K_0(Y) \to K_0(X)$ induced by the inverse image functor f^* provides an isomorphism of complexes

$$\begin{tikzcd}
\cdots \arrow[r, "1-\alpha"] & K_0(X) \arrow[r, "1+\alpha"] & K_0(X) \arrow[r, "1-\alpha"] & \cdots \\
\cdots \arrow[r, "1-\beta"] & K_0(Y) \arrow[u, "f^*"] \arrow[r, "1+\beta"] & K_0(Y) \arrow[u, "f^*"] \arrow[r, "1-\beta"] & \cdots
\end{tikzcd}$$

where $\alpha = {}^{\wedge f^*\mathcal{L}}$ and $\beta = {}^{\wedge\mathcal{L}}$. ∎

3. E-groups of a projective bundle

Theorem 3.1 (Projective bundle theorem). *Let $\mathcal{E}$ be a vector bundle on a scheme S,* rank $\mathcal{E} = n$, *and $X = \mathbb{P}(\mathcal{E}) = \operatorname{Proj}(S(\mathcal{E}^\wedge))$ be the associated projective bundle. Let $\mathcal{O}_X(-1)$ be the tautological line bundle on X and $f : X \to S$ the structure map. Let $\mathcal{L}$ be an arbitrary line bundle on S.*

(i) *If $n = 2k+2$ is even, then there is an exact hexagon*

$$\begin{tikzcd}
& E^+(X, f^*\mathcal{L}) \arrow[dr] & \\
E^+(S,\mathcal{L}) \arrow[ur, "f^*"] & & E^-(S, \bigwedge^n \mathcal{E}^\wedge \otimes \mathcal{L}) \arrow[d, "{[\bigwedge^{k+1}\mathcal{E}]\cdot}"] \\
E^+(S, \bigwedge^n \mathcal{E}^\wedge \otimes \mathcal{L}) \arrow[u, "{[\bigwedge^{k+1}\mathcal{E}]\cdot}"] & & E^-(S,\mathcal{L}) \arrow[dl, "f^*"] \\
& E^+(X, f^*\mathcal{L}) \arrow[ul] &
\end{tikzcd}$$

and $E^\pm(X, f^*\mathcal{L} \otimes \mathcal{O}_X(-1)) = 0$.

(ii) *If $n = 2k+1$ is odd, then $E^\pm(X, f^*\mathcal{L}) \cong E^\pm(S,\mathcal{L})$, and*

$$E^\pm(X, f^*\mathcal{L} \otimes \mathcal{O}_X(-1)) \cong E^\pm(S, \mathcal{L} \otimes \textstyle\bigwedge^n \mathcal{E}^\wedge).$$

Proof. We have

$$K_0(X) \cong (K_0(S))[t]\Big/\Big(\sum_{i=0}^{n} (-1)^i [\textstyle\bigwedge^i \mathcal{E}] t^i\Big)$$

where t corresponds to $\xi = \mathcal{O}_X(-1)$ (see [9], Sect. 8, 1.5) and $\xi^\wedge = \xi^{-1}$. Now $K_0(X)$ is a free $K_0(S)$-module with a free base $\mathcal{O}_X(i) = \xi^{-i}$ for $i = [n/2]-1, \ldots, [n/2]-n$, where [ ] means integer part. There is an identity

$$\sum_{i=-k}^{n-k} (-1)^{i+k} [f^* \textstyle\bigwedge^{i+k} \mathcal{E}] \xi^i = 0$$

in $K_0(X)$ for every integer k. We shall alter the base of $K_0(X)$ to obtain a triangular matrix of the involution under consideration.

• In the case of even $n = 2k+2$ and forms with values in the line bundle $f^*\mathcal{L} \otimes \mathcal{O}_X(-1)$, the initial base ξ^i, $i = -k, 1-k, \dots, -1, 0, 1, \dots, k, k+1$, may be transformed into $\xi^{i+1} + \xi^{-i}$, ξ^{-i} for $i = 0, 1, \dots, k$. If A denotes the span (with coefficients in $K_0(S)$) of all $\xi^{i+1} + \xi^{-i}$ for $i = 0, 1, \dots, k$, then the exact sequence

$$0 \to A \xrightarrow{\epsilon} K_0(X) \xrightarrow{\kappa} K_0(X)/A \to 0,$$

in view of the formulas

$$(f^*(\alpha)(\xi^{i+1} + \xi^{-i}))^\wedge \cdot [f^*\mathcal{L}] \cdot \xi = f^*(\alpha^\wedge \cdot [\mathcal{L}])(\xi^{i+1} + \xi^{-i}),$$
$$(f^*(\alpha)\xi^{-i})^\wedge \cdot [f^*\mathcal{L}] \cdot \xi = f^*(\alpha^\wedge \cdot [\mathcal{L}])(\xi^{i+1} + \xi^{-i}) - f^*(\alpha^\wedge \cdot [\mathcal{L}])\xi^{-i},$$

for all $\alpha \in K_0(S)$, yields an exact Tate cohomology sequence

$$\begin{aligned} \dots \xrightarrow{\wedge\mathcal{L}} E^+(S,\mathcal{L})^{k+1} \xrightarrow{\varepsilon} E^+(X, f^*\mathcal{L} \otimes \mathcal{O}_X(-1)) \xrightarrow{\kappa} E^-(S,\mathcal{L})^{k+1} \\ \xrightarrow{\wedge\mathcal{L}} E^-(S,\mathcal{L})^{k+1} \xrightarrow{\varepsilon} E^-(X, f^*\mathcal{L} \otimes \mathcal{O}_X(-1)) \\ \xrightarrow{\kappa} E^+(S,\mathcal{L})^{k+1} \xrightarrow{\wedge\mathcal{L}} E^+(S,\mathcal{L})^{k+1} \xrightarrow{\varepsilon} \dots \end{aligned}$$

The connecting homomorphisms are induced by the involution ${}^{\wedge\mathcal{L}}$ acting componentwise, and are isomorphisms. Hence $E^{\pm}(X, f^*\mathcal{L} \otimes \mathcal{O}_X(-1)) = 0$.

• In the case of odd $n = 2k+1$ and a line bundle of the form $f^*\mathcal{L}$, the elements $1 = \xi^0, \xi^i + \xi^{-i}, \xi^i$ for $i = 1, \dots, k$ form another base of $K_0(X)$. If we let

$A :=$ span (with coefficients in $K_0(S)$) of $\xi^i + \xi^{-i}$ for $i = 1, \dots, k$,

$B :=$ span of ξ^i for $i = 1, \dots, k$,

$C :=$ span of 1,

then $K_0(X) = A \oplus B \oplus C$, and for any α in $K_0(S)$ we have the formulas

$$(f^*(\alpha)(\xi^i + \xi^{-i}))^\wedge \cdot [f^*\mathcal{L}] = (f^*(\alpha^\wedge \cdot [\mathcal{L}])(\xi^i + \xi^{-i}),$$
$$(f^*(\alpha)\xi^i)^\wedge \cdot [f^*\mathcal{L}] = -f^*(\alpha^\wedge \cdot [\mathcal{L}])\xi^i + f^*(\alpha^\wedge \cdot [\mathcal{L}])(\xi^i + \xi^{-i}).$$

Therefore regarding $A \subset A \oplus B \subset A \oplus B \oplus C$ as a filtration of the complex $C(X,\mathcal{L})$ shows that f^* induces an isomorphism on Tate cohomology: $E^{\pm}(X, f^*\mathcal{L}) \cong E^{\pm}(S,\mathcal{L})$.

• In the case of odd $n = 2k+1$ and a line bundle of the form $f^*\mathcal{L} \otimes \mathcal{O}_X(-1)$, the elements $\xi^{-k}, \xi^i + \xi^{1-i}, \xi^i$ for $i = 1, \dots, k$ form another base of $K_0(X)$. If we let

$A :=$ span (with coefficients in $K_0(S)$) of $\xi^i + \xi^{1-i}$ for $i = 1, \dots, k$,

$B :=$ span of ξ^i for $i = 1, \dots, k$,

$C :=$ span of ξ^{-k},

then $K_0(X) = A \oplus B \oplus C$, and for any α in $K_0(S)$ we have the formulas

$$(f^*(\alpha)(\xi^i + \xi^{1-i}))^\wedge \cdot [f^*\mathcal{L}]\xi = (f^*(\alpha^\wedge \cdot [\mathcal{L}])(\xi^i + \xi^{1-i}),$$

$$(f^*(\alpha)\xi^i)^\wedge \cdot [f^*\mathcal{L}]\xi = -f^*(\alpha^\wedge \cdot [\mathcal{L}])\xi^i + f^*(\alpha^\wedge \cdot [\mathcal{L}])(\xi^i + \xi^{1-i}),$$

$$\begin{aligned}
&(f^*(\alpha)\xi^{-k})^\wedge \cdot [f^*\mathcal{L}] \cdot \xi = f^*(\alpha^\wedge \cdot [\mathcal{L}])\xi^{k+1}\\
&\quad = f^*(a^\wedge \cdot [\mathcal{L}]) \sum_{i=-k}^{k} (-1)^{i+k} [f^* \textstyle\bigwedge^{k+1-i} \mathcal{E}^\wedge] \cdot \xi^i\\
&\quad = f^*(a^\wedge \cdot [\mathcal{L}])\Big([f^* \textstyle\bigwedge^n \mathcal{E}^\wedge] \cdot \xi^{-k} + \displaystyle\sum_{i=1-k}^{k} (-1)^{i+k} [f^* \textstyle\bigwedge^{k+1-i} \mathcal{E}^\wedge] \cdot \xi^i\Big)\\
&\quad = f^*(a^\wedge \cdot [\mathcal{L}])\Big([f^* \textstyle\bigwedge^n \mathcal{E}^\wedge] \cdot \xi^{-k} + \displaystyle\sum_{i=1-k}^{0} (-1)^{i+k} [f^* \textstyle\bigwedge^{k+1-i} \mathcal{E}^\wedge] \cdot \xi^i\Big)\\
&\qquad + f^*(a^\wedge \cdot [\mathcal{L}])\Big(\sum_{i=1}^{k} (-1)^{i+k} [f^* \textstyle\bigwedge^{k+1-i} \mathcal{E}^\wedge] \cdot \xi^i\Big)\\
&\quad = f^*(a^\wedge \cdot [\mathcal{L}])\Big([f^* \textstyle\bigwedge^n \mathcal{E}^\wedge] \cdot \xi^{-k} + \displaystyle\sum_{i=0}^{k-1} (-1)^{k-i} [f^* \textstyle\bigwedge^{k+1+i} \mathcal{E}^\wedge] \cdot \xi^{-i}\Big)\\
&\qquad + f^*(a^\wedge \cdot [\mathcal{L}])\Big(\sum_{i=1}^{k} (-1)^{i+k} [f^* \textstyle\bigwedge^{k+1-i} \mathcal{E}^\wedge] \cdot \xi^i\Big)\\
&\quad = f^*(a^\wedge \cdot [\mathcal{L}])\Big([f^* \textstyle\bigwedge^n \mathcal{E}^\wedge] \cdot \xi^{-k} + \displaystyle\sum_{i=1}^{k} (-1)^{k-i-1} [f^* \textstyle\bigwedge^{k+i} \mathcal{E}^\wedge] \cdot (\xi^{1-i} + \xi^i)\Big)\\
&\qquad + f^*(a^\wedge \cdot [\mathcal{L}])\Big(\sum_{i=1}^{k} (-1)^{i+k} ([f^* \textstyle\bigwedge^{k+1-i} \mathcal{E}^\wedge] + [f^* \textstyle\bigwedge^{k+i} \mathcal{E}^\wedge]) \cdot \xi^i\Big).
\end{aligned}$$

Thus in the E_2 part of the spectral sequence associated with the filtration $A \subset A \oplus B \subset A \oplus B \oplus C = K_0(X)$ of the complex $C(X, f^*\mathcal{L} \otimes \mathcal{O}_X(-1))$ the differentials $d : \mathrm{E}_2^{p,1} = (E^{(-1)^p}(S,\mathcal{L}))^k \to (E^{(-1)^p}(S,\mathcal{L}))^k = \mathrm{E}_2^{p+2,0}$ are isomorphisms induced by ${}^{\wedge\mathcal{L}}$. Therefore $E^{(-1)^p}(X, f^*\mathcal{L} \otimes \mathcal{O}_X(-1)) = \mathrm{E}_2^{p-2,2} \cong E^{(-1)^p}(S, \mathcal{L} \otimes \bigwedge^n \mathcal{E}^\wedge)$.

• In the most complicated case $n = 2k + 2$ and a line bundle of the type $f^*\mathcal{L}$, the initial base ξ^i, $i = -k, \ldots, k + 1$, should be replaced by $1, \xi^{k+1}, \xi^i + \xi^{-i}, \xi^i$ for $i = 1, \ldots, k$.

The formulas

$$(f^*\alpha(\xi^i + \xi^{-i}))^\wedge \cdot [f^*\mathcal{L}] = (f^*\alpha^{\wedge\mathcal{L}})(\xi^i + \xi^{-i}),$$

$$(f^*\alpha\xi^i)^\wedge \cdot [f^*\mathcal{L}] = (f^*\alpha^{\wedge\mathcal{L}})\xi^{-i} = -(f^*\alpha^{\wedge\mathcal{L}})\xi^i + (f^*\alpha^{\wedge\mathcal{L}})(\xi^i + \xi^{-i}),$$

$$
\begin{aligned}
&(f^*\alpha\xi^{k+1})^\wedge \cdot [f^*\mathcal{L}]\\
&\quad = (f^*\alpha^{\wedge\mathcal{L}})\xi^{-k-1} = f^*\alpha^{\wedge\mathcal{L}} \sum_{i=-k}^{k+1} (-1)^{k+i}[f^* \textstyle\bigwedge^{k+1+i} \mathcal{E}] \cdot \xi^i\\
&\quad = f^*\alpha^{\wedge\mathcal{L}}\Big(- [f^* \textstyle\bigwedge^n \mathcal{E}] \cdot \xi^{k+1} + \displaystyle\sum_{i=1}^{k}(-1)^{k-i}[f^* \textstyle\bigwedge^{k+1-i} \mathcal{E}] \cdot \xi^{-i}\Big)\\
&\qquad + f^*\alpha^{\wedge\mathcal{L}}\Big(\sum_{i=1}^{k}(-1)^{k+i}[f^* \textstyle\bigwedge^{k+1+i} \mathcal{E}] \cdot \xi^i + (-1)^k[f^* \textstyle\bigwedge^{k+1} \mathcal{E}]\Big)\\
&\quad = \; - f^*\alpha^{\wedge\mathcal{L}}[f^* \textstyle\bigwedge^n \mathcal{E}] \cdot \xi^{k+1}\\
&\qquad + f^*\alpha^{\wedge\mathcal{L}} \displaystyle\sum_{i=1}^{k}(-1)^{k-i}[f^* \textstyle\bigwedge^{k+1-i} \mathcal{E}] \cdot (\xi^i + \xi^{-i})\\
&\qquad + f^*\alpha^{\wedge\mathcal{L}} \displaystyle\sum_{i=1}^{k}(-1)^{k+i}([f^* \textstyle\bigwedge^{k+1+i} \mathcal{E}] - [f^* \textstyle\bigwedge^{k+1-i} \mathcal{E}]) \cdot \xi^i\\
&\qquad + (-1)^k f^*\alpha^{\wedge\mathcal{L}}[f^* \textstyle\bigwedge^{k+1} \mathcal{E}]
\end{aligned}
$$

allow us to define a filtration $A \subset A \oplus B \subset A \oplus B \oplus C \subset A \oplus B \oplus C \oplus D = K_0(X)$ of the complex $C(X, f^*\mathcal{L})$ where

$A :=$ span of $\xi^i + \xi^{-i}$ for $i = 1, \dots, k$,
$B :=$ span of ξ^i for $i = 1, \dots, k$,
$C := f^*K_0(S) \cdot 1$,
$D := f^*K_0(S) \cdot \xi^{k+1}$.

In the E_2-term of the associated spectral sequence the differentials $d_2^{\cdot,1} : \mathrm{E}_2^{\cdot,1} = (E^\pm(S, \mathcal{L}))^k \to (E^\pm(S, \mathcal{L}))^k = \mathrm{E}_2^{\cdot+2,0}$ are the isomorphisms induced by ${}^{\wedge\mathcal{L}}$, while the differentials $d_2^{\cdot,3} : \mathrm{E}_2^{\cdot,3} = E^\pm(S, \mathcal{L} \otimes \bigwedge^n \mathcal{E}) \to E^\pm(S, \mathcal{L}) = \mathrm{E}_2^{\cdot+2,2}$ are induced by multiplication by $[\bigwedge^{k+1} \mathcal{E}]$, hence the theorem follows. ■

If the bundle $\mathcal{E}$ in the theorem is trivial, then $[\bigwedge^{k+1} \mathcal{E}]$ is in the image of $1 + {}^{\wedge\mathcal{L}}$, so the maps $[\bigwedge^{k+1} \mathcal{E}]\cdot$ in the hexagon of the theorem are zero maps. In this case a more detailed description of the E-groups of a projective space may be given.

Proposition 3.2. *For every scheme S let $X = \mathbb{P}^d_S$ and let $p_1 : X \to \mathbb{P}^d$ and $p_2 : X \to S$ be the projections. Then for every line bundle $\mathcal{L}$ on S and every line bundle $\mathcal{M}$ on $\mathbb{P}^d$,*

$$E^{+}(X,\mathcal{M}\boxtimes\mathcal{L}) = E^{+}(\mathbb{P}^d,\mathcal{M})\boxtimes E^{+}(S,\mathcal{L}) \oplus E^{-}(\mathbb{P}^d,\mathcal{M})\boxtimes E^{-}(S,\mathcal{L}),$$
$$E^{-}(X,\mathcal{M}\boxtimes\mathcal{L}) = E^{+}(\mathbb{P}^d,\mathcal{M})\boxtimes E^{-}(S,\mathcal{L}) \oplus E^{-}(\mathbb{P}^d,\mathcal{M})\boxtimes E^{+}(S,\mathcal{L}),$$

where $\boxtimes$ *is induced by the operation* $\mathcal{F}\boxtimes\mathcal{G} = p_1^*(\mathcal{F})\otimes p_2^*(\mathcal{G})$.

P r o o f. By the projective bundle theorem for K-theory the maps p_1^*, p_2^* yield an identification $K_0(X) = K_0(\mathbb{P}^d)\otimes K_0(S)$. Define

$$\begin{aligned} A &= \operatorname{Ker}(K_0(\mathbb{P}^d) \stackrel{1-{}^{\wedge\mathcal{M}}}{\longrightarrow} K_0(\mathbb{P}^d)),\\ B &= (1-{}^{\wedge\mathcal{M}})K_0(\mathbb{P}^d),\\ \mathcal{K} &= p_1^*\mathcal{M}\otimes p_2^*\mathcal{L}. \end{aligned}$$

The complex (2.1)

$$\ldots \to K_0(X) \stackrel{1+{}^{\wedge\mathcal{K}}}{\to} K_0(X) \stackrel{1-{}^{\wedge\mathcal{K}}}{\to} K_0(X) \stackrel{1+{}^{\wedge\mathcal{K}}}{\longrightarrow} K_0(X) \to \ldots$$

for $X = \mathbb{P}^d\times S$ fits into the short exact sequence of complexes

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
A\otimes K_0(S) \arrow[r, tail] \arrow[u] & K_0(X) \arrow[r, two heads, "(1-{}^{\wedge\mathcal{M}})\otimes 1"] \arrow[u] & B\otimes K_0(S) \arrow[u, "1-{}^{\wedge\mathcal{K}}"'] \\
A\otimes K_0(S) \arrow[r, tail] \arrow[u, "1-{}^{\wedge\mathcal{K}}"] & K_0(X) \arrow[r, two heads, "(1-{}^{\wedge\mathcal{M}})\otimes 1"] \arrow[u, "1-{}^{\wedge\mathcal{K}}"] & B\otimes K_0(S) \arrow[u, "1-{}^{\wedge\mathcal{K}}"'] \\
A\otimes K_0(S) \arrow[r, tail] \arrow[u, "1-{}^{\wedge\mathcal{K}}"] & K_0(X) \arrow[r, two heads, "(1-{}^{\wedge\mathcal{M}})\otimes 1"] \arrow[u, "1-{}^{\wedge\mathcal{K}}"] & B\otimes K_0(S) \arrow[u, "1-{}^{\wedge\mathcal{K}}"'] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

Note that $1\pm{}^{\wedge\mathcal{K}}$ restricted to $A\otimes K_0(S)$ coincides with $1\otimes(1\pm{}^{\wedge\mathcal{L}})$ and induces $1\otimes(1\mp{}^{\wedge\mathcal{L}})$ on $B\otimes K_0(S)$. Therefore the exact hexagon in homology breaks into short split exact sequences

$$\begin{aligned} 0 \to E^{+}(\mathbb{P}^d,\mathcal{M})\otimes E^{-}(S,\mathcal{L}) \to E^{-}(X,\mathcal{K})&\\ \to E^{-}(\mathbb{P}^d,\mathcal{M})\otimes E^{+}(S,\mathcal{L}) \to 0&,\\ 0 \to E^{+}(\mathbb{P}^d,\mathcal{M})\otimes E^{+}(S,\mathcal{L}) \to E^{+}(X,\mathcal{K})&\\ \to E^{-}(\mathbb{P}^d,\mathcal{M})\otimes E^{-}(S,\mathcal{L}) \to 0&. \ \blacksquare \end{aligned}$$

To identify explicit generators of groups under consideration, for the absolute projective space $Y = \mathbb{P}^d$, we define

$$
(3.1)\qquad \begin{aligned} 1 &= [\mathcal{O}_Y], \text{ the unit element in } K_0(Y),\\ H &= 1-[\mathcal{O}_Y(-1)], \text{ the class of a hyperplane section in } K_0(Y). \end{aligned}
$$

We summarize some technicalities as follows:

LEMMA 3.3. *If* $Y=\mathbb{P}^d$, *then*

(i) $H^{d+1}=0$;

(ii) $[\mathcal{O}_Y(1)]=(1-H)^{-1}=\sum_{i=0}^{d} H^i$ *in* $K_0(Y)$ (*here* $H^0=1$);

(iii) $H^\wedge=\dfrac{-H}{1-H}=-\sum_{i=1}^{d} H^i$;

(iv) $(H^k)^\wedge=\left(\dfrac{-H}{1-H}\right)^k=(-1)^kH^k\sum_{i=0}^{d-k}\binom{k+i-1}{i}H^i$;

(v) $(H^d)^\wedge=(-1)^dH^d$ *is the class of a rational point.*

Proof. $H=1-[\mathcal{O}_Y(-1)]$, so $[\mathcal{O}_Y(-1)]=1-H$, $[\mathcal{O}_Y(1)]=(1-H)^{-1}$, H being nilpotent. Thus $H^\wedge=1-[\mathcal{O}_Y(1)]=([\mathcal{O}_Y(-1)]-1)[\mathcal{O}_Y(1)]=-H(1-H)^{-1}$ and $(H^k)^\wedge=(1-H)^{-k}(-H)^k$. ∎

COROLLARY 3.4. *If* $Y=\mathbb{P}^d$, *the projective space*, *then*

$$
\begin{aligned}
E^+(Y)&=E^+(Y,\mathcal{O}_Y)=\mathbb{Z}/2\mathbb{Z}[\mathcal{O}_Y],\\
E^-(Y)&=E^-(Y,\mathcal{O}_Y)=\begin{cases}0 & \text{for even } d,\\ \mathbb{Z}/2\mathbb{Z}[H^d] & \text{for odd } d,\end{cases}\\
E^+(Y,\mathcal{O}_Y(-1))&=\begin{cases}\mathbb{Z}/2\mathbb{Z}[H^d] & \text{for even } d,\\ 0 & \text{for odd } d,\end{cases}\\
E^-(Y,\mathcal{O}_Y(-1))&=0.
\end{aligned}
$$

4. Swan K-theory of projective quadrics. To compute the E-groups of affine quadrics we need some facts on dualization of vector bundles on projective quadrics. All the information needed is known, since indecomposable components of the Swan sheaf correspond to spinor representations. Nevertheless, we give here complete proofs of the facts needed. We shall apply results of [11] in the simplest possible case of a split quadric: X is a projective quadric hypersurface over a field F, $\operatorname{char} F\neq 2$, defined by the quadratic form of maximal index. Consider a vector space V with base $v_0,v_1,\ldots,v_{d+1}$ over a field F with $\operatorname{char} F\neq 2$. Let $z_0,z_1,\ldots,z_{d+1}$ be the dual base of $V^\wedge$ and let q be the quadratic form

$$
q=\sum_{i=0}^{d+1}(-1)^i z_i^2 .
$$

Moreover, let $e_i = \frac{1}{2}(v_{2i} - v_{2i+1})$ and $f_i = \frac{1}{2}(v_{2i} + v_{2i+1})$ for all possible values of i. Thus if d is even, $d = 2m$, then $e_0, f_0, e_1, f_1, \ldots, e_m, f_m$ form a base of V with the dual base $x_0, y_0, x_1, y_1, \ldots, x_m, y_m$ and

$$q = \sum_{i=0}^{m} x_i y_i.$$

If d is odd, $d = 2m+1$, then $e_0, f_0, e_1, f_1, \ldots, e_m, f_m, v_{d+1}$ form a base of V with the dual base $x_0, y_0, x_1, y_1, \ldots, x_m, y_m, z_{d+1}$ and

$$q = \sum_{i=0}^{m} x_i y_i + z_{d+1}^2.$$

We shall prove several properties of the dualization functor on the category of vector bundles on the d-dimensional projective quadric X defined by the equation $q = 0$ in $\mathbb{P}_F^{d+1}$, i.e. for

$$X = \operatorname{Proj} \mathrm{S}(V^\wedge)/(q) \cong \operatorname{Proj} F[z_0, z_1, \ldots, z_{d+1}]/(q),$$

to compute the E-groups of an affine part of this quadric in the next section.

In the case of odd $d = 2m+1$ the even part $C_0 = C_0(q)$ of the Clifford algebra $C(q)$ is isomorphic to the matrix algebra $M_{2N}(F)$, where $N = 2^m$. In particular, $K_p(C_0) \cong K_p(F)$. In the case of even $d = 2m$, the algebra C_0 has the centre $F \oplus F\delta$, where $d = v_0 \cdot v_1 \cdot \ldots \cdot v_{d+1}$ and $\delta^2 = 1$. Thus $\frac{1}{2}(1+\delta)$ and $\frac{1}{2}(1-\delta)$ are orthogonal central idempotents of C_0, so

$$C_0 = \tfrac{1}{2}(1+\delta)C_0 \oplus \tfrac{1}{2}(1-\delta)C_0,$$

where each direct summand is isomorphic to the matrix algebra $M_N(F)$. In fact, in this case there exists an isomorphism $C(q) \cong M_{2N}(F)$ of algebras which identifies C_0 with the subalgebra of block-diagonal matrices $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ and maps $\frac{1}{2}(1+\delta)C_0$ onto the set of matrices of the form $\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$, and $\frac{1}{2}(1-\delta)C_0$ onto the set of matrices of the form $\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}$. This observation provides some motivation for what follows. Such a matrix representation of a Clifford algebra may be found in [11], Lemma 4.3. A more "classical" construction, based upon minimal orthogonal idempotents, may be easily deduced from the proof of Proposition 5.6 below.

For even $d = 2m$ consider the principal antiautomorphism $\Im : C_0 \to C_0$ given by

$$\Im(w_1 \cdot w_2 \cdot \ldots \cdot w_k) = (-1)^k w_k \cdot w_{k-1} \cdot \ldots \cdot w_1$$

for $w_1, w_2, \ldots, w_k \in V$. Note that

$$\Im(\delta) = (-1)^{m+1}\delta.$$

Moreover, for every anisotropic vector $w \in V$ the reflection $\alpha \mapsto -w\alpha w^{-1}$ in V induces an automorphism ϱ_w of C_0, which interchanges the δ with its opposite for even d:

$$\varrho_w(\delta) = (-1)^{d-1}\delta.$$

Regarding subscripts i mod 2 define

$$P_i = (1 + (-1)^i\delta)C_0 \qquad \text{for even } d.$$

LEMMA 4.1. *For even $d = 2m$,*

(i) *the involution $\Im$ of the algebra C_0 provides an identification of the left C_0-module $P_i^\wedge = \mathrm{Hom}_F(P_i, F)$ with the right C_0-module P_{i+m+1};*

(ii) *for any anisotropic vector $w \in V$ the reflection ϱ_w interchanges P_i's: $\varrho_w(P_i) = P_{i+1}$.*

Note that as *left* C_0-modules P_0 and P_1 are *not* isomorphic.

Recall the basic facts and notation of [11]. Denote by C_1 the odd part of the Clifford algebra $C(q)$. We shall use mod2 subscripts in C_i. Recall the definition of the Swan bundle $\mathcal{U}$. Put $\varphi = \sum_{i=0}^{d+1} z_i v_i \in \Gamma(X, \mathcal{O}_X \otimes V)$. The complex

$$\ldots \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n) \otimes C_{n+d+1} \xrightarrow{\varphi\cdot} \mathcal{O}_X(1-n) \otimes C_{n+d} \xrightarrow{\varphi\cdot} \mathcal{O}_X(2-n) \otimes C_{n+d-1} \xrightarrow{\varphi\cdot} \ldots \tag{4.1}$$

is exact and locally splits ([11], Prop. 8.2(a)).

DEFINITION 4.1.

$$\mathcal{U}_n = \mathrm{Coker}(\mathcal{O}_X(-n-2) \otimes C_{n+d+3} \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n-1) \otimes C_{n+d+2}),$$
$$\mathcal{U} = \mathcal{U}_{d-1}.$$

Since the complex (4.1) is (up to twist) periodic with period two, it follows that

$$\mathcal{U}_{n+2} = \mathcal{U}_n(-2).$$

Consider the exact sequences $\mathcal{O}_X(-n-2) \otimes C_{n+d+3} \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n-1) \otimes C_{n+d+2} \to \mathcal{U}_n \to 0$ for two consecutive values of n; twist the first one by 1. For any anisotropic vector $w \in V$ the isomorphism given by right multiplication by $1 \otimes w$ fits into the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X(a) \otimes C_{n+d+4} \arrow[r, "\varphi\cdot"] \arrow[d, "\cdot 1\otimes w"] & \mathcal{O}_X(a+1) \otimes C_{n+d+3} \arrow[r] \arrow[d, "\cdot 1\otimes w"] & \mathcal{U}_{n+1}(1) \arrow[r] & 0 \\
\mathcal{O}_X(a) \otimes C_{n+d+3} \arrow[r, "\varphi\cdot"] & \mathcal{O}_X(a+1) \otimes C_{n+d+2} \arrow[r] & \mathcal{U}_n \arrow[r] & 0
\end{tikzcd}$$

where $a = -n-2$. Thus we have proved the following lemma:

LEMMA 4.2. *$\mathcal{U}_{n+1} \cong \mathcal{U}_n(-1)$ and $\mathcal{U}_n \cong \mathcal{U}_0(-n)$ for every integer n.*

There is an exact sequence

$$0 \to \mathcal{U}_0 \xrightarrow{\varphi\cdot} \mathcal{O}_X \otimes C_0 \to \mathcal{U}_{-1} \to 0 \tag{4.2}$$

where the isomorphism $\cdot 1 \otimes w$ was used to replace $\mathcal{O}_X \otimes C_1$ by $\mathcal{O}_X \otimes C_0$ for even d.

LEMMA 4.3. $\mathrm{End}_X(\mathcal{U}_n) \cong C_0$ *acts on* $\mathcal{U}_n$ *from the right.*

Proof. [11], Lemma 8.7. ∎

The main Theorem 9.1 of [11] states that for every regular ring R and every generalized Azumaya algebra A over R and for each projective quadric X of dimension d over R, defined by a nonsingular quadratic form q, the family of functors

$$u_i(M) = M \otimes \mathcal{O}_X(-i) \quad \text{for } i = 0, 1, \ldots, d-1,$$
$$u(M) = \mathcal{U} \otimes_{C_0(q)} M$$

defines an isomorphism

$$(u_0, u_1, \ldots, u_{d-1}, u) : K_*(A)^d \oplus K_*(A \otimes C_0(q)) \to K_*(X).$$

An important argument is that for a large enough class of sheaves (namely regular sheaves) $\mathcal{F}$ there exists a truncated canonical resolution

$$TCan.(\mathcal{F}) : 0 \to \mathcal{U} \otimes_{C_0(q)} T(\mathcal{F}) \to \mathcal{O}_X(1-d) \otimes T_{d-1}(\mathcal{F}) \to \ldots$$
$$\ldots \to \mathcal{O}_X \otimes T_0(\mathcal{F}) \to \mathcal{F} \to 0$$

([11], Section 6). In the special case when $R = A$ is a field F with char $F \neq 2$, $T_i(\mathcal{F})$ are vector spaces over F, so $\mathcal{O}_X(-i) \otimes T_i(\mathcal{F})$ is a direct sum of $\dim T_i(\mathcal{F})$ copies of $\mathcal{O}_X(-i)$. Thus in $K_0(X)$ we have the equality

$$[\mathcal{F}] = \dim T_0(\mathcal{F})[\mathcal{O}_X] - \dim T_1(\mathcal{F})[\mathcal{O}_X(-1)] + \ldots$$
$$+ (-1)^{d-1} \dim T_{d-1}(\mathcal{F})[\mathcal{O}_X(1-d)] + (-1)^d[\mathcal{U} \otimes_{C_0(q)} T(\mathcal{F})].$$

We are now ready to compute $\mathcal{U}_n^\wedge$.

LEMMA 4.4. $\mathcal{U}_n^\wedge \cong \mathcal{U}_n(2n+1)$, *in particular* $\mathcal{U}^\wedge \cong \mathcal{U}(2d-1)$.

Proof. We have chosen a base $v_0, v_1, \ldots, v_{d+1}$ of V above. The set of naturally ordered products of an even number of v_i's forms a base of C_0. Define a quadratic form Q on C_0 as follows: let distinct base products be orthogonal to each other and

$$Q(v_{i_1} \cdot \ldots \cdot v_{i_k}) = q(v_{i_1}) \cdot \ldots \cdot q(v_{i_k}).$$

The form Q is nonsingular and defines (by scalar extension) a nonsingular symmetric bilinear form on $\mathcal{O}_X \otimes C_0$. Since $(q(v_i))^2 = 1$, so that

$$Q(v_{i_1} \cdot \ldots \cdot v_{i_l} \cdot \ldots \cdot v_{i_k}) = Q(v_{i_1} \cdot \ldots \cdot v_{i_l}) Q(v_{i_{l+1}} \cdot \ldots \cdot v_{i_k}),$$

direct computation shows that $\mathrm{Im}(\mathcal{O}_X(-1) \otimes C_0 \xrightarrow{\varphi\cdot} \mathcal{O}_X \otimes C_0) = \varphi\mathcal{U}_0 \cong \mathcal{U}_0$ is a totally isotropic subspace of $\mathcal{O}_X \otimes C_0$. Therefore

$$\mathcal{U}_0 \cong \varphi\mathcal{U}_0 = (\varphi\mathcal{U}_0)^\perp \cong ((\mathcal{O}_X \otimes C_0)/(\varphi\mathcal{U}_0))^\wedge \cong \mathcal{U}_{-1}^\wedge.$$

Thus

$$\mathcal{U}_0^\wedge \cong \mathcal{U}_{-1} \cong \mathcal{U}_0(1)$$

and, in general

$$\mathcal{U}_n^\wedge \cong (\mathcal{U}_0(-n))^\wedge \cong \mathcal{U}_0^\wedge(n) \cong \mathcal{U}_0(n+1) \cong \mathcal{U}_n(2n+1). \ \blacksquare$$

COROLLARY 4.5. (i) $[\mathcal{U}^\wedge] = [\mathcal{U}(2d-1)]$ *and* $[\mathcal{U}(d-1)] + [\mathcal{U}(d-1)]^\wedge = 2d+1$ *in* $K_0(X)$;

(ii) $\operatorname{rank}\mathcal{U} = \frac{1}{2}\dim C_0 = 2^d$.

In the case of even $d = 2m$ the algebra $\operatorname{End}_X(\mathcal{U}) = C_0$ splits into the direct product of the algebras P_i defined above: $C_0 = P_0 \times P_1$.

DEFINITION 4.2. For even d, set

$$\mathcal{U}_n' = \mathcal{U}_n \otimes_{C_0} P_0, \qquad \mathcal{U}' = \mathcal{U} \otimes_{C_0} P_0,$$
$$\mathcal{U}_n'' = \mathcal{U}_n \otimes_{C_0} P_1, \qquad \mathcal{U}'' = \mathcal{U} \otimes_{C_0} P_1.$$

Note that $\mathcal{U}_n = \mathcal{U}_n' \oplus \mathcal{U}_n''$ and $\mathcal{U} = \mathcal{U}' \oplus \mathcal{U}''$. $\mathcal{U}_0'$ and $\mathcal{U}_0''$ correspond to spinor representations and we shall reproduce here a standard dualization argument (compare [5], Sect. 4.3). In the case of even $d = 2m$ another property of φ and the quadratic form Q introduced in the proof of Lemma 4.4 may be verified by direct computation:

LEMMA 4.6. *For even* $d = 2m$,

(i) *if* m *is even, then* $P_i = (1 \pm \delta)C_0$ *are orthogonal to each other, hence self-dual*;

(ii) *if* m *is odd, then* $P_i = (1 \pm \delta)C_0$ *are totally isotropic, hence dual to each other*;

(iii) $\varphi(1 \pm \delta) = (1 \mp \delta)\varphi$.

COROLLARY 4.7. *For even* $d = 2m$,

(i) $\mathcal{U}'^\wedge \cong \mathcal{U}'(2d-1)$ *and* $\mathcal{U}''^\wedge \cong \mathcal{U}''(2d-1)$ *for even* m;

(ii) $\mathcal{U}'^\wedge \cong \mathcal{U}''(2d-1)$ *and* $\mathcal{U}''^\wedge \cong \mathcal{U}'(2d-1)$ *for odd* m;

(iii) $\operatorname{End}_X(\mathcal{U}') \cong \operatorname{End}_X(\mathcal{U}'') \cong M_{2^m}(F)$;

(iv) *the exact sequence* (4.2) *splits into two exact parts*

$$0 \to \mathcal{U}_0' \xrightarrow{\varphi\cdot} \mathcal{O}_X \otimes P_0 \to \mathcal{U}_0''(1) \to 0,$$
$$0 \to \mathcal{U}_0'' \xrightarrow{\varphi\cdot} \mathcal{O}_X \otimes P_1 \to \mathcal{U}_0'(1) \to 0.$$

The standard way to determine indecomposable components is tensoring with a simple left module over an appropriate endomorphism algebra. We will use superscripts as a notation for direct sums of identical objects.

DEFINITION 4.3. (i) For odd $d = 2m+1$, set

$$\mathcal{V} = \mathcal{U} \otimes_{C_0} F^{2N}.$$

(ii) For even $d = 2m$, set

$$\mathcal{V}_0 = \mathcal{U}' \otimes_{M_N(F)} F^N, \qquad \mathcal{V}_1 = \mathcal{U}'' \otimes_{M_N(F)} F^N$$

where $N = 2^m$.

For convenience we will use mod 2 subscripts in $\mathcal{V}_i$. Since $M_N(F) = (F^N)^N$ as left $M_N(F)$-modules, indecomposable components inherit the properties of the Swan bundle. We have

Proposition 4.8. (a) *For odd* $d = 2m + 1$ *we have*:

(i) $\mathcal{U} = \mathcal{V}^{2N}$, *where* $N = 2^m$;
(ii) $\mathcal{V}^\wedge = \mathcal{V}(2d - 1)$;
(iii) $\operatorname{End}_X(\mathcal{V}) \cong F$ *and* $\operatorname{rank} \mathcal{V} = 2^m$.

(b) *For even* $d = 2m$ *we have*:

(i) $\mathcal{U}' = \mathcal{V}_0^N$ *and* $\mathcal{U}'' = V_1^N$, *where* $N = 2^m$;
(ii) $\mathcal{V}_i^\wedge = \mathcal{V}_{i+m}(2d - 1)$;
(iii) $\operatorname{End}_X(\mathcal{V}_i) \cong F$ *and* $\operatorname{rank}\ \mathcal{V}_i = 2^{m-1}$,
(iv) $[\mathcal{V}_i(d - 1)] + [\mathcal{V}_{i+1}(d)] = 2^m$ *in* $K_0(X)$.

In particular, there is no global morphism $\mathcal{V}_i \to \mathcal{V}_{i+1}$, since

$$\operatorname{End}_X(\mathcal{U}) = \operatorname{End}_X(\mathcal{V}_0^N \oplus \mathcal{V}_1^N) = M_N(\operatorname{End}_X(\mathcal{V}_0)) \times M_N(\operatorname{End}_X(\mathcal{V}_1)).$$

Example 4.8.1. The split projective quadric $X = \operatorname{Proj} S$,

$$S = F[x_0, y_0, x_1, y_1, \ldots, x_m, y_m] / \Big(\sum_{i=0}^{m} x_i y_i \Big),$$

of dimension $d = 2m$ contains two projective spaces of dimension m:

- $Y = \operatorname{Proj} F[y_0, y_1, \ldots, y_m]$ defined by $x_0 = x_1 = \ldots = x_m = 0$,

and

- $Z = \operatorname{Proj} F[x_0, y_1, \ldots, y_m]$ defined by $y_0 = x_1 = \ldots = x_m = 0$.

These subvarieties are not rationally equivalent. It may be shown that their structural sheaves define two distinct elements of $K_0'(X) = K_0(X)$:

$$\sum_{i=0}^{d-1} \Big(\sum_{p=0}^{m} \binom{m}{p} \Big) (-1)^i [\mathcal{O}_X(-i)] + [\mathcal{V}_0]$$

and

$$\sum_{i=0}^{d-1} \Big(\sum_{p=0}^{m} \binom{m}{p} \Big) (-1)^i [\mathcal{O}_X(-i)] + [\mathcal{V}_1]$$

(see [12], Theorem 4.1). If, in particular, $d = 2$, then X is isomorphic to the product of two projective lines, and Y and Z are generatrices.

5. E-groups of affine quadrics. The following theorem due to Swan [11] (Theorem 10.5, Corollary 10.7) describes the K-theory of a (relative) affine quadric.

THEOREM 5.1. *Let R be a regular ring, $q \cong \langle -1\rangle \oplus q'$ a nonsingular quadratic form over R defined on the projective R-module $M = M' \oplus R$, S the coordinate ring of the affine quadric $q' = 1$ over R, $S = S(M^\wedge)/(q' - 1)$, and A a generalized Azumaya algebra over R. Let, moreover,*

- $g : K_*(A \otimes_R C_0(q')) \to K_*(A)$ *be the norm map,*
- $h : K_*(A \otimes_R C_0(q')) \to K_*(A \otimes_R C_0(q))$ *be the scalar extension map,*
- $r : K_*(A) \to K_*(A \otimes_R S)$ *be the scalar extension map, and*
- $s : K_*(A \otimes_R C_0(q)) \to K_*(A \otimes_R S)$ *be the map induced by the exact functor $\mathcal{U}(\operatorname{Spec} S) \otimes_{C_0(q)} -$.*

Then the sequence

$$\ldots \to K_{i+1}(A \otimes_R S) \xrightarrow{\partial} K_i(A \otimes_R C_0(q'))$$
$$\xrightarrow{\alpha} K_i(A) \oplus K_i(A \otimes_R C_0(q)) \xrightarrow{[r,s]} K_i(A \otimes_R S) \xrightarrow{\partial} \ldots,$$

where $\alpha = \begin{bmatrix} -g \\ h \end{bmatrix}$, is exact.

Let X be the affine quadric $\operatorname{Spec} S$.

PROPOSITION 5.2. *Under the assumptions of Theorem 5.1, if in addition $A = R = F$ is a field of characteristic different from 2, $d = 2m$ is even and $q = \sum_{i=0}^{d+1}(-1)^i z_i^2$, $q' = \sum_{i=0}^{d}(-1)^i z_i^2$, then*

- $g : K_0(C_0(q')) \to K_0(F)$ *is the map $2^m\cdot : \mathbb{Z} \to \mathbb{Z}$,*
- $h : K_0(C_0(q')) \to K_0(C_0(q)) = K_0(P_0) \oplus K_0(P_1)$ *is the diagonal map $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$,*
- $r : K_0(F) \to K_0(S)$ *maps the generator $[F^1]$ onto the generator $[S^1]$,*
- $s : K_0(C_0(q)) = K_0(P_0) \oplus K_0(P_1) \to K_0(S)$ *maps the generators $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ onto $[V_0]$, $[V_1]$ respectively, where $V_i = \mathcal{V}_i(\operatorname{Spec} S)$ for $i = 0, 1$.*

Proof. This is a direct computation on the level of functors. The (reduced) norm map g is induced by the forgetful functor from $C_0(q')$-modules to F-modules, which maps the generator of $K_0(C_0(q'))$ (the class of the simple $C_0(q')$-module) onto the class of the same module. The scalar extension map h is induced by the scalar extension functor, which in turn produces identity if composed with projections onto P_i's. The map r is induced by the scalar extension functor $S \otimes_F -$. The definition of s is in fact the definition of V_0, V_1. ■

PROPOSITION 5.3. *Under the assumptions of Proposition 5.2, $K_0(S) \cong \mathbb{Z} \oplus \mathbb{Z}$ has generators $1 = [S^1], [V_0], [V_1]$ subject to the defining identity $[V_0] + [V_1] = 2^m[S^1]$.*

Proof. This is an immediate consequence of Proposition 5.2. ∎

PROPOSITION 5.4. *Under the assumptions of Proposition* 5.2,

(i) *if m is even*, *then*

$$E^+(X) = E^+(S) = \mathbb{Z}/2\mathbb{Z}[S^1] \oplus \mathbb{Z}/2\mathbb{Z}(2^{m-1}[S^1] - [V_0]),$$
$$E^-(X) = E^-(S) = 0;$$

(ii) *if m is odd*, *then*

$$E^+(X) = E^+(S) = \mathbb{Z}/2\mathbb{Z}[S^1],$$
$$E^-(X) = E^-(S) = \mathbb{Z}/2\mathbb{Z}(2^{m-1}[S^1] - [V_0]).$$

Proof. By Proposition 4.8(b)(ii),(iv), the involution $\wedge$ on $K_0(X)$ has the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & (-1)^m \end{bmatrix}$$

with respect to the base $[S^1], 2^{m-1}[S^1] - [V_0]$ of the group $K_0(X)$. ∎

EXAMPLE 5.4.1. For $d = 4$ we consider the affine quadric defined by the equation $x_0y_0 + x_1y_1 + z^2 = 1$. The sequence

$$0 \leftarrow F[y_0, y_1] \leftarrow S \xleftarrow{\alpha} S^3 \xleftarrow{\beta} S^4 \xleftarrow{\gamma} S^4 \leftarrow V_0 \leftarrow 0,$$

where

$$\alpha = [1 - z, -x_1, x_0],$$
$$\beta = \begin{bmatrix} 1+z & x_1 & -x_0 & 0 \\ y_1 & 1-z & 0 & x_0 \\ -y_0 & 0 & 1-z & x_1 \end{bmatrix},$$
$$\gamma = \frac{1}{2}\begin{bmatrix} 1-z & -x_1 & x_0 & 0 \\ -y_1 & 1+z & 0 & -x_0 \\ y_0 & 0 & 1+z & -x_1 \\ 0 & -y_0 & -y_1 & 1-z \end{bmatrix},$$

is exact and is the sequence of sections over the complement to a hyperplane section of the truncated canonical resolution for the structural sheaf of Y from Example 4.8.1. The map γ is chosen to be a projection of S^4 onto a direct summand. Hence V_0 is a submodule of S^4 generated by columns of the matrix

$$1 - \gamma = \frac{1}{2}\begin{bmatrix} 1+z & x_1 & -x_0 & 0 \\ y_1 & 1-z & 0 & x_0 \\ -y_0 & 0 & 1-z & x_1 \\ 0 & y_0 & y_1 & 1+z \end{bmatrix}.$$

Note that $m = 2$, so $E^{-}(S)$ is trivial. In fact, $\operatorname{rank} V_0 = 2$, so the module V_0 carries the canonical nonsingular skew-symmetric bilinear form, defined by the exterior multiplication $V_0 \times V_0 \to \bigwedge^2 V_0 \cong S^1$.

PROPOSITION 5.5. *Under the assumptions of Proposition* 5.2, *in the ring* $K_0(S) = \mathbb{Z}[S^1] \oplus \mathbb{Z}(2^{m-1}[S^1] - [V_0])$ *we have the equality*

$$(2^{m-1}[S^1] - [V_0])^2 = 0.$$

Proof. The rank map $K_0(S) \to \mathbb{Z}$ is a ring homomorphism with kernel $\mathbb{Z}(2^{m-1}[S^1] - [V_0])$. Moreover, $(2^{m-1}[S^1] - [V_0])^\wedge = -(2^{m-1}[S^1] - [V_0])$, so $(2^{m-1}[S^1] - [V_0])^2$ is a fixed point of $^\wedge$, hence it is a multiple of $[S^1]$, and it is an element of rank zero, hence it is a multiple of $2^{m-1}[S^1] - [V_0]$. ∎

It will be crucial for the construction in the next section that the identity $[V_0] + [V_1] = 2^m[S^1]$ arises from a direct decomposition with special properties.

PROPOSITION 5.6. *Under the assumptions of Proposition* 5.2,

(a) $S^N \cong V_0 \oplus V_1$;

(b) *if* $m \equiv 2 \bmod 4$, *then there exists a direct decomposition* $S^N = S^{N/2} \oplus S^{N/2\wedge}$ *such that for the associated symmetric hyperbolic form* $\overline{\chi}$ *on* S^N *the direct summands* V_0, V_1 *are totally isotropic.*

Proof. In the ring S the last variable z_{d+1} is set to be 1:

$$z_{d+1} = 1.$$

More precisely, the affine coordinates z_i are obtained as z_i/z_{d+1} from the homogoneous coordinates z_i. Therefore the multiplier $\varphi \in S \otimes_F V$ is equal to $\sum_{i=0}^{d} z_i \otimes v_i + v_{d+1}$. The exact sequence (4.1) restricted to $\operatorname{Spec} S$ reduces to the following exact sequence of free S-modules:

$$\ldots \xrightarrow{\varphi\cdot} S \otimes_F C_{n+d+1} \xrightarrow{\varphi\cdot} S \otimes_F C_{n+d} \xrightarrow{\varphi\cdot} S \otimes_F C_{n+d-1} \xrightarrow{\varphi\cdot} \ldots$$

There is one projective S-module $U = \Gamma(\operatorname{Spec} S, \mathcal{U}_{-1})$ such that

$$U \cong \operatorname{Coker}(S \otimes_F C_0 \xrightarrow{\varphi\cdot} S \otimes_F C_1) \cong \operatorname{Coker}(S \otimes_F C_1 \xrightarrow{\varphi\cdot} S \otimes_F C_0)$$

instead of the sequence $\mathcal{U}_n$. Moreover, $U' = \Gamma(\operatorname{Spec} S, \mathcal{U}'_{-1}) = U \otimes_{C_0} P_0$ and $U'' = \Gamma(\operatorname{Spec} S, \mathcal{U}''_{-1}) = U \otimes_{C_0} P_1$. In addition to the central idempotents $\frac{1}{2}(1 \pm \delta)$, consider a family of minimal orthogonal idempotents of the even Clifford algebra C_0 defined for sequences $I = (i_0, i_1, \ldots, i_m)$ of ± 1's as follows:

$$e_I = \frac{1}{2^{m+1}}(1 + i_0 v_0 v_1) \cdot \ldots \cdot (1 + i_m v_{2m} v_{2m+1}).$$

It follows that

$$\begin{aligned}
\delta\varepsilon_I &= v_0 \cdot v_1 \cdot \ldots \cdot v_{2m} \cdot v_{2m+1} \cdot \frac{1}{2^{m+1}}(1 + i_0 v_0 v_1) \cdot \ldots \cdot (1 + i_m v_{2m} v_{2m+1}) \\
&= \frac{1}{2^{m+1}} v_0 \cdot v_1 \cdot (1 + i_0 v_0 v_1) \cdot \ldots \cdot v_{2m} \cdot v_{2m+1} \cdot (1 + i_m v_{2m} v_{2m+1}) \\
&= \frac{1}{2^{m+1}} (v_0 v_1 + i_0) \cdot \ldots \cdot (v_{2m} v_{2m+1} + i_m) \\
&= i_0 i_1 \ldots i_m \frac{1}{2^{m+1}} (1 + i_0 v_0 v_1) \cdot \ldots \cdot (1 + i_m v_{2m} v_{2m+1}).
\end{aligned}$$

Thus

- $\frac{1}{2}(1+\delta)\varepsilon_I = \varepsilon_I$ and $\frac{1}{2}(1-\delta)\varepsilon_I = 0$ if the number of -1's in the sequence I is even,
- $\frac{1}{2}(1+\delta)\varepsilon_I = 0$ and $\frac{1}{2}(1-\delta)\varepsilon_I = \varepsilon_I$ if the number of -1's in the sequence I is odd.

Each left ideal $C_0 \cdot \varepsilon_I$ is a minimal left ideal of C_0, and it is isomorphic to F^N as a C_0-module. Moreover, P_0 is a direct sum of half of these ideals, while P_1 is the direct sum of the other half. Write for short $\varepsilon_0 = \varepsilon_{(1,1,\ldots,1)}$, $\varepsilon_1 = \varepsilon_{(1,1,\ldots,-1)}$, $\varepsilon_{-1} = \varepsilon_{(-1,-1,\ldots,-1)}$. Then

$$S^N = S \otimes_F F^N \cong S \otimes_F C_0\varepsilon_0 \cong S \otimes_F C_0 \otimes_{C_0} P_0 \otimes_{P_0} F^N.$$

The identity $a = \big(1 + \frac{1}{2}\varphi \cdot v_{d+1}\big)a - \frac{1}{2}\varphi \cdot v_{d+1} \cdot a$ yields a direct sum decomposition

$$S^N \cong S \otimes_F C_0\varepsilon_0 = \big(1 + \tfrac{1}{2}\varphi \cdot v_{d+1}\big) \cdot (S \otimes_F C_0\varepsilon_0) \oplus \varphi \cdot (S \otimes_F C_1\varepsilon_0)$$

since $v_{d+1}C_0 = C_1$. Therefore

$$\begin{aligned}
V_0 &= \operatorname{Coker}(S \otimes_F C_1 \xrightarrow{\varphi\cdot} S \otimes_F C_0) \otimes_F C_0 \otimes_{C_0} P_0 \otimes_{P_0} F^N \\
&\cong (1 + \tfrac{1}{2}\varphi \cdot v_{d+1}) \cdot (S \otimes_F C_0\varepsilon_0),
\end{aligned}$$

so

$$S^N \cong V_0 \oplus \varphi \cdot (S \otimes_F C_1\varepsilon_0).$$

Analogously,

$$S^N \cong S \otimes_F C_0\varepsilon_1 = \big(1 + \tfrac{1}{2}\varphi \cdot v_{d+1}\big) \cdot (S \otimes_F C_0\varepsilon_1) \oplus \varphi \cdot (S \otimes_F C_1\varepsilon_1)$$

where $\big(1 + \frac{1}{2}\varphi \cdot v_{d+1}\big) \cdot (S \otimes_F C_0\varepsilon_1) \cong V_1$, and

$$S^N = S \otimes_F F^N \cong S \otimes_F C_0\varepsilon_1 \cong S \otimes_F C_0 \otimes_{C_0} P_1 \otimes_{P_1} F^N.$$

Left multiplication by φ and right multiplication by v_{d+1} yield

$$\begin{aligned}
V_1 &\cong \varphi V_1 v_{d+1} \cong \varphi\big(1 + \tfrac{1}{2}\varphi v_{d+1}\big) \cdot (S \otimes_F C_0\varepsilon_1) \cdot v_{d+1} \\
&= \varphi \cdot (S \otimes_F C_0\varepsilon_1) \cdot v_{d+1} \\
&= \varphi \cdot (S \otimes_F C_0 v_{d+1}\varepsilon_0) = \varphi \cdot (S \otimes_F C_1\varepsilon_0),
\end{aligned}$$

since $v_{d+1}\varepsilon_0 = \varepsilon_1 v_{d+1}$. Thus

$$S^N \cong V_0 \oplus V_1.$$

To prove claim (b) consider the principal antiautomorphism $\Im$ of $S \otimes_F C_0$. First of all note that $\Im(\varepsilon_0) = \varepsilon_{-1}$. Next, the set of all products $v_{2i_1+1} \cdot v_{2i_2+1} \cdot \ldots \cdot v_{2i_k+1}$ with $0 \leq i_1 < i_2 < \ldots < i_k \leq m$ forms a base of the minimal ideal $C\varepsilon_0$ of the Clifford algebra $C = C(q)$, and the set of such products with an even number of factors forms a base of $C_0\varepsilon_0$. Moreover,

$$\begin{aligned}
&\Im\Big(\Big(\prod_{i\in A} v_{2i+1}\Big) \cdot \varepsilon_0\Big) \cdot \Big(\prod_{i\in B} v_{2i+1}\Big) \cdot \varepsilon_0 \\
&\quad = \begin{cases} 0 & \text{if } A \cap B \neq \emptyset \text{ or } A \cup B \neq \{0, 1, \ldots, m\}, \\ \pm v_1 v_3 \cdot \ldots \cdot v_{d+1}\varepsilon_0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus for odd m the map $(\alpha\varepsilon_0, \beta\varepsilon_0) \mapsto$ coefficient of $v_1 \cdot v_3 \cdot \ldots \cdot v_{d+1}\varepsilon_0$ in $\Im(\alpha\varepsilon_0) \cdot \beta\varepsilon_0$ defines a nonsingular pairing on S^N, whose adjoint is an isomorphism of direct summands in a suitably chosen direct sum decomposition $S^N = S^{N/2} \oplus S^{N/2}$. This pairing is symmetric for $m \equiv 3 \bmod 4$, and skew-symmetric for $m \equiv 1 \bmod 4$.

Finally,

$$\begin{aligned}
&\Im(\varphi \cdot (S \otimes_F C_1\varepsilon_0)) \cdot \varphi \cdot (S \otimes_F C_1\varepsilon_0) = (\varepsilon_{-1} S \otimes_F C_1)\varphi^2 (S \otimes_F C_1\varepsilon_1) = 0, \\
&\Im\big(\big(1 + \tfrac{1}{2}\varphi \cdot v_{d+1}\big) \cdot (S \otimes_F C_0\varepsilon_0)\big) \cdot \big(1 + \tfrac{1}{2}\varphi \cdot v_{d+1}\big) \cdot (S \otimes_F C_0\varepsilon_0) \\
&\qquad = (\varepsilon_{-1} S \otimes_F C_0)\big(1 + \tfrac{1}{2} v_{d+1} \cdot \varphi\big)\big(1 + \tfrac{1}{2}\varphi \cdot v_{d+1}\big) \cdot (S \otimes_F C_0\varepsilon_0) \\
&\qquad = (\varepsilon_{-1} S \otimes_F C_0)\big(1 + \tfrac{1}{2}(v_{d+1} \cdot \varphi + \varphi \cdot v_{d+1})\big) \cdot (S \otimes_F C_0\varepsilon_0) = 0
\end{aligned}$$

so V_0 and V_1 are totally isotropic. ∎

Example 5.6.1. Consider the case $d = 2$. As in Example 5.4.1 there is an exact sequence

$$0 \leftarrow F[y_0] \leftarrow S \xleftarrow{\alpha} S^2 \xleftarrow{\beta} S^2 \xleftarrow{\gamma} S^2 \xleftarrow{\beta} \ldots$$

where

$$\begin{aligned}
S &= F[x_0, y_0, z_2]/(x_0 y_0 + z_2^2 - 1), \\
\alpha &= [z_2 - 1, x_0], \\
\beta &= \frac{1}{2}\begin{bmatrix} 1 + z_2 & x_0 \\ y_0 & 1 - z_2 \end{bmatrix}, \\
\gamma &= \frac{1}{2}\begin{bmatrix} 1 - z_2 & -x_0 \\ -y_0 & 1 + z_2 \end{bmatrix}.
\end{aligned}$$

Thus $\beta + \gamma = 1$, $\beta\gamma = \gamma\beta = 0$, hence $\beta^2 = \beta$, $\gamma^2 = \gamma$. Now V_0 is the submodule of S^2 spanned by the columns of γ, V_1 is the submodule of S^2

spanned by the columns of β and the form $\overline{\chi}$ has the matrix $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$:

$$\begin{bmatrix} 1+z_2 & x_0 \\ y_0 & 1-z_2 \end{bmatrix}^T \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1+z_2 & x_0 \\ y_0 & 1-z_2 \end{bmatrix} = \begin{bmatrix} 0 & -y_0x_0 - z_2^2 + 1 \\ y_0x_0 + z_2^2 - 1 & 0 \end{bmatrix} = 0.$$

EXAMPLE 5.6.2. An interesting case of smallest dimension is the one for $d = 6$. Then $N = 2^m = 8$ and

$$S = F[x_0, y_0, x_1, y_1, x_2, y_2, z_6]/(x_0y_0 + x_1y_1 + x_2y_2 + z_6^2 - 1).$$

Now V_0 and V_1 are submodules of the free module S^8 spanned by the columns of the matrices

$$\beta = \frac{1}{2}\begin{bmatrix} 1-z_6 & 0 & 0 & 0 & -y_2 & y_1 & -y_0 & 0 \\ 0 & 1-z_6 & 0 & 0 & -x_1 & -x_2 & 0 & y_0 \\ 0 & 0 & 1-z_6 & 0 & x_0 & 0 & -x_2 & y_1 \\ 0 & 0 & 0 & 1-z_6 & 0 & x_0 & x_1 & y_2 \\ -x_2 & -y_1 & y_0 & 0 & 1+z_6 & 0 & 0 & 0 \\ x_1 & -y_2 & 0 & y_0 & 0 & 1+z_6 & 0 & 0 \\ -x_0 & 0 & -y_2 & y_1 & 0 & 0 & 1+z_6 & 0 \\ 0 & x_0 & x_1 & x_2 & 0 & 0 & 0 & 1+z_6 \end{bmatrix}$$

and

$$\gamma = \frac{1}{2}\begin{bmatrix} 1+z_6 & 0 & 0 & 0 & y_2 & -y_1 & y_0 & 0 \\ 0 & 1+z_6 & 0 & 0 & x_1 & x_2 & 0 & -y_0 \\ 0 & 0 & 1+z_6 & 0 & -x_0 & 0 & x_2 & -y_1 \\ 0 & 0 & 0 & 1+z_6 & 0 & -x_0 & -x_1 & -y_2 \\ x_2 & y_1 & -y_0 & 0 & 1-z_6 & 0 & 0 & 0 \\ -x_1 & y_2 & 0 & -y_0 & 0 & 1-z_6 & 0 & 0 \\ x_0 & 0 & y_2 & -y_1 & 0 & 0 & 1-z_6 & 0 \\ 0 & -x_0 & -x_1 & -x_2 & 0 & 0 & 0 & 1-z_6 \end{bmatrix},$$

which satisfy analogous conditions: $\beta+\gamma = 1$, $\beta\gamma = \gamma\beta = 0$. The symmetric bilinear form on S^8 which has totally isotropic submodules V_0 and V_1 has the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. Witt ring of the projective line. Consider the ring

$$S = F[z_0, z_1, \dots, z_d]/\Big(\sum_{i=0}^{d}(-1)^i z_i^2 - 1\Big)$$

and the projective line $X = \mathbb{P}^1_S = \operatorname{Proj} S[t_0, t_1]$. By Proposition 3.2, Corollary 3.4 and Proposition 5.4 above, for $d \equiv 2 \bmod 4$ the group $E^+(X)$ has four elements:

$$E^+(X) = \mathbb{Z}/2\mathbb{Z}[\mathcal{O}_X] \oplus \mathbb{Z}/2\mathbb{Z}(2^{m-1}[S^1] - [V_0]) \boxtimes H,$$

where $\boxtimes$ denotes the tensor product of inverse images by projections of X onto $\operatorname{Spec} S$ and onto $\mathbb{P}^1$. We need to find a self-dual locally free sheaf $\mathcal{M}$ on X which maps onto $(2^{m-1}[S^1]-[V_0])\boxtimes H$ in $E^+(X)$ and a symmetric bilinear form on $\mathcal{M}$. One may translate every statement on finitely generated S-modules into an analogous statement on sheaves of coherent $\mathcal{O}_Y$-modules by means of the canonical equivalence of categories of finitely generated S-modules and of sheaves of coherent $\mathcal{O}_Y$-modules, given by the exact functors $\Gamma(\operatorname{Spec} S, -)$ and $\tilde{}$, for example $\mathcal{O}_Y = S\tilde{}$. So we will treat S-modules as sheaves on $\operatorname{Spec} S$. Note that in $K_0(X)$ the element $(2^{m-1}[S^1]-[V_0])\boxtimes H$ is the direct image of $2^{m-1}[S^1] - [V_0]$ under the immersion ι of an S-rational point Y into X since there is an exact sequence

$$0 \leftarrow \iota_*\mathcal{O}_Y \leftarrow \mathcal{O}_X \xleftarrow{t_0\cdot} \mathcal{O}_X(-1) \leftarrow 0$$

of sheaves. The twist does not change the sheaf $\iota_*\mathcal{O}_Y$, since for every integer k the sequence

$$0 \leftarrow \iota_*\mathcal{O}_Y \leftarrow \mathcal{O}_X(k) \xleftarrow{t_0\cdot} \mathcal{O}_X(k-1) \leftarrow 0$$

is exact.

THEOREM 6.1. *If* $d = 2m$, $m = 4k+3$ *and*

$$S = F[z_0, z_1, \dots, z_d]/\Big(\sum_{i=0}^{d}(-1)^i z_i^2 - 1\Big),$$

then there exists a vector bundle $\mathcal{M}$ *on* $\mathbb{P}^1_S$ *and a symmetric bilinear form* $\phi : \mathcal{M} \to \mathcal{M}^\wedge$ *such that*

$$e^0(\mathcal{M}, \phi) = (2^{m-1}[S^1] - [V_0]) \boxtimes H.$$

In particular, the Witt ring $W(\mathbb{P}^1_S)$ is larger than the Witt ring $W(S)$.

P r o o f. Let, as in Proposition 5.6, $N = 2^m$. Fix a direct sum decomposition

$$S^N = S^{N/2} \oplus S^{N/2\wedge} = V_0 \oplus V_1$$

such that V_0 is a totally isotropic subbundle of the hyperbolic space $(S^N, \overline{\chi})$. These data define a symmetric formation $(\iota_*\mathcal{O}_Y^N, \overline{\chi}; \iota_*\mathcal{O}_Y^{N/2}, \iota_* V_0)$ in $\mathbb{M}_1$ in the sense of [3]. This formation has a resolution (in the sense of Definition 15, p. 464 of [3])—we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X(-1)^{N/2} \arrow[r] \arrow[d] & \mathcal{O}_X(-1)^{N/2} \oplus \mathcal{O}_X^{N/2} \arrow[r] \arrow[d, "h"] & \mathcal{O}_X^{N/2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{N/2} \arrow[r] \arrow[d] & \mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2} \arrow[r] \arrow[d, "\pi"] & \mathcal{O}_X(1)^{N/2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \iota_*\mathcal{O}_Y^{N/2} \arrow[r] \arrow[d] & \iota_*\mathcal{O}_Y^{N/2} \oplus \iota_*(\mathcal{O}_Y^{N/2\wedge}) \arrow[r] \arrow[d] & \iota_*(\mathcal{O}_Y^{N/2\wedge}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that each column is a direct sum of resolutions $0 \leftarrow \iota_*\mathcal{O}_Y \leftarrow \mathcal{O}_X(k) \leftarrow \mathcal{O}_X(k-1) \leftarrow 0$ for an appropriate integer k. Moreover, there is a nonsingular $\mathcal{O}_X(1)$-valued symmetric bilinear form

$$\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2} \xrightarrow{\chi} (\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2})^\wedge \otimes \mathcal{O}_X(1)$$

which reduces to $\overline{\chi}$ on $\iota_*\mathcal{O}_Y^N$. To fit into the setup of Definition 15 of a resolution of a formation in [3], only one accomodation is needed: let $\mathcal{K}$ be the constant sheaf defined by the function field of X. There exists a canonical map $\mathcal{O}_X(1) \to \mathcal{K}$ given by the inclusion $\Gamma(U, \mathcal{O}_X(1)) \rightarrowtail \Gamma(U, \mathcal{K})$ over each affine open subset U, and composing with this map allows us to change the $\mathcal{O}_X(1)$-valued form into the $\mathcal{K}$-valued form

$$\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2} \xrightarrow{\chi'} (\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2})^\wedge \otimes \mathcal{K}$$

which reduces to $\overline{\chi}$ on $\iota_*\mathcal{O}_Y^N$ and which is nonsingular at the generic point. The following is Lemma 16 of [3] with its proof (and adjusted notation):

LEMMA 6.2. *Let* (r) *be a resolution of a formation as above. Let* $\mathcal{M}$ *be the subsheaf of* $\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2}$ *defined by* $\pi^{-1}(V_0)$; *this means that* $\mathcal{M}$ *is the pullback*

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[d] & \mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2} \arrow[d] \\ \iota_* V_0 \arrow[r] & \iota_* \mathcal{O}_Y^{N/2} \oplus \iota_*(\mathcal{O}_Y^{N/2\wedge}) \end{tikzcd}$$

Then $\mathcal{M}$ is a locally free sheaf, and the restriction ϕ of the form χ to $\mathcal{M}$ defines a nonsingular symmetric bilinear form $\mathcal{M} \times \mathcal{M} \to \mathcal{O}_X$.

Proof. Since this is a local statement, the proof is the same as in [8] 7.3, p. 374. ∎

It remains to compute $e^0(\mathcal{M}, \phi)$. As $\mathcal{M}$ is a product of $\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2}$ and $\iota_* V_0$ over $\iota_* \mathcal{O}_Y^{N/2} \oplus \iota_*(\mathcal{O}_Y^{N/2\wedge})$, in the group $K_0(X)$ we have

$$\begin{aligned} [\mathcal{M}] &= [\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2}] + [\iota_* V_0] - [\iota_* \mathcal{O}_Y^{N/2} \oplus \iota_*(\mathcal{O}_Y^{N/2\wedge})] \\ &= N/2[\mathcal{O}_X] + N/2[\mathcal{O}_X(1)] + [V_0]([\mathcal{O}_X] - [\mathcal{O}_X(-1)]) \\ &\quad - N/2([\mathcal{O}_X] - [\mathcal{O}_X(-1)]) - N/2([\mathcal{O}_X(1)] - [\mathcal{O}_X]) \\ &= [V_0]([\mathcal{O}_X] - [\mathcal{O}_X(-1)]) + N/2[\mathcal{O}_X(-1)] + N/2[\mathcal{O}_X] \\ &= N[\mathcal{O}_X] + ([V_0] - N/2[\mathcal{O}_X])H. \end{aligned}$$

Since $N = 2^{4k+3}$ is even, we get $e^0(\mathcal{M}, \phi) = ([V_0] - 2^{m-1}[S]) \boxtimes H$. ∎

REFERENCES

[1] J. K. Arason, *Der Wittring projektiver Räume*, Math. Ann. 253 (1980), 205–212.

[2] J. K. Arason, R. Elman and B. Jacob, *On the Witt ring of elliptic curves*, in: K-theory and Algebraic Geometry: Connections with Quadratic Forms and Division Algebras, Proc. Sympos. Pure Math. 58, Part 2, Amer. Math. Soc., 1995, 1–25.

[3] F. Fernández-Carmena, *On the injectivity of the map of the Witt group of a scheme into the Witt group of its function field*, Math. Ann. 277 (1987), 453–468.

[4] P. Jaworski, *On the Witt rings of function fields of quasihomogeneous varieties*, Colloq. Math. 73 (1997), 195–219.

[5] M. M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. 92 (1988), 479–508.

[6] M. Knebusch, *Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen*, Sitzungsber. Heidelb. Akad. Wiss., Springer, Berlin, 1970, 93–157.

[7] —, *Symmetric bilinear forms over algebraic varieties*, in: Conference on Quadratic Forms (Kingston 1976), Queen's Papers in Pure and Appl. Math. 46, 1977, 102–283.

[8] W. Pardon, *The exact sequence of localization for Witt groups*, in: Lecture Notes in Math. 551, Springer, Berlin, 1976, 336–379.

[9] D. Quillen, *Higher algebraic K-theory I*, in: Lecture Notes in Math. 341, Springer, Berlin, 1973, 85–147.

[10] W. Scharlau, *Quadratic and Hermitian Forms*, Springer, Berlin, 1985.

[11] R. Swan, *K-theory of quadric hypersurfaces*, Ann. of Math. 122 (1985), 113–153.

[12] M. Szyjewski, *An invariant of quadratic forms over schemes*, Documenta Math. J. DMV 1(1996), 449–478.

[13] —, *Witt rings of Grassmann varieties*, in: Proc. Conf. on Algebraic K-Theory, Poznań, September 4–8, 1995, Contemp. Math. 199, Amer. Math. Soc., 1996, 185–210.

Institute of Mathematics
Silesian University
Bankowa 14
40-007 Katowice, Poland
E-mail: szyjewski@gate.math.us.edu.pl

Received 5 September 1996;
revised 3 December 1996 and 14 March 1997