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WEAK BAER MODULES OVER GRADED RINGS

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In [2], Fuchs and Viljoen introduced and classified the B^*-modules for a valuation ring R: an R-module M is a B^*-module if $\mathrm{Ext}^1_R(M,X)=0$ for each divisible module X and each torsion module X with bounded order. The concept of a B^*-module was extended to the setting of a torsion theory over an associative ring in [14]. In the present paper, we use categorical methods to investigate the B^*-modules for a group graded ring. Our most complete result (Theorem 4.10) characterizes B^*-modules for a strongly graded ring R over a finite group G with $|G|^{-1} \in R$. Motivated by the results of [8], [9], [10] and [15], we also study the condition that every non-singular R-module is a B^*-module with respect to the Goldie torsion theory; for the case in which R is a strongly graded ring over a group, extensive information is obtained for group rings of abelian, solvable and polycyclic-by-finite groups.

1. Notation and preliminaries. Throughout this paper, all rings R will be associative and with identity, and all modules will be left R-modules. The category of left R-modules will be denoted by R-Mod. Let G be a (multiplicative) group with identity. A *G-graded ring* is a ring together with a direct sum decomposition $R = \bigoplus_{g\in G} R_g$ (as additive subgroups) such that

$$R_g R_h \subseteq R_{gh} \quad \text{for all } g,h \in G. \tag{1}$$

It is well known that R_1 is a subring of R, and $1 \in R_1$. If in (1) we have equality, i.e. $R_g R_\tau = R_{g\tau}$ for all $g,\tau \in G$, then R is called a *strongly graded ring*. It is easy to see that R is strongly graded if and only if $R_g R_{g^{-1}} = R_1$ for any $g \in G$. If for any $g \in G$, R_g contains an invertible element, then R is called a *crossed product*. It is obvious that if R is a crossed product, then R is strongly graded. By a *left G-graded R-module* we mean a left R-module M plus an internal direct sum decomposition $M = \bigoplus_{g\in G} M_g$ (as additive

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subgroups) such that

$$R_g M_\tau \subseteq M_{g\tau} \quad \text{for all } g, \tau \in G.$$

Denote by R-gr the category of left G-graded R-modules. If $M = \bigoplus_{g\in G} M_g$ and $N = \bigoplus_{g\in G} N_g$ are two G-graded modules, then $\mathrm{Hom}_{R\text{-gr}}(M, N)$ consists of the R-homomorphisms $f : M \to N$ such that $f(M_g) \subseteq N_g$ for every $g \in G$. As is well known [6], R-gr is a Grothendieck category.

If $M = \bigoplus_{g\in G} M_g$ is a graded R-module, then $h(M)$ will stand for the set of all homogeneous elements of M; i.e. $h(M) = \bigcup_{g\in G} M_g \setminus \{0\}$. If $m \in M$, $m \neq 0$, then we can write $m = \sum_{g\in G} m_g$, where $m_g \in M_g$; the finite set $\{m_g \mid g \in G,\, m_g \neq 0\}$ is called the *set of homogeneous components* of m. If $M = \bigoplus_{\lambda\in G} M_\lambda$ is a graded R-module and $g \in G$, then the *g-suspension* of M is defined as the graded module $M(g)$ obtained from M by setting $M(g)_\lambda = M_{\lambda g}$. The g-suspension functor

$$T_g : R\text{-gr} \to R\text{-gr}$$

defined by $T_g(M) = M(g)$ is an isomorphism of categories.

2. Divisible graded modules. Several concepts of relative divisibility have been introduced in the literature. In this section we use the definition of divisibility from [14] and study the preservation of relative injectivity and divisibility by some nice functors. Then applications are made to group graded rings.

Let $\mathcal{C}$, $\mathcal{C}'$ be two abelian categories and let $D : \mathcal{C} \to \mathcal{C}'$ be a covariant functor. Let $\mathcal{T}'$ be a (hereditary) torsion class for $\mathcal{C}'$; i.e. $\mathcal{T}'$ is closed under subobjects, quotient objects, extensions and under arbitrary direct sums. We define $\mathcal{T} = \{X \in \mathcal{C} \mid D(X) \in \mathcal{T}'\}$. Our first result is an easy consequence of the definition.

LEMMA 2.1. *If D is exact and preserves direct sums, then $\mathcal{T}$ is a torsion class for $\mathcal{C}$.*

EXAMPLES 2.2. (i) If we consider the functor $(-)_e : R\text{-gr} \to R_e\text{-Mod}$ and τ is a torsion theory on R_e-Mod, we can induce a torsion theory τ^g in R-gr by taking the torsion class to be $\{M \in R\text{-gr} \mid M_e \text{ is } \tau\text{-torsion}\}$.

(ii) Let R be strongly graded. Then $R \otimes_{R_e} - : R_e\text{-Mod} \to R\text{-gr}$ is exact and preserves direct sums. We can define for any torsion theory τ in R-gr a torsion theory τ_e in R_e.

(iii) We denote by $U : R\text{-gr} \to R\text{-Mod}$ the forgetful functor; U is an exact functor and preserves direct sums. Hence given a torsion theory τ in R-Mod, we define $\tau^* = \{X \in R\text{-gr} \mid U(X) \text{ is } \tau\text{-torsion}\}$.

(iv) It is well known [6] that U has a right adjoint $F : R\text{-Mod} \to R\text{-gr}$ which is defined as follows: if $M \in R$-Mod, then $F(M)$ is the additive group $\bigoplus_{g\in G}({}^gM)$ (where each ${}^gM$ is a copy of M, ${}^gM = \{{}^gx \mid x \in M\}$) with the

R-module structure given by $a *^g x = {}^{hg}(ax)$ for $a \in R_h$. Obviously, the gradation of $F(M)$ is given by $F(M)_g = {}^gM$, $g \in G$, and if $f \in \mathrm{Hom}_R(M,N)$, then $F(f) \in \mathrm{Hom}_{R\text{-gr}}(F(M),F(N))$ is given by $F(f)({}^gx) = {}^gf(x)$. We remark that F is an exact functor and, by [4, Proposition 4.1], F commutes with direct sums. Hence, given a torsion theory τ in R-gr, we can define $\overline{\tau}$ in R-Mod with torsion class $\{X \in R\text{-Mod} \mid F(X) \text{ is } \tau\text{-torsion}\}$.

Note also that $U(F(M))$ need not be a direct sum of copies of M, since the component ${}^gM$ is not an R-submodule, but just an R_e-submodule of $F(M)$. If $M \in R$-Mod, we have the canonical epimorphism

$$F(M) \xrightarrow{\alpha} M \to 0$$

in R-Mod such that $\alpha({}^gx) = x$, $x \in M$.

PROPOSITION 2.3. *Let $L : \mathcal{C}' \to \mathcal{C}$ a left adjoint exact functor of D preserving direct sums. Let $\mathcal{T}'$ be a torsion class in $\mathcal{C}'$ such that $DL(\mathcal{T}') \subseteq \mathcal{T}'$ and $\mathcal{T}$ be the torsion class induced in $\mathcal{C}$. If $X \in \mathcal{C}$ is $\mathcal{T}$-torsionfree, then $D(X)$ is $\mathcal{T}'$-torsionfree.*

Proof. By the adjointness we have $\mathrm{Hom}_{\mathcal{C}'}(T, D(X)) \cong \mathrm{Hom}_{\mathcal{C}}(L(T), X)$ with $T \in \mathcal{C}'$ and $X \in \mathcal{C}$. If T is $\mathcal{T}'$-torsion, then $L(T)$ is $\mathcal{T}$-torsion by hypothesis. Hence the last term is zero and $D(X)$ is $\mathcal{T}$-torsionfree. ∎

Recall that a torsion theory τ in R-gr is said to be *rigid* if $M(g)$ is τ-torsion for any τ-torsion module M for all $g \in G$. By [4, Proposition 4.2] if τ is rigid, then $\overline{\tau}$ is the smallest torsion theory of R-Mod containing the τ-torsion modules. As an easy consequence of this definition and Proposition 2.3, we have the following result:

COROLLARY 2.4. *Let $R = \bigoplus_{g \in G} R_g$ with G finite and let τ be a rigid torsion theory in R-gr. If M is a τ-torsionfree graded R-module, then M is a $\overline{\tau}$-torsionfree R-module.*

Proof. Since G is finite, F is also a left adjoint of U. Let T be $\overline{\tau}$-torsion; then $F(T)$ is τ-torsion. Since $\overline{\mathcal{T}}$ is the smallest torsion class in R-Mod containing $\mathcal{T}$, it follows that $U(F(T))$ is $\overline{\tau}$-torsion. Now, we can apply Proposition 2.3. ∎

COROLLARY 2.5. *Let τ be a rigid torsion theory in R-gr. If M is a $\overline{\tau}$-torsionfree R-module, then $F(M)$ is τ-torsionfree.*

Proof. It is easy to see that if $T \in R$-gr, then $F(U(T)) = \bigoplus_{g \in G} T(g)$ (see [5, Lemma 3.1]). Since τ is rigid, $FU(T)$ is τ-torsion for any τ-torsion T. Therefore we can apply Proposition 2.3. ∎

COROLLARY 2.6. *$F(\overline{\tau}(M)) = \tau(F(M))$ for any $M \in R$-Mod.*

Our aim now is to study the injectivity relative to the torsion theories we have described. We recall that an object $E \in \mathcal{C}$ is called $\mathcal{T}$-*injective* if $\operatorname{Ext}^1_{\mathcal{C}}(T,E) = 0$ for all $T \in \mathcal{T}$. The next result is a relative version of the well-known result that right adjoint functors of exact functors preserve injectivity.

PROPOSITION 2.7. *Let $\mathcal{C}$, $\mathcal{C}'$ be two abelian categories and $\mathcal{T}$ (resp. $\mathcal{T}'$) be a torsion class in $\mathcal{C}$ (resp. $\mathcal{C}'$). If an additive functor $D : \mathcal{C} \to \mathcal{C}'$ is right adjoint to an exact functor L with the property that $L(\mathcal{T}') \subseteq \mathcal{T}$, then $D(E)$ is $\mathcal{T}'$-injective for any $\mathcal{T}$-injective object E.*

Proof. Consider $0 \to A \to B \to C \to 0$ in $\mathcal{C}'$ with $C \in \mathcal{T}'$. By adjointness

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{D}}(B, D(E)) & \xrightarrow{\theta} & \operatorname{Hom}_{\mathcal{D}}(A, D(E)) \\ \| & & \| \\ \operatorname{Hom}_{\mathcal{C}}(L(B), E) & \xrightarrow[\varphi]{} & \operatorname{Hom}_{\mathcal{C}}(L(A), E) \end{array}$$

But $0 \to L(A) \to L(B) \to L(C) \to 0$ is exact, and $L(C) \in \mathcal{T}$. By hypothesis E is $\mathcal{T}$-injective; thus φ is onto. Hence θ is onto. ■

COROLLARY 2.8. *Let $\mathcal{C}$, $\mathcal{C}'$ be two abelian categories and let $\mathcal{T}'$ be a torsion class in $\mathcal{C}'$. Let D be an exact functor that preserves direct sums and that is right adjoint to an exact functor L with the property that $DL(\mathcal{T}') \subseteq \mathcal{T}'$. Then $D(E)$ is $\mathcal{T}'$-injective for any $\mathcal{T}$-injective object E, where $\mathcal{T}$ is the induced torsion theory.*

COROLLARY 2.9. *Let $R = \bigoplus_{g\in G} R_g$ with G finite and let τ be a rigid torsion theory in R-gr. If M is a τ-injective graded R-module, then M is $\overline{\tau}$-injective as an R-module.*

PROPOSITION 2.10. *Let $R = \bigoplus_{g\in G} R_g$ and let τ be a rigid torsion theory in R-gr. If M is a $\overline{\tau}$-injective R-module, then $F(M)$ is τ-injective.*

The following result is an easy consequence of the equivalence of categories for strongly graded rings.

COROLLARY 2.11. *Let R be strongly graded, let τ be a rigid torsion theory in R-gr and let $M \in R$-gr. Then M is τ-injective if and only if M_e is τ_e-injective.*

PROPOSITION 2.12. *Let $E \in R$-gr. If E is $\overline{\tau}$-injective as an R-module, then E is τ-injective in R-gr.*

Proof. Consider the exact sequence in R-gr: $0 \to A \to B \to C \to 0$ where C is τ-torsion. Let $h : A \to E$ be a graded morphism. Since C is $\overline{\tau}$-torsion and E is $\overline{\tau}$-injective, we can extend h to an R-homomorphism $f : B \to E$. But then it is also possible to extend h by a graded morphism (see [6, Lemma I.2.1]). ■

Let $X \in \mathcal{C}$. X is called *$\mathcal{T}$-divisible* [14] if it is a quotient of a direct sum of $\mathcal{T}$-injective objects.

PROPOSITION 2.13. *Let $\mathcal{C}, \mathcal{C}'$ be two abelian categories and $\mathcal{T}$ (resp. $\mathcal{T}'$) be a torsion class in $\mathcal{C}$ (resp. $\mathcal{C}'$). If an exact and direct sum preserving functor $D : \mathcal{C} \to \mathcal{C}'$ is right adjoint to an exact functor L with the property that $L(\mathcal{T}') \subseteq \mathcal{T}$, then $D(X)$ is $\mathcal{T}'$-divisible for any $\mathcal{T}$-divisible object X.*

Proof. Since X is $\mathcal{T}$-divisible, we have the exact sequence $\bigoplus E_\alpha \to X \to 0$ where E_α is $\mathcal{T}$-injective. Hence $\bigoplus D(E_\alpha) \cong D(\bigoplus E_\alpha) \to D(X) \to 0$ is exact and $D(E_\alpha)$ is $\mathcal{T}'$-injective by Proposition 2.7. Therefore $D(X)$ is $\mathcal{T}'$-divisible. ∎

COROLLARY 2.14. *Let $\mathcal{C}, \mathcal{C}'$ be two abelian categories and $\mathcal{T}'$ be a torsion class in $\mathcal{C}'$. If an exact additive functor D, which preserves direct sums, is right adjoint to an exact functor L with the property that $DL(\mathcal{T}') \subseteq \mathcal{T}'$, then $D(X)$ is $\mathcal{T}'$-divisible for any $\mathcal{T}$-divisible object X.*

COROLLARY 2.15. *Let R be strongly graded, M be a graded R-module and τ be a rigid torsion theory in R-*gr. *Then M is a graded τ-divisible module if and only if M_e is a τ_e-divisible R_e-module.*

COROLLARY 2.16. *Let $R = \bigoplus_{g\in G} R_g$ and let τ be a rigid torsion theory in R-*gr. *If M is $\overline{\tau}$-divisible, then $F(M)$ is τ-divisible.*

COROLLARY 2.17. *Let $R = \bigoplus_{g\in G} R_g$ with G finite and let τ be a rigid torsion theory. If M is a τ-divisible graded R-module, then M is a $\overline{\tau}$-divisible R-module.*

PROPOSITION 2.18. *Let $R = \bigoplus_{g\in G} R_g$ and let τ be a rigid torsion theory in R-*gr. *If M is a τ-divisible graded module, then $M(g)$ is τ-divisible.*

Proof. Let $\bigoplus E_\alpha \to M \to 0$ be exact where E_α is τ-injective. By suspension $\bigoplus(E_\alpha(g)) \cong (\bigoplus E_\alpha)(g) \to M(g) \to 0$ is also exact. Since τ is rigid, T_g preserves τ-torsion modules and Proposition 2.7 implies that $E_\alpha(g)$ is also τ-injective. ∎

PROPOSITION 2.19. *Let $R = \bigoplus_{g\in G} R_g$. If M is a divisible graded R-module, then $F(M)$ is a* gr-*divisible graded R-module.*

Proof. Since F is a right adjoint, it preserves injectivity. Now, F is exact and therefore F preserves divisibility. ∎

3. Graded D^*-modules. In this section we assume that τ is a rigid torsion theory in R-gr. We recall that an object $X \in \mathcal{C}$ is said to be a *D^*-object* for $\mathcal{T}$ if $\mathrm{Ext}^1_{\mathcal{C}}(X, D) = 0$ for any $\mathcal{T}$-divisible object D. We now study the properties of D^*-modules (i.e., the D^*-objects) for the category

R-gr; this can be viewed as a continuation of the study of D^*-modules that was begun in [14] and [10].

Our next result is an easy consequence of the equivalence between the categories R_e-Mod and R-gr.

PROPOSITION 3.1. *Let R be a strongly graded ring and $M \in R$-gr. Then M is a D^*-module for τ if and only if M_e is a D^*-module for τ_e.*

PROPOSITION 3.2. *Let $R = \bigoplus_{g\in G} R_g$ with G finite and $|G|^{-1} \in R$. If $M \in R$-gr is a D^*-module for τ, then $M \in R$-Mod is a D^*-module for $\overline{\tau}$.*

Proof. Consider an exact sequence $0 \to A \to B \to M \to 0$ with A $\overline{\tau}$-divisible. Applying F, we obtain

$$0 \to F(A) \to F(B) \to F(M) \cong \bigoplus_{g\in G} M(g) \to 0$$

exact where $F(A)$ is τ-divisible by Corollary 2.16. We claim that $M(g)$ is also a D^*-module for τ. Consider $0 \to D \to X \to M(g) \to 0$, where D is τ-divisible. Then $0 \to D(g^{-1}) \to X(g^{-1}) \to M \to 0$ is exact and $D(g^{-1})$ is τ-divisible by Proposition 2.18; so this sequence splits. Hence by applying the g-suspension, the sequence $0 \to D \to X \to M(g) \to 0$ also splits. Apply [5, Theorem 3.10.1] to deduce that $0 \to A \to B \to M \to 0$ splits. ■

PROPOSITION 3.3. *Let $R = \bigoplus_{g\in G} R_g$ with G finite. If $M \in R$-gr is a D^*-module for $\overline{\tau}$, then M is a D^*-module for τ.*

Proof. Let $0 \to D \to X \to M \to 0$ be an exact sequence in R-gr, where D is τ-divisible. By Corollary 2.17, D is $\overline{\tau}$-divisible. Since M is a D^*-module for $\overline{\tau}$, the exact sequence splits in R-Mod. Thus the sequence splits in R-gr. ■

4. Graded B^*-modules. Let $\mathcal{C}$ be a Grothendieck category with a finitely generated generator V. We say that an object $X \in \mathcal{T}$ of $\mathcal{C}$ has *bounded order* in case there is a $\mathcal{T}$-dense subobject K of V such that X embeds in a factor of $(V/K)^{(\mathcal{A})}$, the direct sum of cardinal of $\mathcal{A}$ copies of V/K, for some set $\mathcal{A}$. In this section, we investigate the preservation of the bounded order property by nice functors. Again, we are particularly interested in applications to a strongly graded ring R and its functor Ext^1_R. This type of investigation is closely related to the Bounded Splitting Problem that has been studied by many authors; see [14] for some background references on splitting problems.

Remarks. (i) It is easy to show that the definition of bounded order does not depend on the finitely generated generator of $\mathcal{C}$.

(ii) This concept of bounded order clearly coincides with the usual one when $\mathcal{C} = R$-Mod. It also makes sense for R-gr when either G is finite or R is strongly graded, since in these cases R-gr has a finitely generated generator.

(iii) If we consider Goldie's torsion theory, this concept coincides with the definition of bounded order introduced in [15].

(iv) Let $M \in R$-gr. It is clear that if M has τ-bounded order, then M has $\overline{\tau}$-bounded order.

PROPOSITION 4.1. *Let $D : \mathcal{C} \to \mathcal{C}'$ be a functor between two Grothendieck categories and let $\mathcal{T}$ (resp. $\mathcal{T}'$) be a torsion class in $\mathcal{C}$ (resp. $\mathcal{C}'$). If D is exact, preserves direct sums, sends a finitely generated generator to a finitely generated generator and $D(\mathcal{T}) \subseteq \mathcal{T}'$, then D sends $\mathcal{T}$-bounded order objects to $\mathcal{T}'$-bounded order objects.*

Proof. Straightforward. ■

COROLLARY 4.2. *Let R be strongly graded and let $X \in R$-gr. If X has $\overline{\tau}$-bounded order, then $F(X)$ has τ-bounded order.*

COROLLARY 4.3. *Let R be strongly graded and let τ be a rigid torsion theory. If $X \in R$-gr has τ-bounded order, then $X(g)$ has τ-bounded order.*

COROLLARY 4.4. *Let R be a strongly graded ring and let τ be a rigid torsion theory in R-gr. Then $M \in R$-gr has τ-bounded order if and only if $M_e \in R_e$-Mod has τ_e-bounded order.*

Following the notation of [14], we say that an object $M \in \mathcal{C}$ is a *B^*-object* for $\mathcal{T}$ if $\mathrm{Ext}^1_{\mathcal{C}}(M, X) = 0$ for each $\mathcal{T}$-divisible X and each X with $\mathcal{T}$-bounded order.

From Proposition 3.1, Corollary 4.4, and the equivalence of the categories R_e-Mod and R-gr for strongly graded rings, we obtain the following result.

PROPOSITION 4.5. *Let R be a strongly graded ring, and let τ be a rigid torsion theory in R-gr. Let $M \in R$-gr. Then M is a B^*-module for τ if and only if M_e is a B^*-module for τ_e.*

PROPOSITION 4.6. *Let $M \in R$-gr. If $\mathrm{Ext}^1_R(M, T) = 0$ for all T with $\overline{\tau}$-bounded order, then $\mathrm{Ext}^1_{R\text{-gr}}(M, T) = 0$ for all T with τ-bounded order.*

Proof. Consider an exact sequence $0 \to T \to X \to M \to 0$ in R-gr, where T has τ-bounded order. Since T has also $\overline{\tau}$-bounded order, the preceding exact sequence splits in R-Mod and therefore in R-gr. ■

PROPOSITION 4.7. *Let $M \in R$-gr. If M is a B^*-module for $\overline{\tau}$, then M is a B^*-module for τ.*

Proof. This follows from Propositions 3.3 and 4.5. ■

PROPOSITION 4.8. *Let* $R = \bigoplus_{g\in G} R_g$ *with* G *finite and* $|G|^{-1} \in R$ *and let* τ *be a rigid torsion theory in* R-gr. *If* $\mathrm{Ext}^1_{R\text{-gr}}(M,T) = 0$ *for all* T *with* τ*-bounder order*, *then* $\mathrm{Ext}^1_R(M,T) = 0$ *for all* T *with* $\overline{\tau}$*-bounded order.*

Proof. Let $0 \to T \to X \to M \to 0$ be an exact sequence in R-Mod with T having $\overline{\tau}$-bounded order. Then $0 \to F(T) \to F(X) \to F(M) \cong \bigoplus_{g\in G} M(g) \to 0$ is exact and $F(T)$ has τ-bounded order by Corollary 4.2. Now $\mathrm{Ext}^1_{R\text{-gr}}(\bigoplus_{g\in G} M(g), F(T)) \cong \prod \mathrm{Ext}^1_{R\text{-gr}}(M(g), F(T))$. Next we show that $\mathrm{Ext}^1_{R\text{-gr}}(M(g), F(T)) = 0$.

Consider the exact sequence $0 \to F(T) \to Y \to M(g) \to 0$. Apply the g^{-1}-suspension to get the exact sequence $0 \to F(T)(g^{-1}) \to Y(g^{-1}) \to M \to 0$. By Corollary 4.3, $F(T)(g^{-1})$ has τ-bounded order. By hypothesis, we now find that $0 \to F(T)(g^{-1}) \to Y(g^{-1}) \to M \to 0$ splits. Applying the g-suspension, we find that $0 \to F(T) \to Y \to M(g) \to 0$ splits, as desired. So $0 \to F(T) \to F(X) \to F(M) \to 0$ splits. Since $|G| < \infty$ and $|G|^{-1} \in R$, from [5, Theorem 3.10.1] we deduce that $0 \to T \to X \to M \to 0$ splits. ■

PROPOSITION 4.9. *Let* $R = \bigoplus_{g\in G} R_g$ *with* G *finite and* $|G|^{-1} \in R$ *and let* τ *be a rigid torsion theory in* R-gr. *Let* $M \in R$-gr. *If* M *is a* B^**-module for* τ, *then* M *is a* B^**-module for* $\overline{\tau}$.

Proof. This follows from Propositions 3.2 and 4.7. ■

We summarize our results in the following:

THEOREM 4.10. *Let* R *be a strongly graded ring over a finite group* G *with* $|G|^{-1} \in R$, *let* τ *be a rigid torsion theory in* R-gr *and let* $X \in R$-gr. *Then the following conditions are equivalent*:

(i) X *is a* B^**-module for* τ.
(ii) X *is a* B^**-module for* $\overline{\tau}$.
(iii) X_e *is a* B^**-module for* τ_e.

5. Non-singular B^*-modules. In this section we study B^*-modules for the Goldie torsion theory. We say that a category $\mathcal{C}$ has the (*ND*) (resp. (*NB*)) *property* if every non-singular object is a D^*-object (resp. B^*-object). Motivated by the work in [8] and [9], we study the conditions under which every non-singular R-module has (NB), where R is a strongly graded ring. Extensive information is obtained for the group ring case.

The following result is an easy consequence of the equivalence between the categories R_e-Mod and R-gr.

PROPOSITION 5.1. *Let* R *be a strongly graded ring. Then* R-gr *has* (*ND*) (*resp.* (*NB*)) *if and only if* R_e-Mod *has* (*ND*) (*resp.* (*NB*)).

THEOREM 5.2. *Let R be a strongly graded ring over a finite group G with $|G|$ invertible in R. If R is non-singular, then the following conditions are equivalent*:

(i) R-Mod *has* (*NB*).
(ii) R-gr *has* (*NB*).
(iii) R_e-Mod *has* (*NB*).

Proof. We only have to show the equivalence of (i) and (ii). Assume that R-Mod has (NB) and let $X \in R$-gr be a gr-non-singular module. Then by [15, Lemma 2.7], $U(X)$ is non-singular as an R-module. Hence $U(X)$ is a B^*-module as an R-module. Let

$$0 \to A \to B \to X \to 0 \tag{1}$$

be an exact sequence of graded R-modules. If A is gr-bounded, we apply the exact functor U to obtain

$$0 \to U(A) \to U(B) \to U(X) \to 0 \tag{2}$$

in R-Mod. Since U preserves essentiality, we apply [15, Proposition 2.3] to deduce that $U(A)$ is bounded. If A is gr-divisible, then $U(A)$ is divisible, since E being gr-injective implies $U(E)$ is injective [3, Theorem 4.7]. In both cases the exact sequence (2) splits, and hence (1) splits in R-gr.

Assume now that R-gr has (NB) and let Y be a non-singular R-module. Let

$$0 \to A \to B \to Y \to 0 \tag{3}$$

be an exact sequence of R-modules. We apply the exact functor F to obtain

$$0 \to F(A) \to F(B) \to F(X) \to 0. \tag{4}$$

If A is bounded, then by [15, Proposition 2.2], $F(A)$ is gr-bounded; and if A is divisible, then $F(A)$ is gr-divisible by Proposition 2.19. Moreover, [15, Proposition 2.2] implies that $F(Y)$ is gr-non-singular and so the hypothesis yields that (4) splits. By [5, Theorem 3.10.1], (3) splits. ∎

PROPOSITION 5.3. *If the group ring $R[G]$ has* (*NB*), *then either G is finite or else $R \cong M_{n_1}(D_1) \times \ldots \times M_{n_t}(D_t)$ for some division rings D_i such that $Z(D_i)[G]$ has BSP* (*Bounded Splitting Property*) *for $i = 1, \ldots, t$.*

Proof. If all the non-singular $R[G]$-modules are B^*-modules, then $R[G]$ has BSP. By [15, Proposition 3.3], the result follows. ∎

From [15, Corollary 3.5], we also have the following result.

PROPOSITION 5.4. *Let G be an infinite group and $G_0 = \{1\} \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_m \lhd G$ be a finite subnormal chain in G. If $R[G]$ has* (*NB*), *then exactly one factor of the chain, say G_k/G_{k-1}, is infinite and each $R[G_i]$ has BSP.*

By Proposition 5.3 and Morita equivalence, there is no loss of generality in assuming that R is a division ring D when we study (NB) for $R[G]$ with G infinite. We wish to show that if $D[G]$ has (NB), then $K[G]$ has (NB), where K is the center of D. We then use this fact and Proposition 5.4 to analyze the (NB) property when G is abelian-by-finite or polycyclic-by-finite or solvable.

LEMMA 5.5. *Let D be a division ring that is finite-dimensional over its center $K = Z(D)$, and let G be an abelian group such that $G \cong T \oplus Z$ with $K[T]$ semisimple artinian. Then $D[G]$ has (NB).*

Proof. Since $K[T]$ is semisimple, so is $D[T]$. Thus $D[G] \cong D[T \oplus Z] \cong (D[T])[Z]$ is hereditary, and hence $D[G]$ has (ND). It also follows from Morita equivalence and [15, Proposition 3.9] that $D[G]$ has BSP. ■

It is very useful to understand the abelian case.

THEOREM 5.6. *Let K be a field and let G be an infinite abelian group. Then $K[G]$ has (NB) if and only if $G = T \oplus Z$ with $K[T]$ semisimple artinian.*

Proof. If $K[G]$ has (NB), then $K[G]$ has BSP. By [15, Theorem 3.6] it follows that either $G \cong T \oplus Z$ with $K[T]$ semisimple artinian or $K[G] \cong K_1[Z_{p^\infty}] \times \ldots \times K_r[Z_{p^\infty}]$, where the fields K_i are of first kind with respect to p and $\mathrm{char}(K_i) \neq p$ for each i. But in the latter case the ring $K_i[Z_{p^\infty}]$ is a regular non-semisimple ring. But $K[G]$ has (NB) and by [10], $K[G]$ is finite-dimensional. Hence this case is eliminated.

The converse is immediate from Lemma 5.5 with $D = K$. ■

PROPOSITION 5.7. *Let D be a division ring and $K \subseteq Z(D)$ be a field. Assume that $D[G]$ is non-singular. If $D[G]$ has (ND), then $K[G]$ has (ND).*

Proof. Let C be a divisible left $K[G]$-module. We will show that $\mathrm{Hom}_{K[G]}(D[G], C)$ is divisible as a left $D[G]$-module.

We have an exact sequence $\bigoplus E_\alpha \to C \to 0$ with injective modules E_α. The epimorphism $\bigoplus_{\beta_\alpha} K[G] \to E_\alpha$ extends to $\bigoplus_{\beta_\alpha} E(K[G]) \to E_\alpha$. Since $K[G]$ is non-singular, we have $E(K[G]) \cong Q_{\max}(K[G]) = Q$. Since $D[G]$ has (ND), it follows that $D[G]$ is finite-dimensional. Since $D[G]$ is free over $K[G]$, we see that $K[G]$ is finite-dimensional. Hence direct sums of Q's are injective. We obtain $E = \bigoplus_\alpha (\bigoplus_{\beta_\alpha} Q) \to C \to 0$ with E injective. Since $D[G]$ is flat over $K[G]$, it follows that $\mathrm{Hom}_{K[G]}(D[G], E)$ is injective as a $D[G]$-module. The exact sequence

$$\mathrm{Hom}_{K[G]}(D[G], E) \to \mathrm{Hom}_{K[G]}(D[G], C) \to 0$$

yields that $\mathrm{Hom}_{K[G]}(D[G], C)$ is divisible.

By [15, Lemma 3.2], if N is non-singular as a $K[G]$-module, then the extension of scalars $D[G] \otimes_{K[G]} N$ is non-singular as a $D[G]$-module.

Finally, take any non-singular $K[G]$-module N and any divisible $K[G]$-module C. By [1, 4.1.4], we have

$$\mathrm{Ext}^1_{K[G]}(D[G]\otimes_{K[G]} N, C) \cong \mathrm{Ext}^1_{D[G]}(D[G]\otimes_{K[G]} N, \mathrm{Hom}_{K[G]}(D[G], C)),$$

and the last term is zero since $D[G]$ has (ND), $\mathrm{Hom}_{K[G]}(D[G], C)$ is divisible and $D[G]\otimes_{K[G]} N$ is non-singular. Since $K[G]$ is a direct summand of $D[G]$ as a $K[G]$-module, it follows that N is a direct summand of $D[G]\otimes_{K[G]} N$. Hence $\mathrm{Ext}^1_{K[G]}(N, C) = 0$. ∎

Corollary 5.8. *Let D be a division ring and let $K \subseteq Z(D)$ be a field. If $D[G]$ has (NB), then so does $K[G]$.*

Proof. If $D[G]$ has (NB), then $K[G]$ has BSP by [15, Theorem 3.11], and $D[G]$ is non-singular. Now $K[G]$ has also (ND) by Proposition 5.7. ∎

Combining Corollary 5.8, Theorem 5.6, and Lemma 5.5, we have the following result.

Theorem 5.9. *Let D be any division ring that is finite-dimensional over its center $K = Z(D)$ and G be an infinite abelian group. The group ring $D[G]$ has (NB) if and only if $K[G]$ does.*

Proposition 5.10. *Let $R[G]$ be a left non-singular ring. If $R[G]$ has (ND), then $R[H]$ has finite left Goldie dimension for any subgroup H of G.*

Proof. By [10], since $R[G]$ has (ND), it follows that $R[G]$ has finite left Goldie dimension. Since $R[G]$ is a free extension of $R[H]$, we see that $R[H]$ has finite Goldie dimension. ∎

Corollary 5.11. *Assume that $R[G]$ has (NB). If H is a locally finite subgroup of an infinite group G and $o(h)^{-1} \in R$ for all $h \in H$, then H is finite.*

Proof. Since $R[G]$ has (NB), we deduce that $R[G]$ has BSP. By [15, Lemma 3.1] it follows that R is semisimple. By Morita equivalence we can assume that $R = D$, a division ring. Then the hypotheses imply $D[H]$ is von Neumann regular. By Proposition 5.10, $D[H]$ has finite Goldie dimension. Hence $D[H]$ must be semisimple artinian. Therefore, H is finite. ∎

We denote by $\Delta(G)$ the set of all the elements in G with finitely many conjugates.

Proposition 5.12. *Assume that $o(h)^{-1} \in R$ for all $h \in t(\Delta(G))$. If $R[G]$ has (NB) and $|\Delta(G)|$ is infinite, then*

(1) $|G/\Delta(G)| < \infty$,
(2) $|t(\Delta(G))| < \infty$,
(3) $\Delta(G)/t(\Delta(G)) \cong Z$.

P r o o f. (1) By [15, Lemma 3.4], $|G/\Delta(G)| < \infty$.

(2) Since $t(\Delta(G))$ is locally finite, we can apply Corollary 5.10.

(3) By [7, IV, Lemma 1.6], $H = \Delta(G)/t(\Delta(G))$ is abelian. By Proposition 5.4, H must have rank one. Choose $C \subseteq H$ with $C \cong Z$. By Proposition 5.4, H/C is finite, and hence H is finitely generated. Therefore, $H \cong Z$. ■

R e m a r k. When G satisfies the hypothesis of Proposition 5.12, then G has a normal series finite-Z-finite. By [11, Proposition 8.2.], G is polycyclic-by-finite. Since the Hirsch number $h(G)$ of G is 1, G has a normal series Z-finite. (See Section 3 of [15] for more information about these $R[G]$.)

PROPOSITION 5.13. *If G is a infinite solvable group and $R[G]$ has BSP, then G is polycyclic-by-finite with $h(G) = 1$.*

P r o o f. By Proposition 5.4, only one factor H of the commutator series for G is infinite; so we can argue as in Proposition 5.12(3) that $H \cong Z$. ■

If G is abelian-by-finite and $R[G]$ has BSP, a similar argument shows that G has a normal series Z-finite. Therefore to study (NB) for G in the case of Proposition 5.11 or G solvable or G polycyclic-by-finite, it is sufficient to examine a group G with a normal series Z-finite. We are able to do this in a special case.

PROPOSITION 5.14. *Let D be a division ring finite-dimensional over its center. If G has a normal series Z-finite and $(o(x))^{-1} \in D$ for each $x \in G/Z$, then $D[G]$ has (NB).*

P r o o f. By [15, Proposition 3.9], $D[Z]$ has BSP. On the other hand, $D[Z]$ is a hereditarily noetherian ring and any divisible module is injective. Hence $D[Z]$ has (NB). Since $D[G] \cong D[Z]*[G/Z]$ and G/Z has nice inverses, $D[G]$ has (NB) by [15, Corollary 2.6] and Theorem 5.2. ■

We now further examine the role of $\Delta(G)$ in the study of BSP.

PROPOSITION 5.15. *Let G be an infinite group with $\Delta(G)$ finite, and assume $R[G]$ has BSP. Let $\aleph$ be an infinite cardinal. If $x \in G$ has $\aleph$ conjugates, then $|G| = \aleph$. Consequently, any element in $G - \Delta(G)$ has $|G|$ conjugates.*

P r o o f. Let $x \in G - \Delta(G)$. Let $\{x_\alpha\}$ be a set of distinct conjugates of x. Then $|\{x_\alpha\}| = \aleph$ by hypothesis. Let H be the subgroup of G generated by $\{x_\alpha\}$. Since $\{x_\alpha\} \subseteq H$, we have $|H| \geq \aleph$. Since H consists of finite products from a set having $\leq \aleph$ elements, we deduce that $|H| \leq \aleph$. Since the conjugate of a product is the product of conjugates, it follows that $H \lhd G$. By [15, Lemma 3.4], $|G/H| < \infty$. Hence G is the union of a finite number of cosets, each of which has $\aleph$ elements. Thus $|G| = \aleph$. ■

Corollary 5.16. *If $R[G]$ has BSP with $|G| = \infty$ and $|\Delta(G)| < \infty$, then either G is countable or else every element of $G - \Delta(G)$ has uncountably many conjugates.*

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