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SOME QUADRATIC INTEGRAL INEQUALITIES OF FIRST ORDER

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We derive and examine some quadratic integral inequalities of first order of the form

$$\int_I (r\dot{h}^2 + 2sh\dot{h} + uh^2)\,dt \ge 0, \qquad h \in H, \tag{1}$$

where $I = (\alpha, \beta)$, $-\infty \le \alpha < \beta \le \infty$, r, s and u are given real functions of the variable t, H is a given class of absolutely continuous functions and $\dot{h} \equiv dh/dt$. The inequalities of the form (1) comprise as special cases integral inequalities of Sturm–Liouville type examined by Florkiewicz and Rybarski [10] and quadratic integral inequalities of Opial type examined by Kuchta [13]. The method used to obtain the integral inequalities of the form (1) is an extension of the uniform method of obtaining various types of integral inequalities involving a function and its derivative. The method we extend was used in [8]–[10], [13]. The method makes it possible, given a function r and an auxiliary function φ, to determine the functions s and u, and next the class H of the functions h for which (1) holds. In this paper s and u are solutions of a certain differential inequality which makes it possible to obtain a large set of functions s and u for which inequality (1) holds.

Inequalities of the form (1) have been considered by Beesack [2]–[5], Redheffer [16], Yang [18], Benson [6], Boyd [7] and others (for an extensive bibliography see [14]).

The positive definiteness of quadratic functionals of the form (1) is a basic problem of the theory of singular quadratic functionals introduced by Morse and Leighton [15] (cf. [17], [1]). This problem is of significant importance for the oscillation theory for second order linear differential equations on a non-compact interval (see [17]).

Let $I = (\alpha, \beta)$, $-\infty \le \alpha < \beta \le \infty$, be an arbitrary open interval. We denote by $M(I)$ the class of real functions which are defined and Lebesgue

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measurable on I and by $AC(I)$ the class of real functions defined and absolutely continuous on I. Let $r \in AC(I)$ and $\varphi \in AC(I)$ be given functions such that $r > 0$, $\varphi > 0$ on I and $\dot{\varphi} \in AC(I)$. Then $r^{-1} = 1/r \in AC(I)$ and $\varphi^{-1} = 1/\varphi \in AC(I)$. Let $s \in AC(I)$ and $u \in M(I)$ be arbitrary functions satisfying the differential inequality

$$\dot{s} - u + (r\dot{\varphi})^{\cdot}\varphi^{-1} \le 0 \tag{2}$$

almost everywhere on I.

Denote by $\widehat{H}$ the class of functions $h \in AC(I)$ satisfying the following integral and limit conditions:

$$\int_I r\dot{h}^2\,dt < \infty, \qquad \int_I sh\dot{h}\,dt < \infty, \qquad \int_I uh^2\,dt < \infty, \tag{3}$$

$$\liminf_{t\to\alpha} vh^2 < \infty, \qquad \limsup_{t\to\beta} vh^2 > -\infty, \tag{4}$$

where

$$v = r\dot{\varphi}\varphi^{-1} + s. \tag{5}$$

THEOREM 1. *For every $h \in \widehat{H}$ both limits in* (4) *are proper and finite*, *and*

$$\lim_{t\to\beta} vh^2 - \lim_{t\to\alpha} vh^2 \le \int_I (r\dot{h}^2 + 2sh\dot{h} + uh^2)\,dt. \tag{6}$$

If $h \not\equiv 0$, then equality holds in (6) *if and only if s and u satisfy the differential equation*

$$\dot{s} - u + (r\dot{\varphi})^{\cdot}\varphi^{-1} = 0 \tag{7}$$

a.e. on I, $\varphi \in \widehat{H}$ and $h = c\varphi$ with $c = \mathrm{const} \ne 0$.

Proof. Let $h \in AC(I)$. By (5) and our assumptions we have $vh^2 \in AC(I)$ and $\varphi^{-1}h \in AC(I)$ and we easily check that

$$r\dot{h}^2 + 2sh\dot{h} + uh^2 = (vh^2)^{\cdot} + f + g \quad \text{a.e. on } I, \tag{8}$$

where

$$f = -(\dot{s} - u + (r\dot{\varphi})^{\cdot}\varphi^{-1})h^2 \ge 0, \tag{9}$$

$$g = r\varphi^2[(\varphi^{-1}h)^{\cdot}]^2 \ge 0. \tag{10}$$

Now, let $h \in \widehat{H}$. By the first condition of (3) it follows that $r\dot{h}^2$ is summable on I because $r\dot{h}^2 \ge 0$ on I. By the assumptions, all other functions appearing in (8) are summable on each compact interval $[a,b] \subset I$. Thus, by (8),

$$\int_a^b r\dot{h}^2\,dt + 2\int_a^b sh\dot{h}\,dt + \int_a^b uh^2\,dt = vh^2|_a^b + \int_a^b f\,dt + \int_a^b g\,dt \tag{11}$$

for all $\alpha < a < b < \beta$. By (4) there exist two sequences $\{a_n\}$ and $\{b_n\}$ such that $\alpha < a_n < b_n < \beta$, $a_n \to \alpha$, $b_n \to \beta$ and

$$\lim_{n\to\infty} (-vh^2)|_{a_n} > -\infty, \qquad \lim_{n\to\infty} vh^2|_{b_n} > -\infty.$$

Thus there is a constant C such that

$$vh^2|_{a_n}^{b_n} \ge C > -\infty.$$

In view of (9) and (10), from (11) we infer that

$$\int_{a_n}^{b_n} (2sh\dot{h} + uh^2)\,dt \ge -\int_{a_n}^{b_n} r\dot{h}^2\,dt + C \ge -\int_{\alpha}^{\beta} r\dot{h}^2\,dt + C$$

and letting $n \to \infty$ gives

$$\int_I (2sh\dot{h} + uh^2)\,dt \ge -\int_I r\dot{h}^2\,dt + C > -\infty.$$

By this estimate and by the second and third conditions of (3) we easily see that the functions $sh\dot{h}$ and uh^2 are summable on I. In the analogous way we show that f and g are summable on I. Thus all integrals in (11) have finite limits as $a \to \alpha$ or $b \to \beta$. It follows that both limits in (4) are proper and finite. Now, by (11), as $a \to \alpha$ and $b \to \beta$, we obtain

$$\int_I (r\dot{h}^2 + 2sh\dot{h} + uh^2)\,dt = \lim_{t\to\beta} vh^2 - \lim_{t\to\alpha} vh^2 + \int_I f\,dt + \int_I g\,dt \tag{12}$$

whence (6) follows, since $f \ge 0$ and $g \ge 0$ on I.

By (12), equality holds in (6) for a non-vanishing function $h \in \widehat{H}$ if and only if $\int_I f\,dt = 0$ and $\int_I g\,dt = 0$, i.e. $f = 0$ and $g = 0$ a.e. on I. In view of (10), $g = 0$ a.e. on I if and only if $(\varphi^{-1}h)^{\cdot} = 0$ a.e. on I. Hence $h = c\varphi$, where $c = \text{const} \neq 0$, since $\varphi^{-1}h \in AC(I)$ by assumption. Thus $\varphi \in \widehat{H}$. Further, from (9), $f = 0$ a.e. on I if and only if s and u satisfy (7) a.e. on I, because $h^2 = c^2\varphi^2 > 0$ on I. ∎

Denote by $\widetilde{H}$ the class of functions $h \in \widehat{H}$ satisfying additionally the limit condition

$$\liminf_{t\to\alpha} vh^2 \le \limsup_{t\to\beta} vh^2. \tag{13}$$

By Theorem 1 we can write it in the equivalent form

$$\lim_{t\to\alpha} vh^2 \le \lim_{t\to\beta} vh^2. \tag{14}$$

THEOREM 2. *For every* $h \in \widetilde{H}$,

$$\int_I (r\dot{h}^2 + 2sh\dot{h} + uh^2)\,dt \ge 0. \tag{15}$$

If $h \not\equiv 0$, *then equality holds in* (15) *if and only if* $\varphi^{-1}h = \text{const} \neq 0$ *and the additional conditions* (7) *and*

$$\varphi \in \widehat{H}, \quad \lim_{t\to\alpha} v\varphi^2 = \lim_{t\to\beta} v\varphi^2 \tag{16}$$

are satisfied.

Proof. By (14) and Theorem 1, inequality (15) follows from (6). If equality occurs in (15) for some non-vanishing function $h \in \widetilde{H}$, then by (6) and (14) we have $\lim_{t\to\alpha} vh^2 = \lim_{t\to\beta} vh^2$. Using Theorem 1 once again we conclude that (7) holds, $\varphi \in \widehat{H}$ and $h = c\varphi$, where $c = \text{const} \neq 0$, whence we obtain (16). ■

Now we describe the class $\widetilde{H}$ in the cases that occur most frequently. If $ru - s^2 \geq 0$ a.e. on I, then inequality (15) holds for all $h \in AC(I)$. Thus it is natural to consider cases like $ru - s^2 < 0$ a.e. in some interval $(a, b) \subset I$.

Lemma 1. *Let* $\alpha \leq a < b \leq \beta$. *If* $ru - s^2 < 0$ *a.e. on* (a, b), *then the function* v *satisfies the differential inequality*

$$r\dot{v} < 2sv - v^2 \tag{17}$$

a.e. on (a, b).

Proof. By (5) and (2) we have

$$\dot{v} = (r\dot{\varphi})^{\cdot}\varphi^{-1} + \dot{s} - r\dot{\varphi}^2\varphi^{-2} \leq u - r\dot{\varphi}^2\varphi^{-2} \tag{18}$$

a.e. on (a, b). Thus from the assumptions we obtain

$$r\dot{v} \leq ru - r^2\dot{\varphi}^2\varphi^{-2} < s^2 - (r\dot{\varphi}\varphi^{-1})^2,$$

since $r > 0$ on I. Further, by (5) we have $(r\dot{\varphi}\varphi^{-1})^2 = s^2 - 2sv + v^2$ on I, whence (17) follows. ■

We will denote by U_α (resp. U_β) some right-hand (resp. left-hand) neighbourhood of the point α (resp. β). By Lemma 1 it follows that if $ru - s^2 < 0$ a.e. on U_α and $sv \leq 0$ on U_α, then $\dot{v} < 0$ a.e. on U_α and consequently the function v is decreasing on U_α. Thus the limit $v(\alpha) = \lim_{t\to\alpha} v$ exists and $v < v(\alpha)$ on U_α. Analogously, if $ru - s^2 < 0$ a.e. on U_β and $sv \leq 0$ on U_β, then $v(\beta) = \lim_{t\to\beta} v$ exists and $v > v(\beta)$ on U_β.

Lemma 2. *If* $ru - s^2 < 0$ *a.e. on* U_α (*resp.* U_β), $sv \leq 0$ *on* U_α (*resp.* U_β) *and* $v(\alpha) \neq 0$ (*resp.* $v(\beta) \neq 0$), *then* $\int_\alpha^t r^{-1}\,d\tau < \infty$ (*resp.* $\int_t^\beta r^{-1}\,d\tau < \infty$) *for some* $t \in I$. *Moreover*, *if* $v(\alpha) = \infty$ (*resp.* $v(\beta) = -\infty$), *then* $v(t)\int_\alpha^t r^{-1}\,d\tau = O(1)$ *as* $t \to \alpha$ (*resp.* $v(t)\int_t^\beta r^{-1}\,d\tau = O(1)$ *as* $t \to \beta$).

Proof. We prove the lemma only for the point α. The proof for β is analogous.

Let $v(\alpha) \neq 0$ and consider some right-hand neighbourhood $U \subset U_\alpha$ of α such that $v \neq 0$ on U. By the assumptions and Lemma 1, from (17) we get $r\dot{v} < -v^2$ a.e. on U. Then $r^{-1} < -v^{-2}\dot{v}$ a.e. on U, because $r > 0$ on I and we have the estimate

$$\int_a^t r^{-1}\,d\tau \le -\int_a^t v^{-2}\dot{v}\,d\tau = v^{-1}(t) - v^{-1}(a) \tag{19}$$

for $\alpha < a < t < \beta$ on U.

If $v(\alpha) > 0$ (i.e. $v > 0$ on U), then by (19) as $a \to \alpha$ we obtain $\int_\alpha^t r^{-1} d\tau < v^{-1}(t) < \infty$. Hence $0 < v(t)\int_\alpha^t r^{-1} d\tau < 1$ and thus $v(t)\int_\alpha^t r^{-1} d\tau = O(1)$ as $t \to \alpha$.

If $v(\alpha) < 0$ (i.e. $v < 0$ on U), then by (19) we obtain $\int_a^t r^{-1}\,d\tau < -v^{-1}(a)$, whence as $a \to \alpha$ we get $\int_\alpha^t r^{-1}\,d\tau < -v^{-1}(\alpha) < \infty$. ■

We introduce the following terminology:

• a boundary point α (resp. β) of the interval I is of *type I* if $v \le 0$ on U_α (resp. $v \ge 0$ on U_β);

• α (resp. β) is of *type II* if $ru - s^2 < 0$ a.e. on U_α (resp. U_β) and $sv \le 0$ on U_α (resp. U_β) and $0 < v(\alpha) < \infty$ (resp. $-\infty < v(\beta) < 0$);

• α (resp. β) is of *type III* if $ru - s^2 < 0$ a.e. on U_α (resp. U_β) and $sv \le 0$ on U_α (resp. U_β) and $v(\alpha) = \infty$ (resp. $v(\beta) = -\infty$).

We denote by H the class of functions $h \in AC(I)$ satisfying the integral conditions (3), and by H_0 (resp. H^0) the class of functions $h \in H$ satisfying the limit condition

$$\liminf_{t\to\alpha} |h| = 0 \quad \text{(resp. } \liminf_{t\to\beta} |h| = 0\text{)}. \tag{20}$$

In the cases considered in the sequel the condition (20) is equivalent to

$$\lim_{t\to\alpha} h \equiv h(\alpha) = 0 \quad \text{(resp. } \lim_{t\to\beta} h \equiv h(\beta) = 0\text{)}. \tag{21}$$

THEOREM 3. (i) *If both α and β are of type I, then $\widetilde{H} = H$.*

(ii) *If α is of type II and β is of type I, then $\widetilde{H} \supset H_0$.*

(iii) *If α is of type III and β is of type I, then $\widetilde{H} = H_0$.*

(iv) *If α is of type I and β is of type II, then $\widetilde{H} \supset H^0$;*

(v) *If α is of type I and β is of type III, then $\widetilde{H} = H^0$;*

(vi) *If both α and β are of type II or III, then $\widetilde{H} = H_0 \cap H^0$.*

Proof. If α is of type I and $h \in AC(I)$, then $vh^2 \le 0$ on U_α and hence $\liminf_{t\to\alpha} vh^2 \le 0$.

Let α be of type II or III. Then by Lemma 2 we have $\int_\alpha^t r^{-1}\,d\tau < \infty$ for some $t \in I$. Furthermore, if $h \in AC(I)$ and $\int_I r\dot{h}^2\,dt < \infty$, then using

Schwarz's inequality we obtain the estimate

$$|h(b)-h(a)| \le \int_a^b |\dot h|\,dt \le \Big(\int_a^b r^{-1}\,dt\Big)^{1/2}\Big(\int_a^b r\dot h^2\,dt\Big)^{1/2}, \tag{22}$$

where $\alpha < a < b \le t$, and the Cauchy condition for the existence of the limit yields the existence of a finite limit $h(\alpha) = \lim_{t\to\alpha} h$.

If α is of type III and $h \in \widetilde H$, then $v(\alpha) = \infty$ and a finite limit $h(\alpha)$ exists. If $h(\alpha) \ne 0$, then $\lim_{t\to\alpha} vh^2 = \infty$, which contradicts (4). Thus $h(\alpha) = 0$, i.e. $h \in H_0$.

If α is of type II or III, then by Lemma 2 we have $\int_\alpha^t r^{-1}\,d\tau < \infty$ for some $t \in I$ and $v(t)\int_\alpha^t r^{-1}\,d\tau = O(1)$ as $t \to \alpha$. Furthermore, if $h \in H_0$, then from (22) as $a \to \alpha$ and $b = t$ we get the estimate

$$0 \le |vh^2| \le \Big|v(t)\int_\alpha^t r^{-1}\,d\tau\Big| \int_\alpha^t r\dot h^2 d\tau$$

and hence $\lim_{t\to\alpha} vh^2 = 0$.

Similar symmetric conclusions are valid if α is replaced by β and the class H_0 by H^0.

If both α and β are of type II or III and $h \in \widetilde H$, then $\lim_{t\to\alpha} vh^2 \ge 0$ and $\lim_{t\to\beta} vh^2 \le 0$ and by (14) we have

$$\lim_{t\to\alpha} vh^2 = \lim_{t\to\beta} vh^2 = 0. \tag{23}$$

Since $v(\alpha) > 0, v(\beta) < 0$ and the finite values $h(\alpha)$ and $h(\beta)$ exist, it follows from (23) that $h(\alpha) = h(\beta) = 0$, i.e. $h \in H_0 \cap H^0$.

Basing on these considerations we can easily derive the theorem. ∎

Now we prove some new inequalities. According to these examples we see that all cases of Theorem 3 can hold.

Example 1. Take $I = (0,1), r = e^{at}$ and $\varphi = e^{ct}$ where $a \ne 0$ and c are arbitrary constants. Then the functions

$$s = \frac{1-ac-c^2}{a}e^{at} + k,$$

where k is an arbitrary constant and $u = e^{at}$, satisfy equation (7) on I, and inequality (15) takes the form

$$\int_0^1 \Big(e^{at}\dot h^2 + 2\Big(\frac{1-ac-c^2}{a}e^{at} + k\Big)h\dot h + e^{at}h^2\Big)\,dt \ge 0. \tag{24}$$

Denote by $\widetilde a$ the root of the equation $2e^a - a = 2$ such that $-2 < \widetilde a < -1$ and by $\widehat a$ the root of $(2-a)e^a = 2$ such that $1 < \widehat a < 2$. From Theorems 2 and 3(i), (ii), (iv) we obtain:

• *If either* (i) *or* (ii) *holds, where*

(i) $\widetilde{a} < a < 0$ *or* $a > 0$,

$$-1 + \frac{a}{e^a - 1} < c < 1, \qquad \frac{c^2-1}{a} e^a < k < \frac{c^2-1}{a} + c - 1,$$

(ii) $a < 0$ *or* $0 < a < \widehat{a}$,

$$-1 < c < 1 - \frac{ae^a}{e^a - 1}, \qquad \Big(\frac{c^2-1}{a} + c + 1\Big) e^a < k < \frac{c^2-1}{a},$$

then inequality (24) *holds for every* $h \in H$, *i.e. for* h *satisfying only the integral conditions* (3).

• *If*

$$\text{(iii)} \quad a < \widetilde{a}, \quad 1 < c < -1 + \frac{a}{e^a - 1}, \quad \frac{c^2-1}{a} < k < \frac{c^2-1}{a} + c - 1,$$

then (24) *holds for* $h \in H_0$.

• *If*

$$\text{(iv)} \ a > \widehat{a}, \quad 1 - \frac{ae^a}{e^a - 1} < c < -1, \quad \Big(\frac{c^2-1}{a} + c + 1\Big) e^a < k < \frac{c^2-1}{a},$$

then (24) *holds for* $h \in H^0$.

Inequality (24) *is strict for* $h \not\equiv 0$.

The condition $ru - s^2 < 0$ is satisfied on the interval $(0, \tau_0)$ with

$$0 < \tau_0 = \frac{1}{a} \ln \frac{ak}{(c-1)(c+a+1)} < 1$$

in case (i), on $(\tau_1, 1)$ with

$$0 < \tau_1 = \frac{1}{a} \ln \frac{ak}{(c+1)(c+a-1)} < 1$$

in case (ii) and on $(0, 1)$ in cases (iii) and (iv).

Example 2. Let $I = (\alpha, \beta)$, where $0 \le \alpha < \beta \le \infty$. Take $r = t^a$ and $\varphi = t^{(1-a)/2}$ on I, where $a \ne 1$ is an arbitrary constant. Then the functions $s = At^{a-1}$ and $u = \frac{1}{4}(a-1)(6A - a + 1)t^{a-2}$, where A is an arbitrary constant, satisfy equation (7) on I. If (i) $a < 1$ and $(a-1)/2 < A \le 0$ or (ii) $a > 1$ and $0 \le A < (a-1)/2$, then $ru - s^2 < 0$ on I and in case (i) the boundary point α is of type II if $\alpha > 0$ or of type III if $\alpha = 0$ and the boundary point β is of type I, and in case (ii) the point α is of type I and the point β is of type II if $\beta < \infty$ or of type III if $\beta = \infty$.

Applying Theorems 2 and 3(ii), (iii), (iv), (v) we get:

If $0 \le \alpha < \beta \le \infty$ *and either* $a < 1$, $(a-1)/2 < A \le 0$ *or* $a > 1$, $0 \le A < (a-1)/2$, *and* $h \not\equiv 0$, *then*

$$\int_{\alpha}^{\beta} \left[t^a \dot{h}^2 + 2At^{a-1} h\dot{h} + \tfrac{1}{4}(a-1)(6A-a+1)t^{a-2}h^2\right] dt > 0 \tag{25}$$

for every $h \in \widetilde{H}$; *and* $\widetilde{H} = H_0$ *if* $a < 1$ *and* $\widetilde{H} = H^0$ *if* $a > 1$.

Inequality (25) for $A = 0$ was considered in [3] (cf. [13]); if $\alpha = 0, \beta = \infty$ and $a = 0$ we get the well-known Hardy integral inequality ([11, Th. 253]).

EXAMPLE 3. We take $I = (-1, 1)$ and $r = (1-t^2)^a$ on I. We put $\varphi = (1-t^2)^k$ on I and $k = 1-a$ or $k = 1/2 - a$, where a is an arbitrary constant such that $k > 0$. Then the functions $s = At(1-t^2)^b$ and $u = (B - Ct^2)(1-t^2)^{b-1}$, where $b = a, B = A + 2a - 2, C = A(2a+1)$ if $k = 1-a$ or $b = a-1, B = A + 2a - 1, C = A(2a-1)$ if $k = 1/2 - a$ and A is an arbitrary constant, satisfy (7) on I.

If $a < -1/2$, $0 \le A < 1 - 1/a$ or $-1/2 \le a < 1, 0 \le A < 2 - 2a$ in the case $k = 1 - a$; or $a < 0, 0 \le A < 1$ or $0 \le a < 1/2$, $0 \le A < 1 - 2a$ in the case $k = 1/2 - a$, then both boundary points are of type III.

Applying Theorems 2 and 3(vi) we infer the following:

Let $h \in H_0 \cap H^0$.

(i) *If* $a < -1/2$, $0 \le A < 1 - 1/a$ *or* $-1/2 \le a < 1$, $0 \le A < 2 - 2a$, *then*

$$\int_{-1}^{1} [(1-t^2)^a \dot{h}^2 + 2At(1-t^2)^a h\dot{h} + (B - Ct^2)(1-t^2)^{a-1}h^2]\, dt \ge 0, \tag{26}$$

where $B = A + 2a - 2$ *and* $C = A(2a+1)$. *Equality holds in* (26) *if and only if* $h = c(1-t^2)^{1-a}$, *where* $c = \text{const} \ne 0$.

(ii) *If* $a < 0$, $0 \le A < 1$ *or* $0 \le a < 1/2$, $0 \le A < 1 - 2a$, *then*

$$\int_{-1}^{1} [(1-t^2)^a \dot{h}^2 + 2At(1-t^2)^{a-1} h\dot{h} + (B - Ct^2)(1-t^2)^{a-2}h^2]\, dt \ge 0, \tag{27}$$

where $B = A + 2a - 1$ *and* $C = A(2a-1)$. *If* $h \not\equiv 0$, *then for* $a < 0$ *equality holds in* (27) *if and only if* $h = c(1-t^2)^{1/2-a}$, *where* $c = \text{const} \ne 0$, *and for* $0 \le a < 1/2$ *inequality* (27) *is strict.*

The condition $ru - s^2 < 0$ is satisfied on $(-1, 1)$ in both cases.

Inequalities (26) and (27) for $A = 0$ were discussed in [12] and [16] (cf. [10]).

Let $s \in AC(I)$ and $u \in M(I)$ be arbitrary functions satisfying the differential inequality (2) a.e. on I such that $s = 0$ on I and $u < 0$ a.e. on I. Then the second and third conditions of (3) are trivially satisfied and inequality (15) takes the form

$$\int_I |u|h^2\,dt \le \int_I r\dot{h}^2\,dt. \tag{28}$$

Inequalities of the form (28) are the integral inequalities of Sturm–Liouville type which were examined in [10].

In this case we have $ru - s^2 = ru < 0$ a.e. on I and $sv = 0$ on I. Thus the function v is decreasing on I and $v(\alpha) > v(\beta)$. Moreover, α (resp. β) is of type I if $v(\alpha) \le 0$ (resp. $v(\beta) \ge 0$), of type II if $0 < v(\alpha) < \infty$ (resp. $-\infty < v(\beta) < 0$) and of type III if $v(\alpha) = \infty$ (resp. $v(\beta) = -\infty$). Hence α and β cannot be simultaneously of type I. In this way from Theorems 2 and 3 we get Theorems 3 and 4 of [10].

Now, let $s \in AC(I)$ and $u \in M(I)$ be arbitrary functions satisfying the differential inequality (2) a.e. on I such that $u \le 0$ a.e. on I. Then the third of the integral conditions (3) is trivially satisfied and if $s^2 + u^2 > 0$ a.e. on I, then $ru - s^2 < 0$ a.e. on I. Next by (18) we have $\dot{v} \le u - r\dot{\varphi}^2\varphi^{-2} \le 0$ a.e. on I. Thus v is nonincreasing on I and $v(\alpha) > v(\beta)$ except for the trivial case $s \equiv 0$ and $u \equiv 0$. Hence α and β cannot be simultaneously of type I.

THEOREM 4. *Let $u \le 0$ a.e. on I and let $h \in AC(I)$ satisfy the integral condition $\int_I r\dot{h}^2\,dt < \infty$. If $s \le 0$ on I, $v(\beta) \ge 0$ and $h(\alpha) = 0$, or $s \ge 0$ on I, $v(\alpha) \le 0$ and $h(\beta) = 0$, then*

$$2\int_I |sh\dot{h}|\,dt + \int_I |u|h^2\,dt \le \int_I r\dot{h}^2\,dt. \tag{29}$$

If $h \not\equiv 0$, then equality holds in (29) if and only if s and u satisfy the differential equation (7) a.e. on I, $\varphi^{-1}h = \text{const} \ne 0$,

$$\int_I r\dot{\varphi}^2\,dt < \infty, \qquad \lim_{t\to\alpha} v\varphi^2 = \lim_{t\to\beta} v\varphi^2, \tag{30}$$

and $\varphi(\alpha) = 0$, $\dot{\varphi} \ge 0$ on I provided $s \le 0$ on I, or $\varphi(\beta) = 0$, $\dot{\varphi} \le 0$ on I provided $s \ge 0$ on I.

Proof. Let $s \le 0$ on I and $v(\beta) \ge 0$. Then $v(\alpha) > 0$ and $v > 0$ on I, whence $sv \le 0$ on I. Thus α is of the type II or III and β is of type I.

Further, let $h_+ \in AC(I)$ be such that $h_+(\alpha) = 0, h_+ \ge 0$ on I, $\dot{h}_+ \ge 0$ a.e. on I and $\int_I r\dot{h}_+^2\,dt < \infty$. Then $\int_I sh_+\dot{h}_+\,dt \le 0$ and the second of the integral conditions (3) is satisfied. Thus $h_+ \in H_0$ and by Theorem 3(ii)–(iii) we have $h_+ \in \widetilde{H}$. Next by Theorem 2 we get

$$2\int_I |s|h_+\dot{h}_+\,dt + \int_I |u|h_+^2\,dt \le \int_I r\dot{h}_+^2\,dt. \tag{31}$$

Now, let $h \in AC(I)$ be such that $h(\alpha) = 0$ and $\int_I r\dot{h}^2\,dt < \infty$. Put $h_+ = \int_\alpha^t |\dot{h}|\,d\tau$. Then $h_+ \in AC(I)$, $h_+(\alpha) = 0$, $h_+ \ge 0$ on I, $\dot{h}_+ = |\dot{h}| \ge 0$ a.e. on I and

$$\int_I r\dot{h}_+^2\,dt = \int_I r\dot{h}^2\,dt < \infty. \tag{32}$$

Hence h_+ satisfies inequality (31). Notice that

$$|h| = \Big|\int_\alpha^t \dot{h}\,d\tau\Big| \le \int_\alpha^t |\dot{h}|\,d\tau = h_+$$

on I, and equality holds if and only if $\dot{h}$ does not change sign on I. Hence

$$2\int_I |sh\dot{h}|\,dt + \int_I |u|h^2\,dt \le 2\int_I |s|h_+\dot{h}_+\,dt + \int_I |u|h_+^2\,dt \tag{33}$$

and by (31)–(33) we get inequality (29).

If both sides of (29) are equal for some non-vanishing function $h \in AC(I)$ such that $h(\alpha) = 0$ and $\int_I r\dot{h}^2\,dt < \infty$, then by (31)–(33) it follows that for $h_+ = \int_\alpha^t |\dot{h}|\,d\tau$ equality holds in (31) and (33). It follows that $|h| = h_+$ and hence $\dot{h}$ does not change sign on I. Since $h_+ \in \widetilde{H}$ and by Theorem 2, equality occurs in (31) if and only if s and u satisfy (7) a.e. on I, $\varphi^{-1}h_+ = \text{const} > 0$ and conditions (16) are satisfied. Hence $\varphi^{-1}h = \text{const} \ne 0$, $\varphi(\alpha) = 0$ and $\dot{\varphi} \ge 0$ on I.

Let s and u satisfy (7) a.e. on I and φ be such that $\varphi(\alpha) = 0, \dot{\varphi} \ge 0$ and conditions (30) hold. Then we easily check that the function $h = c\varphi$, where $c = \text{const} \ne 0$, satisfies $h(\alpha) = 0$ and $\int_I r\dot{h}^2\,dt < \infty$ and for this function equality holds in (29).

The case when $s \ge 0$ on I, $v(\alpha) \le 0$, $h(\beta) = 0$ can be proved in a similar way considering the function $h_- = \int_t^\beta |\dot{h}|\,d\tau \in \widetilde{H}$. ∎

Inequalities (29) embrace, as a particular case (if $u = 0$ on I), the integral inequalities of Opial type which were examined in [13].

Example 4. Let $I = (\alpha, \beta)$, $-\infty \le \alpha < \beta \le \infty$. Let $r > 0$ and $u \le 0$ be functions absolutely continuous on I such that $\int_I r^{-1}\,dt < \infty$ and

$$\int_I u\,dt \ge -\Big(\int_I r^{-1}\,dt\Big)^{-1}.$$

If we put $\varphi = \int_\alpha^t r^{-1} d\tau$, then the functions u and

$$s = -\int_t^\beta u\,d\tau - \Big(\int_I r^{-1}\,dt\Big)^{-1} \le 0 \tag{34}$$

satisfy equation (7) on I and $v(\beta) = 0$. If we put $\varphi = \int_t^\beta r^{-1} d\tau$, then the functions u and

$$s = \int\limits_\alpha^t u\, d\tau + \Big(\int\limits_I r^{-1}\, dt\Big)^{-1} \geq 0 \tag{35}$$

satisfy (7) on I and $v(\alpha) = 0$.

Now, applying Theorem 4 we get:

If $h \in AC(I)$ satisfies $\int_I r\dot{h}^2\, dt < \infty$ and $h(\alpha) = 0$ or $h(\beta) = 0$, then the inequality of the form (29) *with s defined by* (34) *if $h(\alpha) = 0$ or by* (35) *if $h(\beta) = 0$ is valid. In both cases equality holds only for $h = c\varphi$, where* $c = \mathrm{const}$.

If $u \equiv 0$, then we obtain the inequalities which were considered in [4] (cf. [13]).

In the case when $0 = \alpha < \beta \leq 1, r = 1, u = -1$ on I we obtain the inequality

$$2\int\limits_0^\beta \Big(\frac{1-\beta^2}{\beta} + t\Big)|h\dot{h}|\, dt + \int\limits_0^\beta h^2\, dt \leq \int\limits_0^\beta \dot{h}^2\, dt, \tag{36}$$

which holds for all $h \in AC((0,\beta))$ such that $h(0) = 0$ and $\int_0^\beta \dot{h}^2\, dt < \infty$, and the inequality

$$2\int\limits_0^\beta \Big(\frac{1}{\beta} - t\Big)|h\dot{h}|\, dt + \int\limits_0^\beta h^2\, dt \leq \int\limits_0^\beta \dot{h}^2\, dt, \tag{37}$$

which holds for all $h \in AC((0,\beta))$ such that $h(\beta) = 0$ and $\int_0^\beta \dot{h}^2\, dt < \infty$.

Equality holds in (36) only for $h = ct$, and in (37) only for $h = c(\beta - t)$, where $c = \mathrm{const}$.

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