We derive and examine some quadratic integral inequalities of first order of the form
\[
\int_I \left( r \dot{h}^2 + 2s \dot{h}h + uh^2 \right) dt \geq 0, \quad h \in H,
\]
where \( I = (\alpha, \beta) \), \(-\infty \leq \alpha < \beta \leq \infty\), \( r, s \) and \( u \) are given real functions of the variable \( t \), \( H \) is a given class of absolutely continuous functions and \( \dot{h} \equiv dh/dt \). The inequalities of the form (1) comprise as special cases integral inequalities of Sturm–Liouville type examined by Florkiewicz and Rybarski [10] and quadratic integral inequalities of Opial type examined by Kuchta [13]. The method used to obtain the integral inequalities of the form (1) is an extension of the uniform method of obtaining various types of integral inequalities involving a function and its derivative. The method we extend was used in [8]–[10], [13]. The method makes it possible, given a function \( r \) and an auxiliary function \( \varphi \), to determine the functions \( s \) and \( u \), and next the class \( H \) of the functions \( h \) for which (1) holds. In this paper \( s \) and \( u \) are solutions of a certain differential inequality which makes it possible to obtain a large set of functions \( s \) and \( u \) for which inequality (1) holds.

Inequalities of the form (1) have been considered by Beesack [2]–[5], Redheffer [16], Yang [18], Benson [6], Boyd [7] and others (for an extensive bibliography see [14]).

The positive definiteness of quadratic functionals of the form (1) is a basic problem of the theory of singular quadratic functionals introduced by Morse and Leighton [15] (cf. [17], [1]). This problem is of significant importance for the oscillation theory for second order linear differential equations on a non-compact interval (see [17]).

Let \( I = (\alpha, \beta) \), \(-\infty \leq \alpha < \beta \leq \infty\), be an arbitrary open interval. We denote by \( M(I) \) the class of real functions which are defined and Lebesgue

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measurable on \( I \) and by \( AC(I) \) the class of real functions defined and absolutely continuous on \( I \). Let \( r \in AC(I) \) and \( \varphi \in AC(I) \) be given functions such that \( r > 0, \varphi > 0 \) on \( I \) and \( \varphi \in AC(I) \). Then \( r^{-1} = 1/r \in AC(I) \) and \( \varphi^{-1} = 1/\varphi \in AC(I) \). Let \( s \in AC(I) \) and \( u \in M(I) \) be arbitrary functions satisfying the differential inequality

\[
\dot{s} - u + (r\dot{\varphi})\varphi^{-1} \leq 0
\]

almost everywhere on \( I \).

Denote by \( \hat{H} \) the class of functions \( h \in AC(I) \) satisfying the following integral and limit conditions:

\[
\int_I \dot{r}h^2 \, dt < \infty, \quad \int_I sh\dot{h} \, dt < \infty, \quad \int_I uh^2 \, dt < \infty,
\]

\[
\liminf_{t \to \alpha} \dot{v}h^2 < \infty, \quad \limsup_{t \to \beta} \dot{v}h^2 > -\infty,
\]

where

\[
v = r\dot{\varphi}\varphi^{-1} + s.
\]

**Theorem 1.** For every \( h \in \hat{H} \) both limits in (4) are proper and finite, and

\[
\lim_{t \to \beta} \dot{v}h^2 - \lim_{t \to \alpha} \dot{v}h^2 \leq \int_I (\dot{r}h^2 + 2sh\dot{h} + uh^2) \, dt.
\]

If \( h \neq 0 \), then equality holds in (6) if and only if \( s \) and \( u \) satisfy the differential equation

\[
\dot{s} - u + (r\dot{\varphi})\varphi^{-1} = 0
\]

a.e. on \( I \), \( \varphi \in \hat{H} \) and \( h = c\varphi \) with \( c = \text{const} \neq 0 \).

**Proof.** Let \( h \in AC(I) \). By (5) and our assumptions we have \( \dot{v}h^2 \in AC(I) \) and \( \varphi^{-1}h \in AC(I) \) and we easily check that

\[
\dot{r}h^2 + 2sh\dot{h} + uh^2 = (\dot{v}h^2)^* + f + g \quad \text{a.e. on } I,
\]

where

\[
f = - (\dot{s} - u + (r\dot{\varphi})\varphi^{-1})h^2 \geq 0,
\]

\[
g = r\varphi^2[(\varphi^{-1}h)^*]^2 \geq 0.
\]

Now, let \( h \in \hat{H} \). By the first condition of (3) it follows that \( \dot{r}h^2 \) is summable on \( I \) because \( \dot{r}h^2 \geq 0 \) on \( I \). By the assumptions, all other functions appearing in (8) are summable on each compact interval \([a, b] \subset I \).

Thus, by (8),

\[
\int_a^b \dot{r}h^2 \, dt + 2\int_a^b sh\dot{h} \, dt + \int_a^b uh^2 \, dt = \dot{v}h^2|_a^b + \int_a^b f \, dt + \int_a^b g \, dt
\]
for all $\alpha < a < b < \beta$. By (4) there exist two sequences $\{a_n\}$ and $\{b_n\}$ such that $\alpha < a_n < b_n < \beta$, $a_n \to \alpha$, $b_n \to \beta$ and

$$\lim_{n \to \infty} (-vh^2)|_{a_n} > -\infty, \quad \lim_{n \to \infty} vh^2|_{b_n} > -\infty.$$ 

Thus there is a constant $C$ such that

$$vh^2|_{a_n} \geq C > -\infty.$$ 

In view of (9) and (10), from (11) we infer that

$$\int_{a_n}^{b_n} (2shh + uh^2) dt \geq - \int_{a_n}^{b_n} r\dot{h}^2 dt + C \geq - \int_{\alpha}^{\beta} r\dot{h}^2 dt + C$$

and letting $n \to \infty$ gives

$$\int_I (2shh + uh^2) dt \geq - \int_I r\dot{h}^2 dt + C > -\infty.$$ 

By this estimate and by the second and third conditions of (3) we easily see that the functions $sh\dot{h}$ and $uh^2$ are summable on $I$. In the analogous way we show that $f$ and $g$ are summable on $I$. Thus all integrals in (11) have finite limits as $a \to \alpha$ or $b \to \beta$. It follows that both limits in (4) are proper and finite. Now, by (11), as $a \to \alpha$ and $b \to \beta$, we obtain

$$\int_I (r\dot{h}^2 + 2shh + uh^2) dt = \lim_{t \to \beta} vh^2 - \lim_{t \to \alpha} vh^2 + \int_I f dt + \int_I g dt$$

whence (6) follows, since $f \geq 0$ and $g \geq 0$ on $I$.

By (12), equality holds in (6) for a non-vanishing function $h \in \tilde{H}$ if and only if $\int_I f dt = 0$ and $\int_I g dt = 0$, i.e. $f = 0$ and $g = 0$ a.e. on $I$. In view of (10), $g = 0$ a.e. on $I$ if and only if $(\varphi^{-1}h) = 0$ a.e. on $I$. Hence $h = c\varphi$, where $c = \text{const} \neq 0$, since $\varphi^{-1}h \in AC(I)$ by assumption. Thus $\varphi \in \tilde{H}$. Further, from (9), $f = 0$ a.e. on $I$ if and only if $s$ and $u$ satisfy (7) a.e. on $I$, because $h^2 = c^2\varphi^2 > 0$ on $I$. □

Denote by $\tilde{H}$ the class of functions $h \in \tilde{H}$ satisfying additionally the limit condition

$$\liminf_{t \to \alpha} vh^2 \leq \limsup_{t \to \beta} vh^2.$$ 

By Theorem 1 we can write it in the equivalent form

$$\lim_{t \to \alpha} vh^2 \leq \lim_{t \to \beta} vh^2.$$ 

**Theorem 2.** For every $h \in \tilde{H}$,

$$\int_I (r\dot{h}^2 + 2shh + uh^2) dt \geq 0.$$
If \( h \neq 0 \), then equality holds in (15) if and only if \( \varphi^{-1}h = \text{const} \neq 0 \) and the additional conditions (7) and
\[
\varphi \in \tilde{H}, \quad \lim_{t \to \alpha} v\varphi^2 = \lim_{t \to \beta} v\varphi^2
\]
are satisfied.

**Proof.** By (14) and Theorem 1, inequality (15) follows from (6). If equality occurs in (15) for some non-vanishing function \( h \in \tilde{H} \), then by (6) and (14) we have \( \lim_{t \to \alpha} vh^2 = \lim_{t \to \beta} vh^2 \). Using Theorem 1 once again we conclude that (7) holds, \( \varphi \in \tilde{H} \) and \( h = c\varphi \), where \( c = \text{const} \neq 0 \), whence we obtain (16).

Now we describe the class \( \tilde{H} \) in the cases that occur most frequently. If \( ru - s^2 \geq 0 \) a.e. on \( I \), then inequality (15) holds for all \( h \in AC(I) \). Thus it is natural to consider cases like \( ru - s^2 < 0 \) a.e. in some interval \((a, b) \subset I \).

**Lemma 1.** Let \( \alpha \leq a < b \leq \beta \). If \( ru - s^2 < 0 \) a.e. on \((a, b)\), then the function \( v \) satisfies the differential inequality
\[
rv < 2sv - v^2
\]
a.e. on \((a, b)\).

**Proof.** By (5) and (2) we have
\[
\dot{v} = (r\dot{\varphi})\varphi^{-1} + s - r\varphi^2\varphi^{-2} \leq u - r\varphi^2\varphi^{-2}
\]
a.e. on \((a, b)\). Thus from the assumptions we obtain
\[
rv \leq ru - r^2\varphi^2\varphi^{-2} < s^2 - (r\dot{\varphi}\varphi^{-1})^2,
\]
since \( r > 0 \) on \( I \). Further, by (5) we have \((r\dot{\varphi}\varphi^{-1})^2 = s^2 - 2sv + v^2 \) on \( I \), whence (17) follows.

We will denote by \( U_\alpha \) (resp. \( U_\beta \)) some right-hand (resp. left-hand) neighbourhood of the point \( \alpha \) (resp. \( \beta \)). By Lemma 1 it follows that if \( ru - s^2 < 0 \) a.e. on \( U_\alpha \) and \( sv \leq 0 \) on \( U_\alpha \), then \( \dot{v} < 0 \) a.e. on \( U_\alpha \) and consequently the function \( v \) is decreasing on \( U_\alpha \). Thus the limit \( v(\alpha) = \lim_{t \to \alpha} v \) exists and \( v < v(\alpha) \) on \( U_\alpha \). Analogously, if \( ru - s^2 < 0 \) a.e. on \( U_\beta \) and \( sv \leq 0 \) on \( U_\beta \), then \( v(\beta) = \lim_{t \to \beta} v \) exists and \( v > v(\beta) \) on \( U_\beta \).

**Lemma 2.** If \( ru - s^2 < 0 \) a.e. on \( U_\alpha \) (resp. \( U_\beta \)), \( sv \leq 0 \) on \( U_\alpha \) (resp. \( U_\beta \)) and \( v(\alpha) \neq 0 \) (resp. \( v(\beta) \neq 0 \)), then \( \int_{t}^{\alpha} r^{-1} d\tau < \infty \) (resp. \( \int_{t}^{\beta} r^{-1} d\tau < \infty \)) for some \( t \in I \). Moreover, if \( v(\alpha) = \infty \) (resp. \( v(\beta) = -\infty \)), then \( v(t)\int_{t}^{\alpha} r^{-1} d\tau = O(1) \) as \( t \to \alpha \) (resp. \( v(t)\int_{t}^{\beta} r^{-1} d\tau = O(1) \) as \( t \to \beta \)).

**Proof.** We prove the lemma only for the point \( \alpha \). The proof for \( \beta \) is analogous.
Let \( v(\alpha) \neq 0 \) and consider some right-hand neighbourhood \( U \subset U_\alpha \) of \( \alpha \) such that \( v \neq 0 \) on \( U \). By the assumptions and Lemma 1, from (17) we get \( \dot{r} \leq -v^2 \) a.e. on \( U \). Then \( r^{-1} \leq -v^{-2} \) a.e. on \( U \), because \( r > 0 \) on \( I \) and we have the estimate

\[
(19) \quad \int_a^t r^{-1} \, d\tau \leq - \int_a^t v^{-2} \, d\tau = v^{-1}(t) - v^{-1}(a)
\]

for \( \alpha < a < t < \beta \) on \( U \).

If \( v(\alpha) > 0 \) (i.e. \( v > 0 \) on \( U \)), then by (19) as \( a \to \alpha \) we obtain \( \int_\alpha^t r^{-1} \, d\tau < v^{-1}(t) < \infty \). Hence \( 0 < v(t) \int_\alpha^t r^{-1} \, d\tau < 1 \) and thus \( v(t) \int_\alpha^t r^{-1} \, d\tau = O(1) \) as \( t \to \alpha \).

If \( v(\alpha) < 0 \) (i.e. \( v < 0 \) on \( U \)), then by (19) we obtain \( \int_\alpha^t r^{-1} \, d\tau < -v^{-1}(a) \), whence as \( a \to \alpha \) we get \( \int_\alpha^t r^{-1} \, d\tau < -v^{-1}(\alpha) < \infty \). ■

We introduce the following terminology:

- A boundary point \( \alpha \) (resp. \( \beta \)) of the interval \( I \) is of type I if \( v \leq 0 \) on \( U_\alpha \) (resp. \( v \geq 0 \) on \( U_\beta \));
- \( \alpha \) (resp. \( \beta \)) is of type II if \( ru - s^2 < 0 \) a.e. on \( U_\alpha \) (resp. \( U_\beta \)) and \( sv \leq 0 \) on \( U_\alpha \) (resp. \( U_\beta \)) and \( 0 < v(\alpha) < \infty \) (resp. \( -\infty < v(\beta) < 0 \));
- \( \alpha \) (resp. \( \beta \)) is of type III if \( ru - s^2 < 0 \) a.e. on \( U_\alpha \) (resp. \( U_\beta \)) and \( sv \leq 0 \) on \( U_\alpha \) (resp. \( U_\beta \)) and \( v(\alpha) = \infty \) (resp. \( v(\beta) = -\infty \)).

We denote by \( H \) the class of functions \( h \in AC(I) \) satisfying the integral conditions (3), and by \( H_0 \) (resp. \( H^0 \)) the class of functions \( h \in H \) satisfying the limit condition

\[
(20) \quad \liminf_{t \to \alpha} |h| = 0 \quad (\text{resp.} \quad \liminf_{t \to \beta} |h| = 0).
\]

In the cases considered in the sequel the condition (20) is equivalent to

\[
(21) \quad \lim_{t \to \alpha} h \equiv h(\alpha) = 0 \quad (\text{resp.} \quad \lim_{t \to \beta} h \equiv h(\beta) = 0).
\]

**Theorem 3.** (i) If both \( \alpha \) and \( \beta \) are of type I, then \( \tilde{H} = H \).

(ii) If \( \alpha \) is of type II and \( \beta \) is of type I, then \( \tilde{H} \supset H_0 \).

(iii) If \( \alpha \) is of type III and \( \beta \) is of type I, then \( \tilde{H} = H_0 \).

(iv) If \( \alpha \) is of type I and \( \beta \) is of type II, then \( \tilde{H} \supset H^0 \).

(v) If \( \alpha \) is of type I and \( \beta \) is of type III, then \( \tilde{H} = H^0 \).

(vi) If both \( \alpha \) and \( \beta \) are of type II or III, then \( \tilde{H} = H_0 \cap H^0 \).

**Proof.** If \( \alpha \) is of type I and \( h \in AC(I) \), then \( vh^2 \leq 0 \) on \( U_\alpha \) and hence

\[ \liminf_{t \to \alpha} vh^2 \leq 0. \]

Let \( \alpha \) be of type II or III. Then by Lemma 2 we have \( \int_\alpha^t r^{-1} \, d\tau < \infty \) for some \( t \in I \). Furthermore, if \( h \in AC(I) \) and \( \int_I rh^2 \, dt < \infty \), then using
Schwarz’s inequality we obtain the estimate
\[ |h(b) - h(a)| \leq \int_a^b |h| \, dt \leq \left( \int_a^b r^{-1} \, dt \right)^{1/2} \left( \int_a^b r \hat{h}^2 \, dt \right)^{1/2}, \]
where \( \alpha < a < b \leq t \), and the Cauchy condition for the existence of the limit yields the existence of a finite limit \( h(\alpha) = \lim_{t \to \alpha} h \).

If \( \alpha \) is of type III and \( h \in \tilde{H} \), then \( v(\alpha) = \infty \) and a finite limit \( h(\alpha) \) exists. If \( h(\alpha) \neq 0 \), then \( \lim_{t \to \alpha} v h^2 = \infty \), which contradicts (4). Thus \( h(\alpha) = 0 \), i.e. \( h \in H_0 \).

If \( \alpha \) is of type II or III, then by Lemma 2 we have \( \int_\alpha^t r^{-1} \, d\tau < \infty \) for some \( t \in I \) and \( v(t) \int_\alpha^t r^{-1} \, d\tau = O(1) \) as \( t \to \alpha \). Furthermore, if \( h \in H_0 \), then from (22) as \( a \to \alpha \) and \( b = t \) we get the estimate
\[ 0 \leq |v h^2| \leq \left| v(t) \int_\alpha^t r^{-1} \, d\tau \right| r \hat{h}^2 \, d\tau \]
and hence \( \lim_{t \to \alpha} v h^2 = 0 \).

Similar symmetric conclusions are valid if \( \alpha \) is replaced by \( \beta \) and the class \( H_0 \) by \( H^0 \).

If both \( \alpha \) and \( \beta \) are of type II or III and \( h \in \tilde{H} \), then \( \lim_{t \to \alpha} v h^2 \geq 0 \) and \( \lim_{t \to \beta} v h^2 \leq 0 \) and by (14) we have
\[ \lim_{t \to \alpha} v h^2 = \lim_{t \to \beta} v h^2 = 0. \]
Since \( v(\alpha) > 0, v(\beta) < 0 \) and the finite values \( h(\alpha) \) and \( h(\beta) \) exist, it follows from (23) that \( h(\alpha) = h(\beta) = 0 \), i.e. \( h \in H_0 \cap H^0 \).

Basing on these considerations we can easily derive the theorem.

Now we prove some new inequalities. According to these examples we see that all cases of Theorem 3 can hold.

**Example 1.** Take \( I = (0, 1) \), \( r = e^{at} \) and \( \varphi = e^{ct} \) where \( a \neq 0 \) and \( c \) are arbitrary constants. Then the functions
\[ s = \frac{1 - ac - c^2}{a} e^{at} + k, \]
where \( k \) is an arbitrary constant and \( u = e^{at} \), satisfy equation (7) on \( I \), and inequality (15) takes the form
\[ \int_0^1 \left( e^{at} \hat{h}^2 + 2 \left( \frac{1 - ac - c^2}{a} e^{at} + k \right) \hat{h} + e^{at} \hat{h}^2 \right) \, dt \geq 0. \]

Denote by \( \tilde{a} \) the root of the equation \( 2 e^{a} - a = 2 \) such that \(-2 < \tilde{a} < -1 \) and by \( \hat{a} \) the root of \((2 - a) e^{a} = 2 \) such that \( 1 < \hat{a} < 2 \). From Theorems 2 and 3(i), (ii), (iv) we obtain:
If either (i) or (ii) holds, where

(i) $\tilde{a} < a < 0$ or $a > 0,$

$$-1 + \frac{a}{e^a - 1} < c < 1,$$  

$$\frac{c^2 - 1}{a} e^a < k < \frac{c^2 - 1}{a} + c - 1,$$

(ii) $a < 0$ or $0 < a < \tilde{a},$

$$-1 < c < 1 - \frac{ae^a}{e^a - 1},$$  

$$\left(\frac{c^2 - 1}{a} + c + 1\right) e^a < k < \frac{c^2 - 1}{a},$$

then inequality (24) holds for every $h \in H,$ i.e. for $h$ satisfying only the integral conditions (3).

If

(iii) $a < \tilde{a},$  

$$1 < c < -1 + \frac{a}{e^a - 1},$$  

$$\frac{c^2 - 1}{a} < k < \frac{c^2 - 1}{a} + c - 1,$$

then (24) holds for $h \in H_0.$

If

(iv) $a > \tilde{a},$  

$$1 - \frac{ae^a}{e^a - 1} < c < -1,$$  

$$\left(\frac{c^2 - 1}{a} + c + 1\right) e^a < k < \frac{c^2 - 1}{a},$$

then (24) holds for $h \in H^0.$

Inequality (24) is strict for $h \neq 0.$

The condition $ru - s^2 < 0$ is satisfied on the interval $(0, \tau_0)$ with

$$0 < \tau_0 = \frac{1}{a} \ln \frac{ak}{(c - 1)(c + a + 1)} < 1$$

in case (i), on $(\tau_1, 1)$ with

$$0 < \tau_1 = \frac{1}{a} \ln \frac{ak}{(c + 1)(c + a - 1)} < 1$$

in case (ii) and on $(0, 1)$ in cases (iii) and (iv).

Example 2. Let $I = (\alpha, \beta),$ where $0 \leq \alpha < \beta \leq \infty.$ Take $r = t^a$ and $\varphi = t^{(1-a)/2}$ on $I,$ where $a \neq 1$ is an arbitrary constant. Then the functions $s = At^{a-1}$ and $u = \frac{1}{2}(a - 1)(6A - a + 1)t^{a-2},$ where $A$ is an arbitrary constant, satisfy equation (7) on $I.$ If (i) $a < 1$ and $(a - 1)/2 < A \leq 0$ or (ii) $a > 1$ and $0 \leq A < (a - 1)/2,$ then $ru - s^2 < 0$ on $I$ and in case (i) the boundary point $\alpha$ is of type II if $\alpha > 0$ or of type III if $\alpha = 0$ and the boundary point $\beta$ is of type I, and in case (ii) the point $\alpha$ is of type I and the point $\beta$ is of type II if $\beta < \infty$ or of type III if $\beta = \infty.$

Applying Theorems 2 and 3(ii), (iii), (iv), (v) we get:
If \(0 \leq \alpha < \beta \leq \infty\) and either \(\alpha < 1\), \((\alpha - 1)/2 < A \leq 0\) or \(\alpha > 1\), \(0 \leq A < (\alpha - 1)/2\), and \(h \neq 0\), then

\[
\int_{-1}^{1} \left[ t^\alpha h^2 + 2At^{\alpha-1}h \dot{h} + \frac{1}{4}(a-1)(6A-a+1)t^{a-2}h^2 \right] dt > 0
\]

for every \(h \in \tilde{H}\); and \(\tilde{H} = H_0\) if \(a < 1\) and \(\tilde{H} = H^0\) if \(a > 1\).

Inequality (25) for \(A = 0\) was considered in [3] (cf. [13]); if \(\alpha = 0\), \(\beta = \infty\) and \(a = 0\) we get the well-known Hardy integral inequality ([11, Th. 253]).

**Example 3.** We take \(I = (-1, 1)\) and \(r = (1 - t^2)^a\) on \(I\). We put \(\varphi = (1 - t^2)^b\) on \(I\) and \(k = 1 - a\) or \(k = 1/2 - a\), where \(a\) is an arbitrary constant such that \(k > 0\). Then the functions \(s = At(1 - t^2)^b\) and \(u = (B - Ct^2)(1 - t^2)^{b-1}\), where \(b = a, B = A + 2a - 2, C = A(2a + 1)\) if \(k = 1 - a\) or \(b = a - 1, B = A + 2a - 1, C = A(2a - 1)\) if \(k = 1/2 - a\) and \(A\) is an arbitrary constant, satisfy (7) on \(I\).

If \(a < -1/2, 0 \leq A < 1 - 1/a\) or \(-1/2 \leq a < 1, 0 \leq A < 2 - 2a\) in the case \(k = 1 - a\); or \(a < 0, 0 \leq A < 1\) or \(0 \leq a < 1/2, 0 \leq A < 1 - 2a\) in the case \(k = 1/2 - a\), then both boundary points are of type III.

Applying Theorems 2 and 3(vi) we infer the following:

Let \(h \in H_0 \cap H^0\).

(i) If \(a < -1/2, 0 \leq A < 1 - 1/a\) or \(-1/2 \leq a < 1, 0 \leq A < 2 - 2a\), then

\[
\int_{-1}^{1} \left[ (1 - t^2)^a h^2 + 2At(1 - t^2)^a h \dot{h} + (B - Ct^2)(1 - t^2)^{a-1}h^2 \right] dt \geq 0,
\]

where \(B = A + 2a - 2\) and \(C = A(2a + 1)\). Equality holds in (26) if and only if \(h = c(1 - t^2)^{1-a}\), where \(c = \text{const} \neq 0\).

(ii) If \(a < 0, 0 \leq A < 1\) or \(0 \leq a < 1/2, 0 \leq A < 1 - 2a\), then

\[
\int_{-1}^{1} \left[ (1 - t^2)^a h^2 + 2At(1 - t^2)^{a-1} h \dot{h} + (B - Ct^2)(1 - t^2)^{a-2}h^2 \right] dt \geq 0,
\]

where \(B = A + 2a - 1\) and \(C = A(2a - 1)\). If \(h \neq 0\), then for \(a < 0\) equality holds in (27) if and only if \(h = c(1 - t^2)^{1/2-a}\), where \(c = \text{const} \neq 0\), and for \(0 \leq a < 1/2\) inequality (27) is strict.

The condition \(ru - s^2 < 0\) is satisfied on \((-1, 1)\) in both cases.

Inequalities (26) and (27) for \(A = 0\) were discussed in [12] and [16] (cf. [10]).

Let \(s \in AC(I)\) and \(u \in M(I)\) be arbitrary functions satisfying the differential inequality (2) a.e. on \(I\) such that \(s = 0\) on \(I\) and \(u < 0\) a.e. on \(I\). Then
the second and third conditions of (3) are trivially satisfied and inequality (15) takes the form

$$\int_{I} |u| h^2 \, dt \leq \int_{I} r h^2 \, dt.$$  

Inequalities of the form (28) are the integral inequalities of Sturm–Liouville type which were examined in [10].

In this case we have $ru - s^2 = ru < 0$ a.e. on $I$ and $sv = 0$ on $I$. Thus the function $v$ is decreasing on $I$ and $v(\alpha) > v(\beta)$. Moreover, $\alpha$ (resp. $\beta$) is of type I if $v(\alpha) \leq 0$ (resp. $v(\beta) \geq 0$), of type II if $0 < v(\alpha) < \infty$ (resp. $-\infty < v(\beta) < 0$) and of type III if $v(\alpha) = \infty$ (resp. $v(\beta) = -\infty$). Hence $\alpha$ and $\beta$ cannot be simultaneously of type I.

In this way from Theorems 2 and 3 we get Theorems 3 and 4 of [10].

Now, let $s \in AC(I)$ and $u \in M(I)$ be arbitrary functions satisfying the differential inequality (2) a.e. on $I$ such that $u \leq 0$ a.e. on $I$. Then the third of the integral conditions (3) is trivially satisfied and if $s^2 + u^2 > 0$ a.e. on $I$, then $ru - s^2 < 0$ a.e. on $I$. Next by (18) we have $\dot{v} \leq u - r \varphi^2 \varphi^{-2} \leq 0$ a.e. on $I$. Thus $v$ is nonincreasing on $I$ and $v(\alpha) > v(\beta)$ except for the trivial case $s \equiv 0$ and $u \equiv 0$. Hence $\alpha$ and $\beta$ cannot be simultaneously of type I.

**Theorem 4.** Let $u \leq 0$ a.e. on $I$ and let $h \in AC(I)$ satisfy the integral condition $\int_I r h^2 \, dt < \infty$. If $s \leq 0$ on $I$, $v(\beta) \geq 0$ and $h(\alpha) = 0$, or $s \geq 0$ on $I$, $v(\alpha) \leq 0$ and $h(\beta) = 0$, then

$$2 \int_{I} |shh| \, dt + \int_{I} |u| h^2 \, dt \leq \int_{I} r h^2 \, dt.$$  

If $h \not\equiv 0$, then equality holds in (29) if and only if $s$ and $u$ satisfy the differential equation (7) a.e. on $I$, $\varphi^{-1} h = \text{const} \not\equiv 0$, 

$$\int_{I} r \varphi^2 \, dt < \infty, \quad \lim_{t \to -\alpha} v \varphi^2 = \lim_{t \to -\beta} v \varphi^2,$$

and $\varphi(\alpha) = 0$, $\dot{\varphi} \geq 0$ on $I$ provided $s \leq 0$ on $I$, or $\varphi(\beta) = 0$, $\dot{\varphi} \leq 0$ on $I$ provided $s \geq 0$ on $I$.

**Proof.** Let $s \leq 0$ on $I$ and $v(\beta) \geq 0$. Then $v(\alpha) > 0$ and $v > 0$ on $I$, whence $sv \leq 0$ on $I$. Thus $\alpha$ is of the type II or III and $\beta$ is of type I.

Further, let $h_+ \in AC(I)$ be such that $h_+(\alpha) = 0, h_+ \geq 0$ on $I$, $h_+ \geq 0$ a.e. on $I$ and $\int_I r h_+^2 \, dt < \infty$. Then $\int_I s h_+ h_+ \, dt \leq 0$ and the second of the integral conditions (3) is satisfied. Thus $h_+ \in H_0$ and by Theorem 3(ii)–(iii) we have $h_+ \in H$. Next by Theorem 2 we get

$$2 \int_{I} |s|h_+ h_+ \, dt + \int_{I} |u| h_+^2 \, dt \leq \int_{I} r h_+^2 \, dt.$$  

Now, let \( h \in AC(I) \) be such that \( h(\alpha) = 0 \) and \( \int_I r h^2 dt < \infty \). Put \( h_+ = \int_0^t |h| \, dt \). Then \( h_+ \in AC(I) \), \( h_+(\alpha) = 0 \), \( h_+ \geq 0 \) on \( I \), \( h_+ = |h| \geq 0 \) a.e. on \( I \) and

\[
\int_I r h^2_+ dt = \int_I r h^2 dt < \infty.
\]

Hence \( h_+ \) satisfies inequality (31). Notice that

\[
|h| = \left| \int_\alpha^t \dot{h} \, d\tau \right| \leq \int_\alpha^t |\dot{h}| \, d\tau = h_+
\]
on \( I \), and equality holds if and only if \( \dot{h} \) does not change sign on \( I \). Hence

\[
2 \int_I |sh h| \, dt + \int_I |u| h^2 \, dt \leq 2 \int_I |s| h_+ \dot{h} \, dt + \int_I |u| h^2_+ \, dt
\]

and by (31)–(33) we get inequality (29).

If both sides of (29) are equal for some non-vanishing function \( h \in AC(I) \) such that \( h(\alpha) = 0 \) and \( \int_I r h^2 dt < \infty \), then by (31)–(33) it follows that for \( h_+ = \int_0^t |h| \, d\tau \) equality holds in (31) and (33). It follows that \( |h| = h_+ \) and hence \( \dot{h} \) does not change sign on \( I \). Since \( h_+ \in \bar{H} \) and by Theorem 2, equality occurs in (31) if and only if \( s \) and \( u \) satisfy (7) a.e. on \( I \), \( \varphi^{-1} h_+ = \text{const} > 0 \) and conditions (16) are satisfied. Hence \( \varphi^{-1} h = \text{const} \neq 0 \), \( \varphi(\alpha) = 0 \) and \( \ddot{\varphi} \geq 0 \) on \( I \).

Let \( s \) and \( u \) satisfy (7) a.e. on \( I \) and \( \varphi \) be such that \( \varphi(\alpha) = 0 \), \( \dot{\varphi} \geq 0 \) and conditions (30) hold. Then we easily check that the function \( h = c \varphi \), where \( c = \text{const} \neq 0 \), satisfies \( h(\alpha) = 0 \) and \( \int_I r h^2 dt < \infty \) and for this function equality holds in (29).

The case when \( s \geq 0 \) on \( I \), \( v(\alpha) \leq 0 \), \( h(\beta) = 0 \) can be proved in a similar way considering the function \( h_- = \int_\beta^\alpha |h| \, d\tau \in \bar{H} \).

Inequalities (29) embrace, as a particular case (if \( u = 0 \) on \( I \)), the integral inequalities of Opial type which were examined in [13].

**Example 4.** Let \( I = (\alpha, \beta) \), \( -\infty \leq \alpha < \beta \leq \infty \). Let \( r > 0 \) and \( u \leq 0 \) be functions absolutely continuous on \( I \) such that \( \int_I r^{-1} dt < \infty \) and

\[
\int_I u \, dt \geq -\left( \int_I r^{-1} dt \right)^{-1}.
\]

If we put \( \varphi = \int_\alpha^t r^{-1} \, d\tau \), then the functions \( u \) and

\[
s = -\int_\alpha^\beta u \, dt - \left( \int_I r^{-1} dt \right)^{-1} \leq 0
\]
satisfy equation (7) on \( I \) and \( v(\beta) = 0 \). If we put \( \varphi = \frac{\beta}{t} r^{-1} d\tau \), then the functions \( u \) and
\[
(35) \quad s = \int_{\alpha}^{t} u d\tau + \left( \int_{I}^{t} r^{-1} d\tau \right)^{-1} \geq 0
\]
satisfy (7) on \( I \) and \( v(\alpha) = 0 \).

Now, applying Theorem 4 we get:

If \( h \in AC(I) \) satisfies \( \int_{I} r \dot{h}^2 \, dt < \infty \) and \( h(\alpha) = 0 \) or \( h(\beta) = 0 \), then the inequality of the form (29) with \( s \) defined by (34) if \( h(\alpha) = 0 \) or by (35) if \( h(\beta) = 0 \) is valid. In both cases equality holds only for \( h = c\varphi \), where \( c = \text{const} \).

If \( u \equiv 0 \), then we obtain the inequalities which were considered in [4] (cf. [13]).

In the case when \( 0 = \alpha < \beta \leq 1, r = 1, u = -1 \) on \( I \) we obtain the inequality
\[
(36) \quad 2 \int_{0}^{\beta} \left( \frac{1 - \beta^2}{\beta} + t \right) |h\dot{h}| \, dt + \int_{0}^{\beta} h^2 \, dt \leq \int_{0}^{\beta} \dot{h}^2 \, dt,
\]
which holds for all \( h \in AC((0, \beta)) \) such that \( h(0) = 0 \) and \( \int_{0}^{\beta} \dot{h}^2 \, dt < \infty \), and the inequality
\[
(37) \quad 2 \int_{0}^{\beta} \left( \frac{1}{\beta} - t \right) |h\dot{h}| \, dt + \int_{0}^{\beta} h^2 \, dt \leq \int_{0}^{\beta} \dot{h}^2 \, dt,
\]
which holds for all \( h \in AC((0, \beta)) \) such that \( h(\beta) = 0 \) and \( \int_{0}^{\beta} \dot{h}^2 \, dt < \infty \).

Equality holds in (36) only for \( h = ct \), and in (37) only for \( h = c(\beta - t) \), where \( c = \text{const} \).

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