C O L L O Q U I U M M A T H E M A T I C U M

VOL. 75 1998 NO. 1

SINGULAR INTEGRALS ON THE COMPLEX AFFINE GROUP

BY

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Let $G = \mathbb{C} \rtimes \mathbb{C}^*$ be the complex affine group. We study a canonical right-invariant Laplacian $\Delta = -(X^2 + Y^2 + U^2 + V^2)$. We show that the Newtonian kernel N that defines the operator Δ^{-1} is remarkably simple. It is a rational function of the complex variables $z = x + iy$ and $w = u + iv$:

$$N = \frac{1}{4\pi^2} \cdot \frac{|w|^2}{|z|^2 + |1-w|^2}.$$

There are associated Riesz operators of the form $Z_1\Delta^{-1}Z_2$, $P\Delta^{-1}$ and $\Delta^{-1}P$, where Z_i, $i = 1, 2$, are elements of order 1 of the right-invariant universal enveloping algebra $\mathcal{G}$, and $P \in \mathcal{G}$ is of order at most 2. We give a complete description of the operators of these kinds which are bounded in $L^p(G)$, $1 < p < \infty$, and of weak type (p,p), $1 \le p < \infty$. We prove in particular that the operators $P\Delta^{-1}$ and $\Delta^{-1}P$ that depend on V (in a certain sense) are *bounded*, whereas *all others* are unbounded.

This is a notable contrast with other solvable Lie groups of exponential growth: for instance the groups NA that arise from the Iwasawa decomposition of a rank 1 semisimple group. In that case, the (nonzero) operators $Z_1Z_2\Delta^{-1}$ and $\Delta^{-1}Z_1Z_2$ are *always* unbounded.

1. Introduction. Let G be the complex affine group $\mathbb{C} \rtimes \mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of nonzero complex numbers. The group product is given by

$$(z, w)(z', w') = (z + wz', ww').$$

This corresponds to composition of affine mappings $\zeta \mapsto w\zeta + z$. We write elements z and w of $\mathbb{C}$ and $\mathbb{C}^*$ respectively as

$$z = x + iy = re^{i\theta}, \quad w = u + iv = \varrho e^{i\phi}.$$

1991 *Mathematics Subject Classification*: 43A32, 42B20, 22E30.

The Lie algebra $\mathfrak{g}$ of G can be identified with $\mathbb{C} \oplus \mathbb{C}$; the exponential mapping is given by

$$\exp(\zeta, \tau) = \left(\frac{e^\tau - 1}{\tau} \zeta, e^\tau \right).$$

We take as canonical basis in $\mathfrak{g}$ the elements $X = (1, 0)$, $Y = (i, 0)$, $U = (0, 1)$ and $V = (0, i)$. We regard the elements of $\mathfrak{g}$ as right-invariant vector fields on G. The corresponding right-invariant Laplacian is then the operator

$$\Delta = -(X^2 + Y^2 + U^2 + V^2).$$

For the sake of simplicity, we shall not distinguish between the differential operator Δ and its self-adjoint closure in $L^2(G)$. Here and in the sequel, L^p spaces on G are defined by means of the left Haar measure. Where necessary, we shall use superscripts to distinguish between right-invariant fields (X^r, etc.) and left-invariant fields (X^l, etc.).

The operator Δ is positive and does not have 0 in its point spectrum. Hence it has an inverse, defined by spectral theory. See §4 for details. We identify the kernel N of the operator Δ^{-1}; the expression for N is surprisingly simple, namely,

$$N = \frac{1}{4\pi^2} \cdot \frac{|w|^2}{|z|^2 + |1 - w|^2}.$$

The Riesz operators are defined using the kernel N: for instance, $Z_1 \Delta^{-1} Z_2 f = Z_1 N * (Z_2 f)$.

1.1. *Statement of results.* We shall study second-order Riesz operators on G; they are defined as products, in some order, of Δ^{-1} and two vector fields $Z_1, Z_2 \in \mathfrak{g}$. As remarked below, G is a solvable group of type $N \rtimes A$, where N is nilpotent and A is abelian. In the authors' earlier papers [7] and [6], similar Riesz operators were studied in the case where G is the solvable group $N \rtimes A$ arising from the Iwasawa decomposition of a rank 1 symmetric space. In that case, the operators $Z_1 \Delta^{-1} Z_2$ are bounded. Our first result states that this remains true in the case of the complex affine group.

THEOREM 1. *Every operator $Z_1 \Delta^{-1} Z_2$ with $Z_1, Z_2 \in \mathfrak{g}$ is bounded on $L^p(G)$ for all $p \in (1, \infty)$, and of weak type* $(1, 1)$.

In the setting of [7] and [6], it was found that all of the operators $Z_1 Z_2 \Delta^{-1}$ and $\Delta^{-1} Z_1 Z_2$ (for $Z_1, Z_2 \in \mathfrak{g}$) are unbounded. In our case, however, some of them turn out to be bounded. To state the criterion for boundedness, we need some notation.

DEFINITION 1. Let $\mathcal{G}_2$ be the vector space of all right-invariant differential operators of order at most 2, including the identity operator I.

The space $\mathcal{G}_2$ is spanned by I, $\mathfrak{g}$ and the set of products Z_1Z_2, with $Z_1, Z_2 \in \mathfrak{g}$. Notice that it is a vector subspace of $\mathcal{G}$, the universal enveloping algebra of G. The vector field V plays a special role, and we need two subspaces of $\mathcal{G}_2$ generated by V in the following way.

DEFINITION 2. Denote by $\mathcal{I}_l$ the subspace of $\mathcal{G}_2$ generated by Δ, V and all products ZV with $Z \in \mathfrak{g}$. The subspace $\mathcal{I}_r$ is defined in a similar way, using the products VZ in place of ZV.

THEOREM 2. *Let $P \in \mathcal{G}_2$.*

(a) *If $P \in \mathcal{I}_l$, then the operator $P\Delta^{-1}$ is bounded on $L^p(G)$ for all $p \in (1,\infty)$, and of weak type $(1,1)$. If $P \notin \mathcal{I}_l$, then $P\Delta^{-1}$ is not of weak or strong type (p,p) for any $p \in [1,\infty)$.*

(b) *Similarly, if $P \in \mathcal{I}_r$, then the operator $\Delta^{-1}P$ is bounded on $L^p(G)$ for all $p \in (1,\infty)$, and of weak type $(1,1)$. If $P \notin \mathcal{I}_r$, then $\Delta^{-1}P$ is not of weak or strong type (p,p) for any $p \in [1,\infty)$.*

We refer to [7] and [6] for further background material.

Acknowledgements. We are grateful to Andreas Bergström for computing the kernels that we deal with by using Mathematica, and also for checking consistency of the kernel of Δ^{-1} with the expression for the heat kernel given in §5.

2. Preliminaries

2.1. *Invariant vector fields*

LEMMA 2.1. *Let $X = (1,0)$, $Y = (i,0)$, $U = (0,1)$ and $V = (0,i)$ be the canonical basis elements of $\mathfrak{g}$. The corresponding right-invariant vector fields X^r, etc., are:*

$$X^r = \frac{\partial}{\partial x}, \qquad U^r = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$
$$Y^r = \frac{\partial}{\partial y}, \qquad V^r = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.$$

The commutation relations among the basis vectors are:

$$\begin{aligned} [X,Y] &= 0, & [Y,U] &= Y, \\ [X,U] &= X, & [Y,V] &= -X, \\ [X,V] &= Y, & [U,V] &= 0. \end{aligned} \tag{1}$$

The corresponding left-invariant fields are:

$$X^l = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \qquad U^l = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$
$$Y^l = -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}, \quad V^l = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.$$

Proof. This is routine. ∎

If the spaces $\mathbb{C}$ and $\mathbb{C}^*$ are given polar coordinates (r,θ) and (ϱ,ϕ) respectively, then

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = R_z, \qquad \varrho\frac{\partial}{\partial \varrho} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} = R_w,$$
$$\frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = T_z, \qquad \frac{\partial}{\partial \phi} = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} = T_w, \qquad \text{say.}$$

The operators R_z and R_w are radial operators, and T_z and T_w are tangential operators, in a natural sense. In this notation, the formulas for U^r, V^r and U^l, V^l can be written as follows:

$$U^r = R_z + R_w, \quad U^l = R_w,$$
$$V^r = T_z + T_w, \quad V^l = T_w.$$

2.2. *Relationship between left-invariant, right-invariant and transpose operators.* The left- and right-invariant Haar measures μ_l and μ_r are given by

$$d\mu_l(z,w) = dz\frac{dw}{|w|^4} \quad \text{and} \quad d\mu_r(z,w) = dz\frac{dw}{|w|^2},$$

where dz and dw denote Lebesgue measure in $\mathbb{C}$. The modular function is therefore $m(z,w) = |w|^{-2}$. The following formulas involving the modular function are used implicitly in (2) and Lemma 2.2:

$$\int_G f(ts)\,d\mu_l(t) = m(s)^{-1}\int_G f(t)\,d\mu_l(t),$$
$$\int_G f(t^{-1})m(t)^{-1}\,d\mu_l(t) = \int_G f(t)\,d\mu_l(t).$$

We shall generally use the left-invariant measure. In particular, the Lebesgue spaces $L^p(G)$, $1 \le p < \infty$, are formed relative to left Haar measure.

The various operators which we study are convolution operators, the kernel in each case being a distribution. As shown in [7, §4], it is natural to define the left-invariant derivative of a distribution k as the distribution $Z^l k$ for which

$$\langle Z^l k, g\rangle = \langle k, -Z^l g\rangle - Zm(e)\langle k, g\rangle \quad (g \in C_c^\infty(G)). \tag{2}$$

DEFINITION 3. Let Z be a right-invariant vector field. Define the *transpose operator* Z^{t} by

$$Z^{\mathrm{t}} = Z^{l} + Zm(e)I.$$

In terms of this definition,

$$k * Zf = (Z^{\mathrm{t}}k) * f.$$

See [7, Lemma 8]. It is simple to see that $(Z_1Z_2)^{\mathrm{t}} = Z_2^{\mathrm{t}}Z_1^{\mathrm{t}}$ for all $Z_1, Z_2 \in \mathfrak{g}$.

LEMMA 2.2. *The transpose operators corresponding to the basis vectors* X, Y, U, V *are*:

$$X^{\mathrm{t}} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \qquad U^{\mathrm{t}} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - 2 = R_w - 2,$$
$$Y^{\mathrm{t}} = -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}, \qquad V^{\mathrm{t}} = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} = T_w.$$

3. Representations and fundamental solutions. In order to identify the kernel of the operator Δ^{-1}, it is natural to use the Fourier transform. If σ is a unitary representation of G and f is a suitable function, then the Fourier transform $\widehat{f}(\sigma)$ is given by

$$\widehat{f}(\sigma) = \int_G f(x)\sigma(x)\,d\mu_l(x).$$

3.1. *Representation theory and Fourier transforms.* Since the group G is a semidirect product $G = N \rtimes A$ in which $N = \mathbb{C}$ is a normal, abelian, regularly-embedded subgroup, we may apply Mackey theory ([8, pp. 159–199] and [2, M. Raïs, Ch. V, §1.10]). The nontrivial representations are induced from characters $\chi_b(z) = e^{ib\cdot z}$ of N, where $b = b_1 + ib_2 \in \mathbb{C}$, and $b \cdot z = b_1x + b_2y$.

We realise the representation σ_b, induced from χ_b and acting on $L^2(A) = L^2(\mathbb{C}; d\eta/|\eta|^2)$, by

$$[\sigma_b(z,w)F](\eta) = e^{ib\cdot z\eta}F(w\eta).$$

The orbit of b is $\{bw : w \in \mathbb{C}^*\}$. By Mackey theory, if $b \neq 0$, there is a single equivalence class of unitary representations corresponding to that orbit; each representation is equivalent to the one where $b = 1$. Let $\sigma = \sigma_1$, so that

$$[\sigma(z,w)F](\eta) = e^{i\overline{z}\cdot\eta}F(w\eta).$$

The representations obtained by extending the characters of the subgroup A to the whole of G do not appear in the Plancherel formula, and will be ignored.

Notice that if we identify A with $\mathbb{R}^2$, and write $\eta = s + it$, then the images under $d\sigma$ of the Lie algebra basis vectors X, Y, U, V are

$$d\sigma(X) = isI, \qquad d\sigma(U) = s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t},$$
$$d\sigma(Y) = -itI, \qquad d\sigma(V) = -t\frac{\partial}{\partial s} + s\frac{\partial}{\partial t},$$

each operator being an operator on $L^2(A)$. It follows that

$$d\sigma(\Delta) = |\eta|^2(I - \Delta_\eta), \tag{3}$$

where Δ_η is the standard Laplacian on $\mathbb{R}^2$. The formula (3) may also be interpreted as saying that the Fourier transform of Δf is

$$|\eta|^2(I - \Delta_\eta)\widehat{f}(\sigma),$$

assuming that f is a test function.

3.2. *Heuristic determination of the fundamental solution.* In the quest for fundamental solutions for Δ, it is natural, in view of the remark above, to attempt to "invert" the operator $|\eta|^2(I - \Delta_\eta)$. Formally speaking, this gives the operator

$$F \mapsto \int_{\mathbb{R}^2} J(\eta - w)\frac{F(w)}{|w|^2}\,dw$$

where J is the convolution kernel corresponding to $(I - \Delta_\eta)^{-1}$.

Suppose that N is a convolution kernel on G which is a fundamental solution of Δ. Then, taking Fourier transforms formally on G, at the representation σ, we get

$$\iint N(z, w)e^{iz\cdot\overline{\eta}} f(w\eta)\,\frac{dw}{|w|^4}\,dz = \int J(\eta - w)\frac{f(w)}{|w|^2}\,dw \tag{4}$$

for suitable functions f. Writing $\widehat{N}_1$ for the Euclidean Fourier transform with respect to the first variable, and changing variables in the right-hand side, we may write (4) in the form

$$\int \widehat{N}_1(-\overline{\eta}, w)f(w\eta)\,\frac{dw}{|w|^4} = \int J(\eta(1 - w))\frac{f(w\eta)}{|w|^2}\,dw.$$

So it is reasonable to require that

$$\widehat{N}_1(-\overline{\eta}, w) = |w|^2 J(\eta(1 - w)). \tag{5}$$

If we invert the Fourier transform formally in (5), we get

$$N(z, w) = \frac{1}{4\pi^2}\cdot\frac{|w|^2}{1 + |z|^2/|1 - w|^2}\cdot\frac{1}{|1 - w|^2} = \frac{1}{4\pi^2}\cdot\frac{|w|^2}{|z|^2 + |1 - w|^2}. \tag{6}$$

We shall see in the next section that this is the kernel of the operator Δ^{-1} on an appropriate domain.

4. Spectral theory and the kernel of Δ^{-1}. The spectral measure dE_s of the operator Δ has no mass at 0, and the operator Δ^{-1} is well-defined, via functional calculus, by

$$\Delta^{-1} = \int_0^\infty s^{-1}\, dE_s.$$

Its domain is

$$\mathcal{D}(\Delta^{-1}) = \Big\{ f : \int_0^\infty s^{-2} d\langle E_s f, f\rangle < \infty \Big\};$$

and for $f \in \mathcal{D}(\Delta^{-1})$ and $g \in L^2(G)$,

$$\langle \Delta^{-1} f, g\rangle = \int_0^\infty \langle e^{-t\Delta} f, g\rangle\, dt. \tag{7}$$

Further, the space

$$V = \{\Delta h : h \in C_{\mathrm{c}}^\infty(G)\}$$

is dense in $L^2(G)$. For these facts, see for instance [7, Sect. 3.3].

LEMMA 4.1. (i) *The operator Δ^{-1} is given on V by convolution with the kernel $N = \int_0^\infty h_t\, dt$, where $(h_t)_{t>0}$ are the kernels of the heat semigroup $e^{-t\Delta}$.*

(ii) *The kernel N is a fundamental solution for Δ.*

Proof. (i) We first verify that $\int_0^\infty h_t\, dt$ converges locally in $L^1(G)$. In view of [4, T7, p. 54], the solvable group G has no recurrent random walk, so that $\sum_{k=1}^\infty h_k$ converges locally in L^1. Harnack's inequality ([10, Theorem III.2.1, p. 29]) then implies that the above integral also converges in L^1_{loc}. For $f \in V$ and $g \in C_{\mathrm{c}}^\infty(G)$, it follows that

$$\langle \Delta^{-1} f, g\rangle = \int_0^\infty \langle h_t * f, g\rangle\, dt = \langle N * f, g\rangle.$$

(ii) We wish to show that $\langle \Delta N, f\rangle = \delta_e$ in the sense of distributions, i.e., $\langle N, \Delta f\rangle = f(e)$ for any $f \in C_{\mathrm{c}}^\infty(G)$. With $f \in C_c^\infty(G)$ fixed, we have

$$\langle N, \Delta f\rangle = \Big\langle \int_0^\infty h_t\, dt, \Delta f\Big\rangle.$$

As verified above, $\int_0^\infty h_t\, dt$ converges in L^1_{loc}. Fubini's theorem then implies that

$$\begin{aligned} \Big\langle \int_0^\infty h_t\, dt, \Delta f\Big\rangle &= \int_0^\infty \int_G h_t(x)\Delta f(x)\, dx\, dt \\ &= \int_0^\infty \langle h_t, \Delta f\rangle\, dt = \int_0^\infty \Big\langle -\frac{\partial h_t}{\partial t}, f\Big\rangle\, dt. \end{aligned} \tag{8}$$

An elementary argument shows that

$$\left\langle -\frac{\partial h_t}{\partial t}, f \right\rangle = -\frac{\partial}{\partial t}\langle h_t, f\rangle,$$

and so from (8),

$$\begin{aligned}\langle N, \Delta f\rangle &= \int\limits_0^\infty -\frac{\partial}{\partial t}\langle h_t, f\rangle\,dt = \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \int\limits_\varepsilon^k -\frac{\partial}{\partial t}\langle h_t, f\rangle\,dt\\ &= \lim_{\varepsilon\to 0}\langle h_\varepsilon, f\rangle - \lim_{k\to\infty}\langle h_k, f\rangle = f(e),\end{aligned}$$

since $\langle h_\varepsilon, f\rangle \to f(e)$ as $\varepsilon \to 0$ and $\langle h_k, f\rangle \to 0$ as $k\to\infty$, because $\sum_{k=1}^\infty h_k$ converges in L^1_{loc}. Finally then, we see that

$$\langle \Delta N, f\rangle = \langle N, \Delta f\rangle = f(e). \ \blacksquare$$

Remark. An alternative, and more general, approach to the proof of Lemma 4.1 is possible. It applies whenever G satisfies the (NC)-condition of Varopoulos [9]. Cf. the argument in [7, §§3.2–3.3].

5. Riemannian structure and heat kernels. The group G can be realised as a two-stage semidirect product $G = (\mathbb{C} \rtimes \mathbb{R}_+) \rtimes \mathbb{T}$ if we write $w = \varrho e^{i\phi}$ and (z, w) as $(z, \varrho, e^{i\phi})$. It is therefore the semidirect product $G' \rtimes \mathbb{T}$, where $G' = \mathbb{C} \rtimes \mathbb{R}_+$ is identifiable with real hyperbolic 3-space. We shall assume that $\phi \in \mathbb{R}/2\pi\mathbb{Z} \cong [-\pi, \pi)$. The left-invariant distance on G' is given by

$$d_{G'}((z, \varrho), (0, 1)) = \operatorname{arcosh}\left(\frac{|z|^2 + 1 + \varrho^2}{2\varrho}\right),$$

and is derived from the hyperbolic metric

$$ds'^2 = \frac{dx^2 + dy^2 + d\varrho^2}{\varrho^2}.$$

This is invariant under the action of $\mathbb{T}$ on G'. It follows that the left-invariant metric on G is given by

$$ds^2 = ds'^2 + d\phi^2 = \frac{dx^2 + dy^2 + d\varrho^2}{\varrho^2} + d\phi^2.$$

So

$$\begin{aligned}d_G((z, w), (0, 1))^2 &= d_{G'}((z, |w|), (0, 1))^2 + \phi^2\\ &= \left(\operatorname{arcosh}\frac{|z|^2 + 1 + |w|^2}{2|w|}\right)^2 + \phi^2.\end{aligned} \tag{9}$$

LEMMA 5.1. *Let Δ and Δ' be the canonical right-invariant Laplacians on G and G' respectively. Then*

$$\Delta(f_1 f_2) = (\Delta' f_1) f_2 - f_1 \frac{\partial^2 f_2}{\partial \phi^2}$$

if f_1 depends on $r = |z|$ and $\varrho = |w|$ only, and f_2 depends only on ϕ.

Proof. A simple calculation shows that

$$\begin{aligned}\Delta = &-\bigg(X^2+Y^2+r^2\frac{\partial^2}{\partial r^2}+\varrho^2\frac{\partial^2}{\partial\varrho^2}\\&+2r\varrho\frac{\partial^2}{\partial r\partial\varrho}+r\frac{\partial}{\partial r}+\varrho\frac{\partial}{\partial\varrho}+\frac{\partial^2}{\partial\theta^2}+2\frac{\partial^2}{\partial\theta\partial\phi}+\frac{\partial^2}{\partial\phi^2}\bigg).\end{aligned}$$

Hence

$$\begin{aligned}\Delta(f_1f_2) = &-\bigg(X^2+Y^2+r^2\frac{\partial^2}{\partial r^2}+\varrho^2\frac{\partial^2}{\partial\varrho^2}\\&+2r\varrho\frac{\partial^2}{\partial r\partial\varrho}+r\frac{\partial}{\partial r}+\varrho\frac{\partial}{\partial\varrho}+\frac{\partial^2}{\partial\phi^2}\bigg)(f_1f_2)\\ =&\ (\Delta' f_1)f_2-f_1\frac{\partial^2 f_2}{\partial\phi^2}. \ \blacksquare\end{aligned}$$

THEOREM 3. *Let $A = \operatorname{arcosh}((|z|^2+1+|w|^2)/2|w|)$. The heat semigroup kernels on G are the functions*

$$h_t = \frac{1}{16\pi^2t^2}\cdot\frac{|w|A}{\sinh A}\sum_{k\in\mathbb{Z}}\exp\bigg[-\frac{A^2+(\phi-2k\pi)^2}{4t}\bigg], \tag{10}$$

in the sense that $e^{-t\Delta}f = h_t * f$, $t>0$, for all $f\in L^2(G)$.

Proof. The heat kernels for the group G' can be derived from those for the Laplace–Beltrami operator in the real hyperbolic space $\mathbb{H}_3$ (cf. [5, p. 178, (5.7.3)]). To do so, one uses the relationship between the heat kernels on $\mathbb{H}_3$ and G' given in [3, Theorem 5.3]. The heat kernels are also essentially given in [1, Prop. 2.3.2]: one needs to adjust for the factor of $1/2$ in Bougerol's definition of the Laplacian, and for the fact that he does not use left Haar measure. The kernels are

$$q_t = \frac{1}{8\pi^{3/2}t^{3/2}}\cdot\frac{|w|A}{\sinh A}e^{-A^2/(4t)} \qquad (t>0).$$

The functions h_t in (10) are the tensor products of the functions q_t and the heat kernels

$$\frac{1}{2\sqrt{\pi t}}\sum_{k\in\mathbb{Z}}e^{-(\phi-2\pi k)^2/(4t)} \qquad (t>0)$$

on the circle group $\mathbb{T}$ for the Laplacian d^2/dt^2. From Lemma 5.1, the functions $(h_t)_{t>0}$ satisfy the heat equation on G, and also the other characteristic properties of the heat semigroup. This gives the theorem. $\blacksquare$

It is now a simple matter to identify the kernel $N = \int_0^\infty h_t\,dt$. Indeed,

$$\begin{aligned}\int_0^\infty h_t\,dt &= \frac{1}{4\pi^2}\sum_{k\in\mathbb{Z}}\frac{|w|A}{A^2+(\phi+2k\pi)^2}\cdot\frac{1}{\sinh A}\\ &= \frac{1}{8\pi^2}\cdot\frac{(1-e^{-2A})|w|}{|1-e^{-A}e^{i\phi}|^2}\cdot\frac{1}{\sinh A} = \frac{1}{4\pi^2}\cdot\frac{|w|e^{-A}}{|1-e^{-A}e^{i\phi}|^2}\\ &= \frac{1}{4\pi^2}\cdot\frac{|w|^2}{|z|^2+|1-w|^2}\end{aligned}$$

since $|1-w|^2+|z|^2 = |w|e^A|1-e^{-A}e^{i\phi}|^2$. In the calculations just given, we have used Poisson summation and the standard formula for the Fourier transform of the Poisson kernel. Note that the formula just derived agrees with (6).

6. Kernels of the operators. It follows from Lemmas 2.1, 2.2 and 4.1 that the kernels of the Riesz operators are computable by applying right-invariant fields and their transposes to the kernel N.

6.1. *Local parts of the kernels.* Each of the operators in question is defined by convolution with a distribution k, at least on a suitable dense subset of $L^2(G)$. The distributional kernel may be written as the sum of two kernels, $k = k_1 + k_2$, where $k_1 = \psi k$, and the cut-off function ψ is C^∞, supported in a relatively compact set, and equal to 1 on a neighbourhood of the identity. The kernel k_2, the "part at infinity", is zero on a neighbourhood of e.

The results in [7] show that, on any Lie group, the *local part* k_1 defines an operator which is bounded on $L^p(G)$ whenever $1 < p < \infty$, and is of weak type $(1,1)$. This approach relies on pseudodifferential operators. Another possibility is to verify that the kernel k_1 is of Calderón–Zygmund type, and then to use the simple and well-known Lemma 11 of [5] to add the local results obtained. The proofs of Theorems 1 and 2 thus rest on an analysis of the parts at infinity of the respective kernels.

6.2. *Simplification of the kernels for large w.* The following lemma helps to avoid complications due to the term $|w-1|^2$ in the expression for N. We replace it by the simpler $|w|^2$, and let

$$N_\infty = \frac{1}{4\pi^2}\cdot\frac{|w|^2}{|z|^2+|w|^2}. \tag{11}$$

LEMMA 6.1. *The functions $N - N_\infty$, $Z_2^{\mathrm{t}}(N-N_\infty)$, $Z_1^{\mathrm{t}}Z_2^{\mathrm{t}}(N-N_\infty)$, $Z_2(N-N_\infty)$ and $Z_1Z_2(N-N_\infty)$ are integrable in the region where $|w| > 2$, for all $Z_1, Z_2 \in \mathfrak{g}$.*

Proof. Write $|(z,w)|^2 = |z|^2 + |w|^2$. Then

$$N = \frac{|w|^2}{4\pi^2|(z,w)-(0,1)|^2} \quad \text{and} \quad N_\infty = \frac{|w|^2}{4\pi^2|(z,w)|^2}.$$

Hence

$$N - N_\infty = \frac{1}{4\pi^2}|w|^2 \frac{2u-1}{|(z,w)|^2|(z,w)-(0,1)|^2} = |w|^2\frac{P}{Q},$$

where $\deg P = 1$ and $\deg Q = 4$. It follows from the expressions for the transpose operators in Lemma 2.2 that for any $Z \in \mathfrak{g}$ and any function f,

$$Z^{\mathrm{t}}(|w|^2 f) = |w|^2 Z^l f. \tag{12}$$

This implies that all of the functions in the lemma that involve transposed operators will be of the form $|w|^2 P_1/Q_1$, where P_1 and Q_1 are polynomials with $\deg Q_1 - \deg P_1 \geq 3$, and Q_1 behaves like a power of $|(z,w)|$ for large w. Such functions are integrable for $|w| > 2$, since

$$\int\limits_{|w|>2} \frac{dw}{|w|^4} \int |w|^2 \left|\frac{P_1}{Q_1}\right| dz \leq C \int\limits_{|w|>2} \frac{dw}{|w|^2} \int \frac{dz}{|z|^3 + |w|^3} < \infty.$$

The argument for the nontransposed operators is similar. ∎

6.3. *Simplification of the kernels for small w.* The formula (9) shows that points for which w is small, as well as those for which w is large, lie near infinity in the group G. To treat the former, we consider the modified kernel

$$N_0 = \frac{1}{4\pi^2} \cdot \frac{|w|^2}{|z|^2+1}.$$

LEMMA 6.2. *The functions $N - N_0$, $Z_2^{\mathrm{t}}(N-N_0)$, $Z_1^{\mathrm{t}}Z_2^{\mathrm{t}}(N-N_0)$, $Z_2(N-N_0)$ and $Z_1Z_2(N-N_0)$ are integrable in the region where $|w| < 1/2$, for all $Z_1, Z_2 \in \mathfrak{g}$.*

Proof. We have

$$N - N_0 = \frac{1}{4\pi^2} \cdot \frac{|w|^2(2u - |w|^2)}{(|z|^2+1)(|z|^2+|w-1|^2)}.$$

Differentiating, one sees that

$$Z_2(N-N_0) = \frac{O(|w|^3)}{(1+|z|^2)^2}$$

uniformly in z, as $w \to 0$. A similar statement applies to Z_1Z_2 and to the transposed operators. ∎

7. Proof of Theorem 1.

Recall from §6.1 that the local part of the kernel of $Z_1\Delta^{-1}Z_2$ defines a bounded operator. The real issue is therefore to examine the behaviour of the kernel away from the origin.

LEMMA 7.1. *For every $Z \in \mathfrak{g}$,*

$$ZN = \frac{|w|^2 P_1(z,w)}{(|z|^2 + |w-1|^2)^2}$$

where $P_1(z,w)$ is of degree at most 1.

Proof. This is a simple direct calculation. ■

LEMMA 7.2. *For $Z_1, Z_2 \in \mathfrak{g}$, the kernel $Z_2^{\mathrm{t}} Z_1 N$ of the operator $Z_1 \Delta^{-1} Z_2$ satisfies*

$$Z_2^{\mathrm{t}} Z_1 N = \frac{|w|^2 P_3(z,w)}{(|z|^2 + |w-1|^2)^3}, \tag{13}$$

where P_3 is a polynomial of degree at most 3 *satisfying $P_3 = O(|w|(1+|z|^2))$ as $w \to 0$, uniformly in z.*

Proof. This is a consequence of Lemma 7.1, the equality (12) and the expressions in Lemma 2.2. ■

COROLLARY 7.3. *The kernels of the operators $Z_1 \Delta^{-1} Z_2$ are all integrable at infinity.*

8. Proof of Theorem 2(b). We break this up into a set of lemmas.

LEMMA 8.1. *Any linear combination P^{t} of the set of operators $X^{\mathrm{t}} V^{\mathrm{t}}$, $Y^{\mathrm{t}} V^{\mathrm{t}}$, $U^{\mathrm{t}} V^{\mathrm{t}} = V^{\mathrm{t}} U^{\mathrm{t}}$, $V^{\mathrm{t}} V^{\mathrm{t}}$, V^{t} and Δ gives rise to an operator $P^{\mathrm{t}} \Delta^{-1}$ that is bounded on $L^p(G)$ for all $p \in (1,\infty)$ and of weak type* $(1,1)$.

Proof. This follows from §6.1 and Lemmas 6.1 and 6.2 and the observation that $V^{\mathrm{t}} N_\infty = V^{\mathrm{t}} N_0 = 0$. ■

In the next lemma, we use the notation

$$X^{\mathrm{t}} X^{\mathrm{t}} \looparrowright -2(u^2+v^2)(|z|^2+|w|^2) + 8(ux+vy)^2 \tag{14}$$

to mean that $X^{\mathrm{t}} X^{\mathrm{t}} N_\infty$ is the product of the function on the right of (14) and $(4\pi^2)^{-1} |w|^2 (|z|^2+|w|^2)^{-3}$.

LEMMA 8.2. *The transposed operators that do not involve V^{t} satisfy the relations*

$$\begin{aligned} X^{\mathrm{t}} X^{\mathrm{t}} &\looparrowright -2|w|^2(|z|^2+|w|^2) + 8(ux+vy)^2, \\ X^{\mathrm{t}} Y^{\mathrm{t}} = Y^{\mathrm{t}} X^{\mathrm{t}} &\looparrowright 8(ux+vy)(-vx+uy), \\ Y^{\mathrm{t}} Y^{\mathrm{t}} &\looparrowright -2|w|^2(|z|^2+|w|^2) + 8(-vx+uy)^2, \\ U^{\mathrm{t}} &\looparrowright -2|w|^2(|z|^2+|w|^2), \\ I &\looparrowright (|z|^2+|w|^2)^2 \end{aligned} \tag{15}$$

and

$$
\begin{aligned}
U^{\mathrm{t}}X^{\mathrm{t}} &\looparrowright -2(ux+vy)(|z|^2-3|w|^2),\\
U^{\mathrm{t}}Y^{\mathrm{t}} &\looparrowright -2(-vx+uy)(|z|^2-3|w|^2),\\
X^{\mathrm{t}}U^{\mathrm{t}} &\looparrowright 8|w|^2(ux+vy),\\
Y^{\mathrm{t}}U^{\mathrm{t}} &\looparrowright 8|w|^2(-vx+uy).
\end{aligned}
\tag{16}
$$

The 9 *homogeneous polynomials in* (15) *and* (16) *are linearly independent.*

P r o o f. This is by direct calculation. For the second part of the statement, note that the functions in (15), which are of even degree in (x,y), are clearly independent of those in (16). ∎

Lemma 8.3. *Any nontrivial linear combination of the operators in Lemma* 8.2 *gives rise to an operator $\Delta^{-1}P$ that is not of weak type (p,p) for any $p\in[1,\infty)$.*

P r o o f. The kernel of the operator $\Delta^{-1}P$ is $P^{\mathrm{t}}N$; and Lemma 6.1 shows that we may consider the kernel $K=P^{\mathrm{t}}N_\infty$ instead, for large w. It is clear from (15) and (16) that $K(z,w)$ is a rational function of (x,y,u,v) that is nonzero and homogeneous of degree 0. By multiplying K by a scalar if necessary, we may choose $\varepsilon>0$ and a nonempty open set $U\subset G$ such that $K>\varepsilon$ on U. Let V be a relatively compact, symmetric neighbourhood of e. Then

$$K*\chi_V(g)=\int_V K(gg')\,d\mu_l(g').$$

Now choose a nonempty open subset U' of U and the neighbourhood V so that $gg'\in U$ for all $g\in U'$ and $g'\in V$. The positive isotropic dilations $\sigma_r(g)=\sigma_r(z,w)=(rz,rw)$, $r>0$, satisfy $\sigma_r(gg')=\sigma_r(g)g'$. It follows that the set

$$E=\{g\in G: K(gg')>\varepsilon \text{ for all } g'\in V\}$$

is invariant under all σ_r, and it contains a nonempty open set U'. So there exist an open set $A\subset\mathbb{C}$ and a relatively open subset $B\subset\mathbb{T}\subset\mathbb{C}$ such that $E\supset\{\sigma_r(A\times B): r>0\}=\{(z,w): w/|w|\in B,\ z\in|w|A\}$. Hence

$$
\begin{aligned}
\int\limits_{E\cap\{w:|w|>1\}}\frac{dz\,dw}{|w|^4} &\ge \int\limits_{\{w:|w|>1,\,w/|w|\in B\}}\frac{dw}{|w|^4}\int\limits_{z\in|w|A}dz\\
&=|A|\int\limits_{\{w:w/|w|\in B,\,|w|>1\}}\frac{dw}{|w|^2}=\infty.
\end{aligned}
$$

So $K*\chi_V(g)$ is bounded away from 0 on a set of infinite measure. This shows that the operator of convolution by K is not of weak type (p,p) for any $p>0$. ∎

We can now easily finish the proof of Theorem 2(b). It is clear from the commutation relations (1) that the differential operators that are listed in Lemmas 8.1 and 8.2 span the space $\mathcal{G}_2$. If $P \in \mathcal{G}_2$, then it follows from Lemmas 8.1 and 8.3 that the operator $\Delta^{-1}P$ is bounded if and only if $P \in \mathcal{I}_r$. ■

9. Proof of Theorem 2(a). It is clear from Lemmas 6.1 and 6.2 that each of the operators $P\Delta^{-1}$ with $P \in \mathcal{I}_l$ is bounded, since $VN_0 = VN_\infty = 0$.

Assume next that the coefficient c_0 of the identity operator in P is nonzero. By Lemma 6.1, the kernel $P(N - N_\infty)$ is integrable in the region where $|w| > 2$ for every $P \in \mathcal{G}_2$. Thus we can replace N by N_∞. Notice that $UN_\infty = VN_\infty = 0$, so

$$(P - c_0 I)N_\infty = O\left(\frac{1}{|z| + |w|}\right) = o(1)$$

for $|w| \to \infty$. On the other hand, $N_\infty \sim 1$ for $|z| \sim |w|$ large.

Let V be a small compact symmetric neighbourhood of $e \in G$, and consider

$$(PN_\infty) * \chi_V(z, w) = \int PN_\infty(z + wz', ww')\chi_V(z', w')\, \frac{dz'\, dw'}{|w'|^4}.$$

If V is small enough, then $|z + wz'| \sim |w| \sim |ww'|$ for $(z', w') \in V$ provided $|z| \sim |w|$. So $(PN_\infty) * \chi_V(z, w) \sim 1$. Since the measure of the set where w is large and $|z| \sim |w|$ is infinite, we see that $PN_\infty * \chi_V$ is in no weak L^p space.

The remainder of the proof deals with operators that contain no Δ^{-1} term. In the first lemma, we use the notation

$$XX \rightsquigarrow p(z) = -2(1 - 3x^2 + y^2) \tag{17}$$

to mean that

$$XXN_0 = \frac{1}{4\pi^2} \cdot \frac{|w|^2 p(z)}{(1 + |z|^2)^3}.$$

LEMMA 9.1. *In the notation of* (17), *we have*

$$\begin{aligned} XX &\rightsquigarrow -2(1 - 3x^2 + y^2), & YU &\rightsquigarrow -8y, \\ YY &\rightsquigarrow -2(1 + x^2 - 3y^2), & UX &\rightsquigarrow -2x(3 - x^2 - y^2), \\ XY &\rightsquigarrow 8xy, & UY &\rightsquigarrow -2y(3 - x^2 - y^2), \\ XU &\rightsquigarrow -8x, & U &\rightsquigarrow 2(1 + x^2 + y^2). \end{aligned}$$

These polynomials are linearly independent.

Proof. This follows from routine calculation. ■

The proof of Theorem 2(a) will be complete once the following lemma is established.

LEMMA 9.2. *If* $p(z)$ *is a nontrivial linear combination of the polynomials in Lemma* 9.1, *and*

$$W(z,w)=\frac{|w|^2p(z)}{(1+|z|^2)^3}\chi_{\{w:|w|<1/2\}},$$

then the convolution operator $f\mapsto W*f$ *is not of weak type* (p,p) *for any* $p\in[1,\infty)$.

Proof. We follow the pattern of [6, Lemmas 13 and 14]. As in that case, one finds that $W*f$ can be written as a convolution in the variable z only. Indeed, suppose that $f\in L^1(G)$ has compact support in the set $\{w:|w|>2\}$; write

$$F(-z)=\int f((z,w)^{-1})|w|^{-2}\,dw.$$

Let $r(z)=p(z)/(1+|z|^2)^3$, and set $r_w(z)=|w|^{-2}r(z/w)$. Then

$$W*f(z,w)=r_{1/w}*_{\mathbb{C}}F(z/w) \tag{18}$$

for $|w|<1$. In (18), the convolution on the right-hand side is taken over the complex plane, with respect to Lebesgue measure.

To deal with the case $p>1$, we simply let $f=f_T$, where $f_T((z,w)^{-1})=h(z)\psi_T(w)$. Here $0\le h\in C_c(\mathbb{C})$, $\psi_T(w)=T^{-1}\phi(|w|^{1/T})$ and $0\le\phi\in C_c(0,1/2)$. With this choice, one finds that $W*f_T$ does not depend on T in the set where $|w|<1$, whereas $\|f_T\|_p\to 0$ as $T\to\infty$.

In analogy with [6], the case $p=1$ amounts to disproving the inequality

$$\mu_l(\{(z,w):|w|<1,\ |r_{1/w}*F(z/w)|>\lambda\})\le\frac{C}{\lambda}\|F\|_{L^1(\mathbb{C})}. \tag{19}$$

Introducing z/w and $1/w$ as new variables, we see that (19) is equivalent to the inequality

$$\mu_r(\{(z,w):|w|>1,\ |r_w*F(z)|>\lambda\})\le\frac{C}{\lambda}\|F\|_{L^1(\mathbb{C})}. \tag{20}$$

As in [6], one finds that this inequality is self-improving, in the sense that one can delete the condition $|w|>1$.

To disprove (20), we consider two cases. If $\int_{\mathbb{C}} r\,dz\ne 0$, (which happens precisely when P has a nonzero U-component), we choose as F the characteristic function of the unit disc. For $|z|<1/2$ and w small, it is then easy to see that $|r_w*F(z)|>\frac12|\int r\,dz|$. The left-hand side of (20) is therefore infinite.

If $\int r\,dz=0$, we can essentially follow the argument in [6, proof of Lemma 14]. A few details will need to be modified: for instance, the decay of $r(z)$ at infinity is like $|z|^{-3}$; the summation index $k=(k_1,k_2)$ (see [6, (27)]) will now be a pair of natural numbers; and U will be a neighbourhood of $(0,1)$ in $\mathbb{C}^2$. ∎

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Received 14 August 1996;
revised 3 April 1997