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ON $\check{H}^n$-BUBBLES IN N-DIMENSIONAL COMPACTA

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A topological space X is called an $\check{H}^n$-bubble (n is a natural number, $\check{H}^n$ is Čech cohomology with integer coefficients) if its n-dimensional cohomology $\check{H}^n(X)$ is nontrivial and the n-dimensional cohomology of every proper subspace is trivial. The main results of our paper are: (1) *Any compact metrizable $\check{H}^n$-bubble is locally connected*; (2) *There exists a* 2*-dimensional* 2*-acyclic compact metrizable ANR which does not contain any $\check{H}^2$-bubbles*; and (3) *Every n-acyclic finite-dimensional $L\check{H}^n$-trivial metrizable compactum contains an $\check{H}^n$-bubble.*

1. Introduction. Kuperberg [9] introduced the concept of an n-bubble, i.e. an n-dimensional compactum X such that $\check{H}^n(X) \neq 0$ and $\check{H}^n(F) = 0$ for all its *closed* proper subspaces $F \subset X$. A natural development of this concept are $\check{H}^n$-bubbles. Let n be a fixed natural number. A topological space X is said to be an *$\check{H}^n$-bubble* if its n-dimensional Čech cohomology group with integer coefficients $\check{H}^n(X)$ is nontrivial and the cohomologies $\check{H}^n(A)$ of all its proper subsets A are trivial.

Every connected compact metrizable n-dimensional homology or equivalently (for integer coefficients), every cohomology manifold is an $\check{H}^n$-bubble [3, 7, 12]. This follows by the Poincaré duality and from the fact that the cohomology group of any subspace of a metrizable space is isomorphic to the direct limit of cohomology groups of all its open neighborhoods [3, 6].

If we glue two points of the n-dimensional sphere S^n, $n > 1$, we obtain an $\check{H}^n$-bubble which is not a homology manifold. Any n-dimensional compact $\check{H}^n$-bubble is an n-bubble. The Warsaw circle is a 1-bubble, but not an $\check{H}^1$-bubble, because there exists a locally compact subspace with a nontrivial 1-dimensional cohomology [8]. Another difference between n-bubbles and

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$\check{H}^n$-bubbles is the following: by a theorem of Kuperberg, every n-dimensional n-acyclic compactum without small cycles contains an n-bubble. In particular, every n-dimensional n-acyclic compact ANR contains one. We prove that there exists a 2-dimensional 2-acyclic compact ANR which does not contain any $\check{H}^2$-bubbles.

Terminology of the paper agrees in general with [2–5]. The authors wish to thank the referee for several important comments and suggestions.

2. General properties of $\check{H}^n$-bubbles. First of all we remark that the cohomological dimension $\dim_{\mathbb{Z}} X$ of every $\check{H}^n$-bubble X is n and if X is a finite-dimensional metrizable space then $\operatorname{Ind} X = n$. It follows by the Mayer–Vietoris sequence that if $X = X_1 \cup X_2$ and X_1, X_2 are both nonempty closed or both nonempty open subsets of X then $\dim_{\mathbb{Z}}(X_1 \cap X_2) \geq n-1$. In particular, every $\check{H}^n$-bubble is a connected space.

THEOREM 2.1. *Any compact metrizable $\check{H}^n$-bubble X is locally connected.*

To prove this theorem we need some lemmas.

LEMMA 2.2. *Let X be a compact metrizable not locally connected space. Then for some point x and its neighborhood V, the intersection $V \cap C$ of the set V with the component C of the point x in the closure $\overline{V}$ is not open in V.*

Proof. Since the space X is not locally connected, there is a point $x \in X$ and its neighborhood U such that no neighborhood of x in U is connected. Consider a countable system $\{V_i\}_{i=1}^{\infty}$ of neighborhoods of the point x in U for which $\overline{V_i} \subset V_{i+1}$. Suppose that all intersections $C_i \cap V_i$ of the components C_i of the point x in the space $\overline{V_i}$ with V_i are open. Then we have $C_i \subset C_{i+1} \cap V_{i+1} \subset C_{i+1}$ and therefore $\bigcup_{i=1}^{\infty}(C_i \cap V_i) = \bigcup_{i=1}^{\infty} C_i$. The left side of this equality is an open set and the right side is connected, so the point x has a connected neighborhood in U. But this contradicts our assumptions. ∎

LEMMA 2.3. *Let V be the same neighborhood as in Lemma 2.2 and let C be the corresponding component of the point x. Then:*

(i) *There exists a countable disjoint system of clopen sets $\{O_i\}_{i=1}^{\infty}$ in $\overline{V}$ for which $C \cup \bigcup_{i=1}^{\infty} O_i$ is compact;*

(ii) *There exists a countable system of open balls $\{E_i\}_{i=1}^{\infty}$ in V with centers at the point $y \in C \cap V$ for which $\bigcap_{i=1}^{\infty} E_i = \{y\}$ and the intersections $O_i \cap E_j$ are nonempty if and only if $i > j$.*

Proof. Since $\overline{V}$ is compact, there exists a countable decreasing system $\{U_i\}_{i=1}^{\infty}$ of sets, clopen in $\overline{V}$, for which $\bigcap_{i=1}^{\infty} U_i = C$. Then $V_i = U_i \setminus U_{i+1}$ are clopen sets in $\overline{V}$. By Lemma 2.2, we can choose any non-interior point y of the set $V \cap C$.

Construct inductively the systems $\{O_i\}_{i=1}^{\infty}$ and $\{E_i\}_{i=1}^{\infty}$. Let E_1 be any open ball in V of radius smaller than 1, with center at y and such that $E_1 \cap V_1 = \emptyset$. Let k_1 be the maximal index for which $V_1, V_2, \ldots, V_{k_1}$ do not intersect E_1 (such an index exists because y is not an interior point of C). Let $O_1 = \bigcup_{i=1}^{k_1} V_i$. Now suppose that the systems $\{O_i\}_{i=1}^{m}$ and $\{E_i\}_{i=1}^{m}$ have been constructed for some m. Let E_{m+1} be the open ball, not intersecting V_{k_1+1} with center at y and of radius smaller than $1/(m+1)$. Let k_{m+1} be the maximal index for which $V_1, V_2, \ldots, V_{k_{m+1}}$ do not intersect E_{m+1} and let $O_{m+1} = \bigcup_{i=1}^{k_{m+1}} V_i$. Then by induction, we may assume that the systems $\{O_i\}_{i=1}^{\infty}$ and $\{E_i\}_{i=1}^{\infty}$ have been constructed which satisfy conditions of Lemma 2.3. ■

The following lemma is a well-known result from the theory of inverse limits (see, e.g. [7]).

Lemma 2.4. *Let $\{G_i\}_{i=1}^{\infty}$ be an inverse system of countable groups. Then the group $\varprojlim^{(1)} G_i$ is either trivial or has the power of continuum. It is trivial if and only if the inverse system satisfies the Mittag-Leffler condition.*
■

Proof of Theorem 2.1. Suppose that X is not locally connected. Then there exist subsets C, $\{O_i\}_{i=1}^{\infty}$, $\{E_i\}_{i=1}^{\infty}$ of X and a point y as in Lemmas 2.2 and 2.3. We shall prove that then $\check{H}^n((C\backslash y) \cup \bigcup_{i=1}^{\infty} O_i) \neq \emptyset$. There is an obvious equality $(C\backslash y) \cup \bigcup_{i=1}^{\infty} O_i = \bigcup_{j=1}^{\infty}((C \cup \bigcup_{i=1}^{\infty} O_i)\backslash E_j)$. Therefore, from the Milnor generalized exact sequences (see e.g. [7; p. 354]) we have the monomorphism

$$\varprojlim^{(1)} \Big\{ \check{H}^{n-1}\Big(\Big(C \cup \bigcup_{i=1}^{\infty} O_i\Big)\backslash E_j\Big)\Big\}_j \to \check{H}^n\Big((C\backslash y) \cup \bigcup_{i=1}^{\infty} O_i\Big).$$

Čech cohomology groups of compact metrizable spaces are countable. Therefore it suffices by Lemma 2.4 to establish that the inverse spectrum $\{\check{H}^{n-1}((C \cup \bigcup_{i=1}^{\infty} O_i)\backslash E_j)\}_j$ does not satisfy the Mittag-Leffler condition. Consider the following commutative diagram in which the horizontal lines are parts of the Mayer–Vietoris sequences:

$$\begin{array}{ccccc}
\check{H}^{n-1}(O_{j+1}) \oplus \check{H}^{n-1}(X\backslash(O_{j+1} \cap E_1)) & \xrightarrow{\varphi_1} & \check{H}^{n-1}(O_{j+1}\backslash E_1) & \xrightarrow{\varphi_2} & \check{H}^n(X) \\
\uparrow \varphi_3 & & \uparrow \varphi_4 & & \uparrow \varphi_5 \\
\check{H}^{n-1}(O_{j+1}) \oplus \check{H}^{n-1}(X\backslash(O_{j+1} \cap E_j)) & \xrightarrow{\varphi_6} & \check{H}^{n-1}(O_{j+1}\backslash E_j) & \xrightarrow{\varphi_7} & \check{H}^n(X)
\end{array}$$

Obviously the homomorphisms φ_2 and φ_7 are epimorphisms.

As φ_7 is nontrivial there exists $\alpha' \in \check{H}^{n-1}(O_{j+1}\backslash E_j)$ for which $\varphi_7(\alpha') \neq 0$. Since φ_5 is an isomorphism, the element $\varphi_4(\alpha')$ does not belong to $\varphi_1(\check{H}^{n-1}(O_{j+1}))$.

Now consider the following diagram:

$$\begin{array}{ccc}
\check{H}^{n-1}((C\cup\bigcup_{i=1}^{\infty}O_i)\backslash E_1) & \xrightarrow{\psi_1} & \check{H}^{n-1}(O_{j+1}\backslash E_1)\\
\uparrow{\scriptstyle\psi_2} & & \uparrow{\scriptstyle\psi_3}\\
\check{H}^{n-1}((C\cup\bigcup_{i=1}^{\infty}O_i)\backslash E_j) & \xrightarrow{\psi_4} & \check{H}^{n-1}(O_{j+1}\backslash E_j)\\
\uparrow{\scriptstyle\psi_5} & & \uparrow{\scriptstyle\psi_6}\\
\check{H}^{n-1}((C\cup\bigcup_{i=1}^{\infty}O_i)\backslash E_{j+1}) & \xrightarrow{\psi_7} & \check{H}^{n-1}(O_{j+1})
\end{array}$$

The sets $O_{j+1}\backslash E_1$, $O_{j+1}\backslash E_j$, O_{j+1} are clopen in $(C\cup\bigcup_{i=1}^{\infty}O_i)\backslash E_1$, $(C\cup\bigcup_{i=1}^{\infty}O_i)\backslash E_j$, $(C\cup\bigcup_{i=1}^{\infty}O_i)\backslash E_{j+1}$ respectively, therefore ψ_1, ψ_4, ψ_7 are epimorphisms. Let $\psi_4(\alpha) = \alpha'$. Then since $\psi_3(\alpha') = \varphi_4(\alpha')$ does not belong to the group $\varphi_1(\check{H}^{n-1}(O_{j+1})) = \psi_3\psi_6(\check{H}^{n-1}(O_{j+1}))$, the element $\psi_2(\alpha)$ does not belong to the image of $\check{H}^{n-1}((C\cup\bigcup_{i=1}^{\infty}O_i)\backslash E_{j+1})$ in the group $\check{H}^{n-1}((C\cup\bigcup_{i=1}^{\infty}O_i)\backslash E_1)$. Thus it follows that the condition of Mittag-Leffler is not fulfiled and X is a locally connected space. ∎

Remark 2.5. It follows by Theorem 2.1 that every compact metrizable $\check{H}^1$-bubble is homeomorphic to S^1 [10]. But if we glue the endpoints of the "long line" [4] we obtain a compact nonmetrizable $\check{H}^1$-bubble.

Remark 2.6. An $\check{H}^n$-bubble need not be cohomologically locally connected; e.g. the one-point compactification of the plane $\mathbb{R}^2$ to which a countable number of orientable 1-handles have been attached is obviously an $\check{H}^2$-bubble but not a clc space.

QUESTION 2.7. *Is it true that every compact metrizable $\check{H}^n$-bubble with $\check{H}^*(X)$ finitely generated is a clc space?*

QUESTION 2.8. *There exists a compact metrizable $\check{H}^2$-bubble with trivial fundamental group not homeomorphic to S^2 (S. Ferry). Is every such space homotopically equivalent to S^2?*

Remark 2.9. The compactness condition in the formulation of Theorem 2.1 is a consequence of some general conditions. Indeed:

PROPOSITION 2.10. *Let X be a paracompact $\check{H}^n$-bubble and $\check{H}^n(X)$ be a slender group* (*for example the group of integers $\mathbb{Z}$*) [13]. *Then X is a compact space.*

LEMMA 2.11. *Let X be a paracompact noncompact space. Then there exists a countable discrete system $\{U_i\}_{i=1}^{\infty}$ of open nonempty sets in X.*

P r o o f. Since X is a noncompact space there exists an open cover $\mathcal{U}$ which does not contain a finite subcover. Let $\mathcal{F}$ be a closed locally finite cover which is a refinement of $\mathcal{U}$.

Let x_1 be any point of X, $\mathrm{St}(x_1, \mathcal{F})$ be the union of all members of $\mathcal{F}$ which contain x_1. Let U_{x_1} be a neighborhood of x_1 which lies in $\mathrm{St}(x_1, \mathcal{F})$.

Suppose that the points $x_1, \ldots, x_n$ and their neighborhoods $U_{x_1}, \ldots, U_{x_n}$ have been chosen for some n. Then let x_{n+1} be any point which belongs to $X \setminus \bigcup_{i=1}^{n} \mathrm{St}(x_i, \mathcal{F})$ and let $U_{x_{n+1}}$ be a neighborhood of x_{n+1} such that

$$U_{x_{n+1}} \subset \mathrm{St}(x_{i+1}, \mathcal{F}) \quad \text{and} \quad \overline{U}_{x_{n+1}} \cap \bigcup_{i=1}^{n} \mathrm{St}(x_i, \mathcal{F}) = \emptyset.$$

In this way we can construct a sequence of points $\{x_i\}_{i=1}^{\infty}$ and their neighborhoods $\{U_i\}_{i=1}^{\infty}$. It is easy to see that this countable system $\{U_i\}_{i=1}^{\infty}$ is discrete. ∎

P r o o f o f P r o p o s i t i o n 2.10. Suppose that X is not compact. Then by Lemma 2.11 there exists a countable discrete system of open subsets $\{U_i\}_{i=1}^{\infty}$ of X. For every k, we have the following commutative diagram:

$$\begin{array}{ccccc}
\check{H}^n(\overline{U}_k, \mathrm{Fr}\, U_k) & \xleftarrow{\ \varphi_k\ } & \check{H}^n(X, X \setminus U_k) & \xrightarrow{\ \psi_k\ } & \check{H}^n(X) \\
\uparrow & & \downarrow{\scriptstyle \tau_k} & & \uparrow \\
\check{H}^n(\bigcup_{i=1}^{\infty} \overline{U}_i, \bigcup_{i=1}^{\infty} \mathrm{Fr}\, U_i) & \xleftarrow{\ \varphi\ } & \check{H}^n(X, X \setminus \bigcup_{i=1}^{\infty} U_i) & \xrightarrow{\ \psi\ } & \check{H}^n(X)
\end{array}$$

in which all homomorphisms are induced by inclusions. The homomorphisms φ_k and φ are the excision isomorphisms. The homomorphisms ψ_k and ψ are epimorphisms, because the n-dimensional cohomology of a proper subspace of X is trivial and the cohomology sequences of pairs are exact.

There is a natural homomorphism

$$\theta : \check{H}^n\Big(\bigcup_{i=1}^{\infty} \overline{U}_i, \bigcup_{i=1}^{\infty} \mathrm{Fr}\, U_i\Big) \to \prod_{i=1}^{\infty} \check{H}^n(\overline{U}_i, \mathrm{Fr}\, U_i)$$

which is induced by the inclusion of the pairs $(\overline{U}_i, \mathrm{Fr}\, U_i)$ in $(\bigcup_{i=1}^{\infty} \overline{U}_i, \bigcup_{i=1}^{\infty} \mathrm{Fr}\, U_i)$. This homomorphism is an isomorphism [3].

For every k, let η_k be any homomorphism of the infinite cyclic group $\mathbb{Z}$ into the group $\check{H}^n(\overline{U}_k, \mathrm{Fr}\, U_k)$ for which the composition $\psi_k \varphi_k^{-1} \eta_k$ is nontrivial.

Let γ be a homomorphism $\prod_{i=1}^{\infty} \mathbb{Z} \to \prod_{i=1}^{\infty} \check{H}^n(\overline{U}_i, \mathrm{Fr}\, U_i)$ which associates with every $(a_1, a_2, \ldots)$ the element $(\eta_1(a_1), \eta_2(a_2), \ldots)$. Let η be the mapping $\theta^{-1}\gamma : \prod_{i=1}^{\infty} \mathbb{Z} \to \check{H}^n(\bigcup_{i=1}^{\infty} \overline{U}_i, \bigcup_{i=1}^{\infty} \mathrm{Fr}\, U_i)$. Consider the mapping $\psi\varphi^{-1}\eta : \prod_{i=1}^{\infty} \mathbb{Z} \to \check{H}^n(X)$ and any element $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ of $\prod_{i=1}^{\infty} \mathbb{Z}$. Then $\eta(e_k)$ is an element of $\check{H}^n(\bigcup_{i=1}^{\infty} \overline{U}_i, \bigcup_{i=1}^{\infty} \mathrm{Fr}\, U_i)$ such that its projections

$$\check{H}^n(\bigcup_{i=1}^{\infty} \overline{U}_i, \bigcup_{i=1}^{\infty} \operatorname{Fr} U_i) \to \check{H}^n(\overline{U}_j, \operatorname{Fr} U_j)$$

are trivial if $j \neq k$ and are equal to $\eta_k(1) \neq 0$ if $j = k$.

Furthermore, it follows for $k \neq j$, by the commutative diagram below,

$$\begin{array}{ccccc} \check{H}^n(X, X\backslash \bigcup_{i=1}^{\infty} U_i) & \xleftarrow{\tau_k} & \check{H}^n(X, X\backslash U_k) & \longrightarrow & \check{H}^n(\overline{U}_j, \overline{U}_j) = 0 \\ & \searrow & \downarrow & \swarrow & \\ & & \check{H}^n(\overline{U}_j, \operatorname{Fr} U_j) & & \end{array}$$

that for $k \neq j$ all projections $\check{H}^n(X, X\backslash U_k) \to (\overline{U}_j, \operatorname{Fr} U_j)$ are trivial.

Therefore, $\theta\varphi\tau_k\varphi_k^{-1}\eta_k(1) = \gamma(e_k)$ and $\varphi^{-1}\eta(e_k) = \tau_k\varphi_k^{-1}\eta_k(1)$. So we may assume that $\psi\varphi^{-1}\eta(e_k) = \psi\tau_k\varphi_k^{-1}\eta_k(1) = \psi_k\varphi_k^{-1}\eta_k(1)$ and by our choice of the homomorphisms η_k, this element is non-trivial.

Consequently, for every k the homomorphism $\psi\varphi^{-1}\eta : \prod_{i=1}^{\infty} \mathbb{Z} \to \check{H}^n(X)$ maps the elements e_k to nontrivial elements of the group $\check{H}^n(X)$. But this is impossible since by hypothesis the group $\check{H}^n(X)$ is slender [13]. ∎

PROPOSITION 2.12. *Let X be a metrizable locally compact $\check{H}^n$-bubble and let $\check{H}^n(X)$ be a countable group. Then X is compact.*

Proof. Suppose that X is noncompact. Then, because $\check{H}^n$-bubbles are connected it follows that $X = \bigcup_{i=1}^{\infty} K_i$, where K_i, $i \in \mathbb{N}$, are compact spaces and $K_i \subset \operatorname{int} K_{i+1}$, $K_i \neq K_{i+1}$. Consider the generalized Milnor exact sequence

$$0 \to \varprojlim^{(1)}\{\check{H}^{n-1}(K_i)\}_i \to \check{H}^n(X) \to \varprojlim \check{H}^n(K_i) \to 0.$$

For every compact metrizable space K_i, the group $\check{H}^{n-1}(K_i)$ is countable. The group $\check{H}^n(X)$ is countable by hypothesis. Next, $\check{H}^n(K_i) = 0$ because X is an $\check{H}^n$-bubble. By Lemma 2.4, $\varprojlim^{(1)}\{\check{H}^{n-1}(K_i)\}_i = 0$. It follows that $\check{H}^n(X) = 0$. However, X is an $\check{H}^n$-bubble. Contradiction. ∎

QUESTION 2.13. *Is every paracompact $\check{H}^n$-bubble compact?*

QUESTION 2.14. *Let X be an $\check{H}^n$-bubble. To which group can $\check{H}^n(X)$ be isomorphic?*

DEFINITION 2.15. A topological space is said to be *hereditarily $\check{H}^n$-trivial* (n is a fixed number) if the cohomology $\check{H}^n$ of every subspace is trivial. It is said to be *hereditarily locally cohomologically n-trivial* ($L\check{H}^n$-trivial) if every point has a hereditarily $\check{H}^n$-trivial neighborhood.

LEMMA 2.16. *Let X be a compact metrizable $L\check{H}^n$-trivial space. Then for every open subspace U, the cohomology $\check{H}^n(U)$ is at most countable.*

Proof. Let $\{U_i\}_{i=1}^k$ be any finite open cover of X. Since X is compact and $\check{H}^n(X)$ is isomorphic to the direct limit of the n-dimensional cohomology groups of the nerves of finite covers, the group $\check{H}^n(X) = \check{H}^n(\bigcup_{i=1}^k U_i)$ is at most countable. Since X is an $L\check{H}^n$-trivial space we may assume that all subspaces U_i are $\check{H}^n$-trivial and that for some index m ($m \le k$) the group $\check{H}^n(\bigcup_{i=1}^m U_i)$ is at most countable. Then we prove that the group $\check{H}^n(\bigcup_{i=1}^{m-1} U_i)$ is at most countable. For this we consider the Mayer–Vietoris sequence

$$\check{H}^n\Big(\bigcup_{i=1}^m U_i\Big) \to \check{H}^n\Big(\bigcup_{i=1}^{m-1} U_i\Big) \oplus \check{H}^n(U_m) \to \check{H}^n\Big(\Big(\bigcup_{i=1}^{m-1} U_i\Big) \cap U_m\Big).$$

The groups $\check{H}^n(U_m)$ and $\check{H}^n((\bigcup_{i=1}^{m-1} U_i) \cap U_m)$ are trivial and hence the homomorphism $\check{H}^n(\bigcup_{i=1}^m U_i) \to \check{H}^n(\bigcup_{i=1}^{m-1} U_i)$ is an epimorphism. By induction, we may conclude that the n-dimensional Čech cohomology of the union of any number of elements of the family $\{U_i\}_{i=1}^n$ is at most countable (because we can renumber this family). Now let U be any open subspace of X. Then add the family $\{U_i \cap U\}_{i=1}^k$ to $\{U_i\}_{i=1}^k$ to conclude that $\check{H}^n(U) = \check{H}^n(\bigcup_{i=1}^k (U_i \cap U))$ is at most countable. ∎

Lemma 2.17. *Let A be any subspace of a metrizable space X. Then $\check{H}^n(A) = \varinjlim \check{H}^n(U)$ (the direct limit is taken over a directed family of all open neighborhoods of A).*

A proof of this lemma can be found in [3, 6]. ∎

Lemma 2.18. *Let L be an $L\check{H}^n$-trivial compact metrizable space and suppose that $\check{H}^n(L) = 0$. Then L is hereditarily $\check{H}^n$-trivial.*

Proof. For every open subspace U of L there exist the following generalized Milnor exact sequences:

$$0 \to \varprojlim{}^{(1)} \check{H}^{n-1}(K_i) \to \check{H}^n(U) \to \varprojlim \check{H}^n(K_i) \to 0,$$

where $\{K_i\}_{i=1}^\infty$ are compact subspaces, $K_i \subset \operatorname{int} K_{i+1}$, $\bigcup_{i=1}^\infty K_i = U$. It follows by Lemmas 2.4 and 2.16 that $\varprojlim{}^{(1)} \check{H}^{n-1}(K_i) = 0$. By the n-dimensionality of L (since L is $L\check{H}^n$-trivial, $\dim_{\mathbb{Z}} L \le n$) and its acyclicity it follows that $\check{H}^n(K_i) = 0$. Thus $\check{H}^n(U) = 0$ and by Lemma 2.17, $\check{H}^n(A) = 0$ for any subspace A of L. ∎

Lemma 2.19. *Every compact metrizable $L\check{H}^n$-trivial n-bubble C is an $\check{H}^n$-bubble.*

Proof. For every open proper subspace U of an n-bubble C there exist exact sequences

$$0 \to \varprojlim{}^{(1)} \check{H}^{n-1}(C_i) \to \check{H}^n(U) \to \varprojlim \check{H}^n(C_i) \to 0$$

in which $\{C_i\}_{i=1}^{\infty}$ is an increasing system of compact subspaces ($C_i \subset \operatorname{int} C_{i+1}$) and $\bigcup_{i=1}^{\infty} C_i = U$. By Lemmas 2.4 and 2.16 it follows that $\varprojlim^{(1)} \check{H}^{n-1}(C_i) = 0$. The groups $\check{H}^n(C_i)$ are trivial because C is a q-bubble. Therefore $\check{H}^n(U) = 0$ and according to Lemma 2.17, C is an $\check{H}^n$-bubble. ■

3. A 2-dimensional 2-acyclic ANR compactum without $\check{H}^n$-bubbles

THEOREM 3.1. *There exists a* 2*-dimensional* 2*-acyclic compact metrizable ANR which does not contain any* $\check{H}^2$*-bubbles.*

Proof. Let p and q be two relatively prime integers. Let f and g be natural mappings of degree p and q of S^1 to S^1, respectively. Let P_i, $i \in \mathbb{N}$, be CW-complexes homeomorphic to the cylinder of the mapping f. Let P be a 2-dimensional polyhedron which is obtained from the topological sum $\bigoplus P_i$ ($i \in \mathbb{N}$) by attaching for every i, the upper base of P_i to the lower base P_{i+1} [5]. Let Q be a polyhedron constructed similarly to P, except that Q_i are CW-complexes homeomorphic to the cylinder of the mapping g.

Let $X_1 = P^*$ and $X_2 = Q^*$ be the one-point compactifications of P and Q, respectively. Let X be a compact space constructed from X_1 and X_2 by attaching the lower base of $P_1 \subset X_1$ to the lower base $Q_1 \subset X_2$. Because X_1 and X_2 are contractible compact spaces (the upper base of every cylinder is its strong deformation retract) it follows by the Mayer–Vietoris sequence that $\check{H}^2(X) = \mathbb{Z}$. Since X is locally contractible and finite-dimensional, it is an ANR [2].

Let us prove that X does not contain any $\check{H}^2$-bubbles. Indeed, let M be an $\check{H}^2$-bubble of X. Then there are two possibilities:

(1) There exists a natural number n_0 such that either $\bigcup_{i=n_0}^{\infty} P_i$ or $\bigcup_{i=n_0}^{\infty} Q_i$ lies in M;

(2) There exist sequences $\{p_k\}_{k=1}^{\infty}$, $\{q_k\}_{k=1}^{\infty}$ of points in X and increasing index sequences $\{i_k\}_{k=1}^{\infty}$ and $\{j_k\}_{k=1}^{\infty}$ such that $p_k \in P_{i_k} \setminus M$, $q_k \in Q_{j_k} \setminus M$.

Case 1. Let $\bigcup_{i=n_0}^{\infty} P_i$ be a proper subset of M. Then obviously

$$H^2\Big(\bigcup_{i=n_0}^{\infty} P_i\Big) = \varprojlim^{(1)} (\mathbb{Z} \overset{p}{\leftarrow} \mathbb{Z} \overset{p}{\leftarrow} \mathbb{Z} \overset{p}{\leftarrow} \dots)$$

where all projections are multiplications by p. Since these sequences do not satisfy the ML condition, $H^2(\bigcup_{i=n_0}^{\infty} P_i) \neq \emptyset$ and M is not an $\check{H}^2$-bubble.

Case 2. Consider X as the union of the following subsets:

$$X = \Big(\bigcup_{k=1}^{\infty} A_k^-\Big) \cup \Big(\bigcup_{k=1}^{\infty} B_k^-\Big) \cup A_0 \cup \Big(\bigcup_{k=1}^{\infty} A_k^+\Big) \cup \Big(\bigcup_{k=1}^{\infty} B_k^+\Big) \cup \{t^-, t^+\}$$

where all subsets are closed in X, the systems $\{A_k^-, A_k^+, A_0\}_{k=1}^\infty$ and $\{B_k^-, B_k^+\}_{k=1}^\infty$ are disjoint, and the points t^-, t^+ are the compactification points of P and Q, respectively. All intersections $A_k^- \cap B_k^-$, $A_k^- \cap B_{k+1}^-$, $A_k^+ \cap B_k^+$, $A_k^+ \cap B_{k+1}^+$, $A_0 \cap B_1^-$, $A_0 \cap B_1^+$ are homeomorphic to S^1 and no one of them lies in M because each contains some of the points p_k or q_k. Each of those subsets (apart from the points t^- and t^+) lies in some finite polyhedron $\bigcup_{i=1}^m (P_i \cup Q_i)$. Applying the Mayer–Vietoris sequence we can conclude that $\check{H}^2(M)$ is isomorphic to the direct sum

$$\check{H}^2\Big(M \cap \Big(\bigcup_{i=1}^\infty A_k^- \cup \bigcup_{i=1}^\infty A_k^+ \cup A_0 \cup \{t^-, t^+\}\Big)\Big) \oplus \check{H}^2\Big(M \cap \Big(\bigcup_{i=1}^\infty B_k^- \cup \bigcup_{i=1}^\infty B_k^+ \cup \{t^-, t^+\}\Big)\Big).$$

By the Mayer–Vietoris formula

$$\check{H}^1\Big(\bigcup_{i=1}^m P_i\Big) \oplus \check{H}^1\Big(\bigcup_{i=1}^m Q_i\Big) \to \check{H}^1(S^1) \to \check{H}^2\Big(\bigcup_{i=1}^m (P_i \cup Q_i)\Big) \to 0$$

and because p and q are relatively prime it follows that $\bigcup_{i=1}^m (P_i \cup Q_i)$ is acyclic in dimension 2. By Lemma 2.18, all of the groups $\check{H}^2(M \cap A_k^-)$, $\check{H}^2(M \cap A_k^+)$, $\check{H}^2(M \cap B_k^-)$, $\check{H}^2(M \cap B_k^+)$ are trivial. It follows that $\check{H}^2(M) = 0$ (see [3]) and M is not an $\check{H}^2$-bubble. ∎

The following problems are interesting in connection with this theorem:

QUESTION 3.2 (Kuperberg [9]). *Does every $(n+1)$-dimensional compactum contain an n-bubble?*

QUESTION 3.3. *Does there exist an $(n+1)$-dimensional compact ANR without $\check{H}^n$-bubbles?*

4. A sufficient condition for existence of $\check{H}^n$-bubbles in n-dimensional compacta

THEOREM 4.1. *Every n-acyclic finite-dimensional $L\check{H}^n$-trivial metrizable compactum X contains an $\check{H}^n$-bubble.*

LEMMA 4.2. *Let X be an $L\check{H}^n$-trivial space. Let $C = \bigcap_{i=1}^\infty U_i$, $X \supset U_i \supset \overline{U}_{i+1}$, $\dim \operatorname{Fr} U_i \le n-1$ and each U_i is open. Then there exists k such that $\check{H}^n(\overline{U}_i \backslash U_j)$ is trivial for every $j > i > k$.*

Proof. It follows from the exact sequence

$$0 \to \varprojlim{}^{(1)} \check{H}^{n-1}(X \backslash U_i) \to \check{H}^n(X \backslash C) \to \varprojlim \check{H}^n(X \backslash U_i) \to 0,$$

and the countability of $\check{H}^n(X\backslash C)$ (Lemma 2.16) and Lemma 2.4 that $\varprojlim^{(1)}\check{H}^{n-1}(X\backslash U_i) = 0$. Therefore, $\check{H}^n(X\backslash C) = \varprojlim \check{H}^n(X\backslash U_i)$. In the spectrum $\check{H}^n(X\backslash U_1) \leftarrow \check{H}^n(X\backslash U_2) \leftarrow \check{H}^n(X\backslash U_3) \leftarrow \ldots$ only a finite number of projections can be non-monomorphisms. (It follows from dimension theory that all these homomorphisms are epimorphisms; if an infinite number of projections were not monomorphisms then $\varprojlim$ would not be countable, but this would contradict the countability of $\check{H}^n(X\backslash C)$.) Consequently, for some k all homomorphisms $\check{H}^q(X\backslash U_i) \leftarrow \check{H}^q(X\backslash U_j)$ where $j > i > k$ are isomorphisms. It follows from this and from the exact sequence

$$\check{H}^n(X\backslash U_j) \to \check{H}^n(X\backslash U_i) \oplus \check{H}^n(\overline{U}_i\backslash U_j) \to \check{H}^n(\operatorname{Fr} U_i)$$

in which $\check{H}^n(\operatorname{Fr} U_i) = 0$ ($\dim \operatorname{Fr} U_i \le n-1$) that $\check{H}^n(\overline{U}_i\backslash U_j) = 0$. ■

LEMMA 4.3. *Suppose that in the commutative diagram with exact horizontal lines*

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\varphi_1} & A_2 & \xrightarrow{\varphi_2} & A_3 & \longrightarrow & 0 \\
\uparrow{\scriptstyle \varphi_3} & & \uparrow{\scriptstyle \varphi_4} & & \uparrow{\scriptstyle \varphi_5} & & \\
B_1 & \xrightarrow{\varphi_6} & B_2 & \xrightarrow{\varphi_7} & B_3 & \longrightarrow & 0
\end{array}$$

the homomorphism φ_4 is a monomorphism and φ_5 is not. Then $\operatorname{Im}\varphi_1 \ne \operatorname{Im}(\varphi_4\varphi_6)$.

Proof. Let $\alpha \in B_3$, $\alpha \ne 0$ and $\varphi_5(\alpha) = 0$. Let $\beta \in B_2$ and $\varphi_7(\beta) = \alpha$. Then $\varphi_4(\beta) \in \operatorname{Im}\varphi_1$. If $\operatorname{Im}\varphi_1 = \operatorname{Im}(\varphi_4\varphi_6)$ then there exists γ such that $\varphi_4\varphi_6(\gamma) = \varphi_4(\beta)$. Since φ_4 is monomorphism, it follows that $\varphi_6(\gamma) = \beta$ and $\varphi_7\varphi_6(\gamma) = \varphi_7(\beta) = \alpha$. But by the exactness condition $\varphi_7\varphi_6 = 0$, hence $\alpha = 0$, which contradicts the choice of α. ■

LEMMA 4.4. *In the commutative diagram*

$$\begin{array}{ccc}
A_1 & \xleftarrow{\psi_1} & A_2 \\
\uparrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} \\
B_1 & \xleftarrow{\psi_4} & B_2
\end{array}$$

let ψ_1 be an epimorphism, ψ_4 a monomorphism and ψ_3 not an epimorphism. Then ψ_2 is not a monomorphism.

Proof. Let ψ_2 be a monomorphism, $\alpha \in B_2$ and $\alpha \notin \operatorname{Im}\psi_3$. Then for some β, $\psi_2\psi_4(\alpha) = \psi_1(\beta)$. From this and by commutativity of the diagram it follows that $\psi_2\psi_4\psi_3(\beta) = \psi_1(\beta) = \psi_2\psi_4(\alpha)$ and $\psi_3(\beta) = \alpha$ but this contradicts the choice of α. ■

Let $a \in \check{H}^n(X)$. By a *support* of a we mean any closed subspace F of X such that the restriction of a to F is not trivial. The support F_2 is less than F_1 if $F_2 \subset F_1$. This relation is obviously a partial ordering on the set of all supports of the element a. It follows by the continuity property of Čech cohomology that for any linearly ordered set of supports their intersection is a support of a. By the Kuratowski–Zorn Lemma there is a *minimal support* of a.

Proof of Theorem 4.1. Since X is compact, the group $\check{H}^n(X)$ must be countable. Let $\{a_i\}$ be a countable set of generators of $\check{H}^n(X)$ and S_0 be a minimal support of a_1. Let i_1 be the smallest index such that $i_1 > 1$ and the restriction of a_{i_1} to S_0 is nontrivial. If such an index does not exist then obviously S_0 is an n-bubble. Let S_1 be a minimal support of a_{i_1} which is contained in S_0. Suppose that a minimal support S_j of some element a_{i_j} has been constructed. Let i_{j+1} be the smallest index such that $i_{j+1} > i_j$ and the restriction of $a_{i_{j+1}}$ to S_j is not trivial. If such an index does not exist then S_j is an n-bubble. Let S_{j+1} be a minimal support of $a_{i_{j+1}}$ which is contained in S_j. By induction we get a chain $S_0 \supset S_1 \supset S_2 \supset \ldots$ of minimal supports of the elements $a_1, a_{i_1}, a_{i_2}, \ldots$

We will prove that this chain is finite and $\check{H}^n(X)$ is finitely generated, i.e. for some m, $S_k = S_{k+1}$ when $k > m$. Indeed, otherwise we have a decreasing sequence of closed sets $S_0 \supset S_1 \supset S_2 \supset \ldots$ and so a countable number of nontrivial elements $a_{i_j} \in \check{H}^n(S_j)$ such that the restrictions of a_{i_j} to $\check{H}^n(S_{j+1})$ are trivial. By the continuity property of Čech cohomology it follows that for every a_{i_j}, there exists $b_j \in \check{H}^n(U_j)$, where U_j is a neighborhood of S_j, such that:

(1) $b_j|_{S_j} = a_{i_j}$ and thus is not trivial;
(2) $U_j \supset \overline{U}_{j+1}$;
(3) $b_j|_{\overline{U}_{j+1}} = 0$; and
(4) $\operatorname{Ind}\operatorname{Fr} U_j \le n-1$.

According to Lemma 4.2, $\check{H}^n(\overline{U}_k \backslash U_i) = 0$ for some k and all $i > k$. For simplicity, we can pass to subchains and suppose that $k = 0$, $U_0 = U$, $\operatorname{Ind}\operatorname{Fr} U_i \le n-1$, and the homomorphisms $\check{H}^n(\overline{U}_i) \to \check{H}^n(\overline{U}_{i+1})$ are not monomorphisms. Then we prove that the inverse sequences

$$\check{H}^{n-1}(\overline{U}\backslash U_1) \leftarrow \check{H}^{n-1}(\overline{U}\backslash U_2) \leftarrow \check{H}^{n-1}(\overline{U}\backslash U_3) \leftarrow \ldots$$

do not satisfy the Mittag-Leffler condition. It is sufficient to prove that for every i the images of the homomorphisms

$$\check{H}^{n-1}(\overline{U}\backslash U_1) \leftarrow \check{H}^{n-1}(\overline{U}\backslash U_i) \quad \text{and} \quad \check{H}^{n-1}(\overline{U}\backslash U_1) \leftarrow \check{H}^{n-1}(\overline{U}\backslash U_{i+1})$$

do not coincide.

Since $\check{H}^n(\overline{U}\backslash U_1) = 0$ we get for every i the commutative diagram with exact horizontal lines:

$$\begin{array}{ccccccc}
\check H^{n-1}(\overline{U}\backslash U_i) & \xrightarrow{\varphi_1} & \check H^{n-1}(\overline{U}\backslash U_1) & \xrightarrow{\varphi_2} & \check H^{n}(\overline{U}\backslash U_i, \overline{U}\backslash U_1) & \longrightarrow & 0\\
\uparrow{\scriptstyle \varphi_3} & & \uparrow{\scriptstyle \varphi_4} & & \uparrow{\scriptstyle \varphi_5} & & \\
\check H^{n-1}(\overline{U}\backslash U_{i+1}) & \xrightarrow{\varphi_6} & \check H^{n-1}(\overline{U}\backslash U_1) & \xrightarrow{\varphi_7} & \check H^{n}(\overline{U}\backslash U_{i+1}, \overline{U}\backslash U_1) & \longrightarrow & 0
\end{array}$$

To prove that $\operatorname{Im}\varphi_1 \neq \operatorname{Im}\varphi_4\varphi_6$ it is sufficient by Lemma 4.3 to prove that φ_5 is not a monomorphism.

Consider another diagram in which all homomorphisms are induced by inclusions:

$$\begin{array}{ccc}
\check H^{n}(\overline{U}\backslash U_i, \overline{U}\backslash U_1) & \xleftarrow{\psi_1} & \check H^{n}(\overline{U}, (\overline{U}\backslash U_1)\cup \overline{U}_i)\\
\uparrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3}\\
\check H^{n}(\overline{U}\backslash U_{i+1}, \overline{U}\backslash U_1) & \xleftarrow{\psi_4} & \check H^{n}(\overline{U}, (\overline{U}\backslash U_1)\cup \overline{U}_{i+1})
\end{array}$$

The homomorphisms ψ_1 and ψ_4 are excision isomorphisms. To prove that $\varphi_5 = \psi_2$ is not a monomorphism it suffices by Lemma 4.4 to prove that ψ_3 is not an epimorphism.

It follows from the $L\check H^n$-triviality that $\check H^{n+1}(\overline{U}, (\overline{U}\backslash U_1)\cup \overline{U}_i) = 0$ for every i. So we have a commutative diagram with exact lines:

$$\begin{array}{ccccccc}
\check H^{n}(\overline{U}, (\overline{U}\backslash U_1)\cup \overline{U}_i) & \longrightarrow & \check H^{n}(\overline{U}) & \longrightarrow & \check H^{n}((\overline{U}\backslash U_1)\cup \overline{U}_i) & \longrightarrow & 0\\
\downarrow{\scriptstyle \psi_3} & & \downarrow & & \downarrow{\scriptstyle \psi_5} & & \\
\check H^{n}(\overline{U}, (\overline{U}\backslash U_1)\cup \overline{U}_{i+1}) & \longrightarrow & \check H^{n}(\overline{U}) & \longrightarrow & \check H^{n}((\overline{U}\backslash U_1)\cup \overline{U}_{i+1}) & \longrightarrow & 0
\end{array}$$

To prove that ψ_3 is not an epimorphism it suffices by Lemma 4.3 to prove that ψ_5 is not a monomorphism. But this follows by the choice of the decreasing sequences of sets $\{U_i\}$ (recall that $\check H^n(\overline{U}_i) \to \check H^n(\overline{U}_{i+1})$ is not a monomorphism).

Hence in the spectrum

$$\check H^{n-1}(\overline{U}\backslash U_1) \leftarrow \check H^{n-1}(\overline{U}\backslash U_2) \leftarrow \check H^{n-1}(\overline{U}\backslash U_3) \leftarrow \dots$$

the Mittag-Leffler condition is not satisfied.

By Lemma 2.4, the $\varprojlim^{(1)}$ of this spectrum has the power of continuum and by the exactness of

$$0 \to \varprojlim{}^{(1)} \check H^{n-1}(\overline{U}\backslash U_i) \to \check H^n\Big(\overline{U}\backslash \bigcap_{i=1}^{\infty} U_i\Big)$$

it follows that $\check H^n(\overline{U}\backslash \bigcap_{i=1}^{\infty} U_i)$ has the power of continuum. But this contradicts Lemma 2.16, because $\overline{U}$ is obviously an LH^n-trivial compact space.

Therefore there exists m such that the chain $S_0 \supset S_1 \supset S_2 \supset \ldots$ of the minimal supports consists of m elements $S_0 \supset S_1 \supset \ldots \supset S_m$, i.e. for all $k \geq m$, $S_k = S_{k+1}$. Then S_m is an n-bubble. (It follows from dimension theory that for every closed subset F of X, $\check{H}^n(X) \to \check{H}^n(F)$ is an epimorphism and by construction—if F is a proper subset of S_m—this homomorphism is trivial.) By Lemma 2.19, S_m is an $\check{H}^n$-bubble. ∎

PROBLEM 4.5. *Can one omit the condition of finite-dimensionality from Theorem* 4.1*?*

REFERENCES

[1] P. S. Aleksandrov, *Dimensionstheorie. Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Math. Ann. 106 (1932), 161–238.

[2] K. Borsuk, *Theory of Retracts*, Monograf. Mat. 44, PWN, Warszawa, 1967.

[3] G. E. Bredon, *Sheaf Theory*, 2nd ed., Springer, New York, 1997.

[4] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[5] D. B. Fuks and V. A. Rokhlin, *Introductory Course in Topology*: *Geometric Chapters*, Nauka, Moscow, 1977 (in Russian).

[6] R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1958.

[7] A. E. Harlap, *Local homology and cohomology, homological dimension and generalized manifolds*, Mat. Sb. 96 (1975), 347–373 (in Russian); English transl.: Math. USSR-Sb. 25 (1975), 323–349.

[8] U. H. Karimov, *On the generalized homotopy axiom*, Izv. Akad. Nauk Tadžik. SSR Otdel. Fiz.-Mat. Khim. i Geol. Nauk 71 (1979), 83–84 (in Russian).

[9] W. Kuperberg, *On certain homological properties of finite-dimensional compacta. Carries, minimal carries and bubbles*, Fund. Math. 83 (1973), 7–23.

[10] K. Kuratowski, *Topology*, Vol. 2, Academic Press, New York, 1968.

[11] S. Mardešić and J. Segal, *Shape Theory*: *The Inverse System Approach*, North-Holland, Amsterdam 1982.

[12] W. J. R. Mitchell, *Homology manifolds, inverse systems and cohomological local connectedness*, J. London Math. Soc. (2) 19 (1979), 348–358.

[13] E. Sąsiada, *Proof that every countable and reduced torsion-free abelian group is slender*, Bull. Acad. Polon. Sci. 7 (1959), 143–144.

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