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ON AXIAL MAPS OF THE DIRECT PRODUCT OF FINITE SETS

BY

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We show that every function $f : A \times B \to A \times B$, where A and B are finite sets, is a composition of 5 axial functions.

A function $f : A\times B \to A\times B$ is called *vertical* if there exists $g : A\times B \to A$ such that $f(a, b) = (g(a, b), b)$, and is called *horizontal* if $f = (a, g(a, b))$ for some $g : A\times B \to B$. Both types of functions are called *axial*. A function which is one-to-one and onto is called a *permutation*. By $\#A$ we denote the number of elements of the set A.

Ehrenfeucht and Grzegorek in [EG] (Th. iv) proved the following

THEOREM 1. *If A and B are finite sets then every function $f : A \times B \to A \times B$ can be represented as a composition $f = f_1 \circ \ldots \circ f_6$, where f_i are axial functions and f_1 is horizontal.*

In that paper the following problem was stated (P910): Is it possible to decrease the number 6 in the theorem above? The aim of this paper is to show that we can put 5 in place of 6.

For other results concerning axial functions see [G] and the references there.

We shall use the following fact from [EG] (Th. iii):

THEOREM 2. *If $\#B < \aleph_0$ (while A may be of arbitrary finite or infinite cardinality), then every permutation p of $A \times B$ can be represented as a composition $p = p_1 \circ p_2 \circ p_3$, where all p_i are axial permutations of $A \times B$ and p_1 is horizontal.*

Our main result is

THEOREM. *If A, B are finite sets and $f : A \times B \to A \times B$ is arbitrary, then there exist axial functions $f_i : A \times B \to A \times B$ $(i = 1, \ldots, 5)$ such that $f = f_1 \circ \ldots \circ f_5$ and f_1 is horizontal.*

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Lemma 1. *If A, B are finite sets and $f : A\times B \to A\times B$, then there exist axial functions $f_i : A\times B\to A\times B$ $(i=1,2,3)$ such that*

$$\forall_{(a,b)\in A\times B}\quad \#(f_1f_2f_3)^{-1}(a,b)=\#f^{-1}(a,b)$$

and f_1 is a horizontal permutation.

We can rewrite Lemma 1 in another form.

Lemma 2. *Let $\{n_{ab}\}$ be a set of natural numbers indexed by pairs from $A\times B$ such that*

$$\sum_{(a,b)\in A\times B} n_{ab}=\#A\cdot\#B.$$

Then there exist axial functions $f_i : A\times B\to A\times B$ $(i=1,2,3)$ such that

$$\forall_{(a,b)\in A\times B}\quad \#(f_1f_2f_3)^{-1}(a,b)=n_{ab}$$

and f_1 is a horizontal permutation.

We start with a few definitions. If $M=[m_{ab}]$ is a matrix with elements indexed by pairs $(a,b)\in A\times B$ and $f : A\times B\to A\times B$ then $f[M]=[m'_{ab}]$ where $m'_{ab}=m_{f(a,b)}$. With that approach it does not matter what are the elements of the matrix, we are dealing only with coordinates. Note that if $f,g : A\times B\to A\times B$ then $f[g[M]]=(g\circ f)[M]$.

Let $X=[(a,b)]_{(a,b)\in A\times B}$ (i.e. the element (a,b) stands at the place (a,b)) and let $r_a=\{(a,b): b\in B\}$ be the "ath" row in the matrix X. The matrix $f[X]$ determines the function f completely, and to prove the lemma we show that the number of occurrences of the element (a,b) in $f_3[f_2[f_1[X]]]$ is n_{ab} for every $(a,b)\in A\times B$.

Proof of Lemma 2 (induction on $\#A$). For $\#A=1$ the lemma is trivial, since in this case every function is axial (horizontal). Assume that

$$(*)\qquad \forall_{A,B(\#A=n)}\forall_{\{n_{ab}\},\sum n_{ab}=\#A\cdot\#B}\exists_{f_1,f_2,f_3}\text{ axial functions }\forall_{(a,b)\in A\times B}$$
$$\#(f_1f_2f_3)^{-1}(a,b)=n_{ab}\text{ and }f_1\text{ is a horizontal permutation.}$$

Let now $\#A=n+1$. For $a\in A$ let $w_a=\sum_{b\in B}n_{ab}$. Clearly $\sum_{a\in A}w_a=\#A\cdot\#B=(n+1)\#B$. There are $a_1,a_2\in A(a_1\neq a_2)$ such that $w_{a_1}\le\#B$ and $w_{a_2}\ge\#B$. Let $\{b_1,\ldots,b_{\#B}\}=B$ be an ordering such that the numbers $n_{a_2b_i}$ decrease (weakly).

Let

$$k=\min\Big\{m:\sum_{i=1}^{m}n_{a_2b_i}+w_{a_1}\ge\#B\Big\},\qquad s=\sum_{i=1}^{k}n_{a_2b_i}+w_{a_1}-\#B$$

(if $w_{a_1}=\#B$ then $k=0$; note that $n_{a_2b_i}>0$ for $i\le k$).

In the row r_{a_1} there exist at least k "null elements", i.e. elements (a_1,b) such that $n_{a_1b}=0$ (indeed, if there were fewer than k null elements (a_1,b)

then $w_{a_1} \geq \#B - (k-1)$ and

$$\sum_{i=1}^{k-1} n_{a_2 b_i} + w_{a_1} \geq k - 1 + \#B - (k-1) \geq \#B,$$

so k would not be minimal).

Let $A' = A \setminus \{a_1\}$. We define numbers n'_{ab} for $(a,b) \in A' \times B$ by

$$n'_{ab} = \begin{cases} 0 & \text{for } (a,b) = (a_2, b_i),\ 1 \leq i < k, \\ s & \text{for } (a,b) = (a_2, b_k), \\ n_{ab} & \text{otherwise.} \end{cases}$$

It is easy to check that $\sum_{(a,b)\in A'\times B} n'_{ab} = \#A' \cdot \#B$.

Let us assign to each element (a_2, b_i) $(1 \leq i \leq k)$, in a one-to-one way, a null element (a_1, b_{l_i}) and define

$$n'_{a_1 b} = \begin{cases} n_{a_2 b_i} & \text{for } (a_1,b) = (a_1, b_{l_i}), \\ n_{a_2 b_k} - s & \text{for } (a_1,b) = (a_1, b_{l_k}), \\ n_{a_1 b} & \text{for } (a_1,b) \neq (a_1, b_{l_i}). \end{cases}$$

Note that $\sum_{b\in B} n'_{a_1 b} = \#B$.

From $(*)$ there exist axial functions f'_1, f'_2, f'_3 from $A' \times B$ to $A' \times B$ such that the assertion of the lemma holds and f'_1 is a horizontal permutation.

We now construct functions f_1, f_2 and f_3.

We define f_1 as an extension of the permutation f'_1 to $A \times B$. Namely, f_1 acts on r_{a_1} so that in $f_1[X]$ each element (a_1, b_{l_i}) is in the same column as (a_2, b_i), $1 \leq i \leq k$, and the other elements (a_1, b) have arbitrary positions.

f_2 is an extension of f'_2 to $A \times B$. In the row r_{a_1} of $f_1[X]$ we replace every null element (a_1, b_{l_1}) by the element (a_2, b_i) (they are in the same column).

f_2 is defined to act on $f_1[X]$ so that the elements (a_2, b_i), $1 \leq i \leq k$, are "copied" to the places where the elements (a_1, b_{l_i}) stand, more precisely: if the element $(a_2, b_i), 1 \leq i \leq k$, in the matrix $f_1[X]$ stands at place (a_2, y) (and so (a_1, b_{l_i}) stands at place (a_1, y)) then $f_2(a_1, y) = (a_2, y)$, $f_2(a_1, y) = (a_1, y)$ for other elements.

Although in the matrix $f'_2[f'_1[X']]$ there may be no elements (a_2, b_i), $i \leq k$ ($n'_{a_2 b_i} = 0$ for $i < k$), in $f_2[f_1[X]]$ they have been "saved" by moving them to the row r_{a_1}.

Finally, we extend f'_3 to the set $A \times B$ obtaining f_3 as follows: f_3 first permutes the row $f_1 f_2[r_{a_1}]$ so that (a_1, b) stands at place (a_1, b) and (a_2, b_i) stands at place (a_1, b_{l_i}). Then f_3 puts each element standing at place (a_1, b) at $n'_{a_1 b}$ places ($\sum_{b\in B} n'_{a_1 b} = \#B$).

In the matrix $f_3[f_2[f_1[X]]]$ the elements $(a_2, b_i), i < k$, are only in the row r_{a_1}, and they appear at $n'_{a_1 b_{l_i}} = n_{a_2 b_i}$ places. The element (a_2, b_k) appears at $s + (n_{a_2 b_k} - s)$ places and other elements (a, b) appear at $n'_{ab} = n_{ab}$ places. So the lemma is proved. ∎

Proof of the Theorem. There exists a permutation p such that $p[f_3[f_2[f_1[X]]]] = f[X]$. By Theorem 2 we can represent p as $p_1 \circ p_2 \circ p_3$, where all p_i are axial permutations and p_1 is horizontal. Thus the function $F_3 = f_3 \circ p_1$ is axial (horizontal) and $f_1 \circ f_2 \circ F_3 \circ p_2 \circ p_3[X] = f[X]$. ■

Remark. We still do not know whether 5 is a minimal number. We know, however, that number 3 is not enough (a joint result with E. Grzegorek). To see this we note an observation:

(A) *Let M, N be matrices of the same size. The existence of functions $f_1, f_2 : A \times B \to A \times B$, with f_1 vertical and f_2 horizontal, such that $f_2[f_1[M]] = N$ is equivalent to the fact that for each row W of N there exists a selector S from the columns of M such that $W^* \subseteq S$, where W^* is the set of all elements of the row W.*

Obviously, we also have:

(B) *Let M, N be matrices of the same size. The existence of functions $f_1, f_2 : A \times B \to A \times B$, with f_1 horizontal and f_2 vertical, such that $f_2[f_1[M]] = N$ is equivalent to the fact that for each column W of N there exists a selector S from the rows of M such that $W^* \subseteq S$, where W^* is the set of all elements of the column W.*

Thus, the 3×2 matrix

$$\begin{bmatrix} A & B \\ A & D \\ A & C \end{bmatrix} \text{ cannot be obtained from } \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix}$$

using three axial functions f_1, f_2, f_3, where f_1 is horizontal.

Analogously, the 2×3 matrix

$$\begin{bmatrix} A & A & A \\ B & C & D \end{bmatrix} \text{ cannot be obtained from } \begin{bmatrix} A & C & E \\ B & D & F \end{bmatrix}$$

using three axial functions f_1, f_2, f_3, where f_1 is vertical.

The $m \times n$ matrix (where ($m \ge 5$ and $n \ge 4$) or ($m \ge 4$ and $n \ge 5$))

$$X = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1,n-2} & b_{1,n-1} & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2,n-2} & b_{2,n-1} & b_{2n} \\ b_{31} & b_{32} & \ldots & b_{3,n-2} & b_{3,n-1} & b_{3n} \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ b_{m1} & b_{m2} & \ldots & b_{m,n-2} & b_{m,n-1} & b_{mn} \end{bmatrix}$$

cannot be transformed into

$$X' = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1,n-2} & b_{1,n-1} & b_{1n} \\ b_{11} & b_{22} & \ldots & b_{2,n-2} & b_{2,n-1} & b_{2n} \\ b_{11} & b_{21} & \star\star\star & \star & b_{3,n-1} & b_{3n} \\ \star & \star & \star\star\star & \star & \cdot & \cdot \\ \star & \star & \star\star\star & \star & \cdot & \cdot \\ \star & \star & \star\star\star & \star & \cdot & \cdot \\ \star & \star & \star\star\star & b_{m,n-1} & b_{m-1,n-1} & b_{m-1,n} \\ \star & \star & \star\star\star & b_{m,n} & b_{m,n} & b_{m,n} \end{bmatrix}$$

(where the dots stand for the corresponding entries of X and the stars are arbitrary) by a function which is a composition of three axial functions.

This is visible if we look at the first three rows of X' (it is impossible to find a horizontal function f such that $f[X]$ would satisfy the condition from observation (A)), and at the last three columns of X' (it is impossible to find a vertical function f such that $f[X]$ would satisfy the condition from observation (B)). So neither starting with a horizontal nor with a vertical function can we obtain the matrix X' from the matrix X, using only three axial functions.

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