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A DECOMPOSITION THEOREM FOR COMPLETE COMODULE ALGEBRAS OVER COMPLETE HOPF ALGEBRAS

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Introduction. Let k be a commutative ring with unity and let cMod denote the category of all complete k-modules (see Section 2). By applying the complete tensor product $\widehat{\otimes}$ in cMod we define in a natural way complete k-algebras, complete Hopf k-algebras, and complete comodule algebras over a complete Hopf k-algebra (and corresponding morphisms). Let H be a complete Hopf k-algebra. If A is a complete H-comodule algebra, via $\varrho : A \to A \widehat{\otimes} H$, then the subalgebra $A^H = \{a \in A : \varrho(a) = a \widehat{\otimes} 1\}$ of A is called the *algebra of invariants*. The main result of the paper is the following decomposition theorem (see Theorem 3.4).

If A is a complete H-comodule algebra admitting a morphism $f : H \to A$ of complete comodule k-algebras such that $f(h)a = af(h)$ for $h \in H$, $a \in A^H$, then the map $\alpha : A^H \widehat{\otimes} H \to A$, $\alpha(a \otimes h) = af(h)$, is an A^H-linear isomorphism of complete k-algebras.

The above theorem can be vieved as a Hopf-theoretic counterpart of the following well-known fact:

If k is an algebraically closed field and $G \times Y \to Y$ is an algebraic action of an algebraic group G over k on an algebraic variety Y admitting a G-morphism $f : Y \to G$, then the geometric quotient Y/G exists, and there exists an isomorphism of G-varieties $Y \cong Y/G \times G$.

In Section 4 some consequences of the above theorem are given.

One of them is as follows.

Let G be an abstract group and let $A = \bigoplus_{g\in G} A_g$ be a G-graded algebra such that there exists a group homomorphism $t : G \to U(A)$ with $t(g) \in A_g$ for $g \in G$ ($U(A)$ is the group of invertible elements in A). If $t(g)a = at(g)$ for all $g \in G$ and $a \in A_1$, then the map $\alpha : A_1 \otimes_k kG$, $\alpha(a \otimes g) = at(g)$, is an A_1-linear isomorphism of G-graded algebras, where kG denotes the group

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algebra of G over k. In particular, if G is the group of rational integers, then the assumptions on the G-graded algebra A reduce to existence of a $y \in A_1 \cap U(A)$ commuting with every $a \in A_1$.

1. Preliminaries and notation. Let k be a fixed commutative ring with unity. All modules, algebras, and tensor products are defined over k, unless stated otherwise.

By a *topological module* we mean a module M provided with a topology given by a family $\{M_i\}$ of submodules of M (as a fundamental system of neighborhoods of 0). When we want to indicate the topology of M we write $(M, \{M_i\})$. The module k will be viewed as a topological module with the discrete topology. A morphism of topological modules is a continuous morphism of modules. All submodules and quotient modules of a topological module will be viewed as topological modules with the induced topology and the quotient topology, respectively.

If $\{N_j, t_{j,j'}\}$ is an inverse system of topological modules, then the module $M = \lim_j \{M_j, t_{j,j'}\}$ with the topology inherited from the product topology in $\prod_j M_j$ is obviously a topological module. If $(M, \{M_i\})$ and $(N, \{N_s\})$ are topological modules, then the tensor product $M \otimes N$ will be treated as a topological module with the topology defined by the family $\{M_i \otimes N + M \otimes N_s\}$ (precisely, their images in $M \otimes N$).

If $(M, \{M_i\})$ is a topological module, then its *completion* $\widehat{M}$ is defined to be the topological module $\lim_i M/M_i$ (notice that M/M_i's have the discrete topology).

A topological module M is said to be *complete* if the canonical homomorphism $p : M \to \widehat{M}$ is an isomorphism of topological modules. It is easy to show that the topology of $\widehat{M}$ is given by the family of submodules $\{\widehat{M_i}\}$, p induces isomorphisms of modules $M/M_i \cong \widehat{M}/\widehat{M_i}$ for all i, and that $\widehat{M}$ is complete. Moreover, for any complete module L and any morphism $t : M \to L$ of topological modules, there is a unique morphism of topological modules $t' : \widehat{M} \to L$ such that $t'p = t$. If $f : M \to N$ is a morphism of topological modules, then $\widehat{f} : \widehat{M} \to \widehat{N}$ denotes the natural morphism of topological modules induced by f.

The category of all complete modules will be denoted by cMod. Since for every module M the topological module $(M, \{0\})$ is complete, the category of modules will be identified with the full subcategory of cMod formed by all discrete modules.

If M, N are complete modules, then we write $M \mathbin{\widehat{\otimes}} N$ for the completion of $M \otimes N$ and call it the *complete tensor product* of M and N.

A *complete algebra* is a triple (A, m, η), where $m : A \mathbin{\widehat{\otimes}} A \to A$, $\eta : k \to A$ are morphisms in cMod satisfying the appropriate associativity and unity axioms (compare for example [5]).

It is easy to prove that a complete algebra is a topological algebra A such that its topology is equivalent to the topology given by a family of two-sided ideals and A is complete as a topological module. If A and A' are complete algebras, then by a morphism $f : A \to A'$ we mean a morphism in cMod which preserves the multiplication and the unity.

Examples of complete algebras are the formal power series algebras $k[[X_1, \dots, X_n]]$ provided with the $(X_1, \dots, X_n)$-adic topology. If k is a field, then clearly all linearly compact k-algebras [1, Chap. I] are complete algebras. Just as for modules, ordinary algebras will be viewed as complete algebras with the discrete topology.

If A is a topological algebra whose topology is given by a family of two-sided ideals, then $\widehat{A}$ admits a unique complete algebra structure such that $p : A \to \widehat{A}$ is a morphism of complete algebras. In particular, the complete tensor product of complete algebras is a complete algebra. Moreover, if $F : A \to A'$ is a morphism of topological algebras whose topologies are given by families of two-sided ideals, then $\widehat{f} : \widehat{A} \to \widehat{A'}$ is a morphism of complete algebras. Obviously k is a complete algebra and $k \mathbin{\widehat{\otimes}} A \cong \widehat{A} \cong A \mathbin{\widehat{\otimes}} k$ for each topological algebra A. Also it is not difficult to prove (see Theorem 1 in [2, Chap. I, 1.]) that, given a complete algebra $(A, \{I_j\})$ (I_j's are two-sided ideals), the complete algebra $A \mathbin{\widehat{\otimes}} k[[X_1, \dots, X_n]]$ is isomorphic to the complete algebra $A[[X_1, \dots, X_n]]$ with the topology given by the family of two-sided ideals $\{I_j[[X]] + (X_1, \dots, X_n)^m\}_{j,m}$.

If A is a complete algebra, then a (right) *complete A-module* is a complete module M together with a morphism of complete modules $M \mathbin{\widehat{\otimes}} A \to M$, which satisfy the associativity and the unity axioms.

Replacing modules and algebras by complete modules and complete algebras, and also the tensor product by the complete tensor product, we define in exactly the same way as in [5] a complete coalgebra, a complete comodule over a complete coalgebra, and a complete Hopf algebra (and the corresponding morphisms). For example, a *complete Hopf algebra* is a system $(H, \varDelta, S, \varepsilon)$, where H is a complete algebra and $\varDelta : H \to H \mathbin{\widehat{\otimes}} H$, $S : H \to H$, $\varepsilon : H \to k$ are morphisms in cMod satisfying appropriate conditions (see [5, 4]).

Obviously, the ordinary Hopf algebras are complete Hopf algebras with the discrete topology. Examples of complete Hopf algebras provide (smooth) formal groups. Let us recall that an *n-dimensional formal group* (over the basic ring k) is a sequence

$$F = F(X, Y) = (F_1(X, Y), \dots, F_n(X, Y))$$

of formal power series from $k[[X, Y]]$, $X = \{X_1, \dots, X_n\}$, $Y = \{Y_1, \dots, Y_n\}$ such that

(1) $F(X, 0) = X$, $F(0, Y) = Y$,

(2) $F(F(X,Y),Z) = F(X,F(Y,Z))$

(see [2]). If F is such a formal group, then one easily verifies that

$$H(F) = (k[[X]], \Delta, S, \varepsilon)$$

with $\Delta(g(X)) = g(F(X,Y))$, $\varepsilon(X_i) = 0$, $i = 1,\ldots,n$, and S constructed as in [2, Chap. I, 3.] is a complete Hopf algebra. Moreover, if k is a field, then each complete Hopf algebra "living" on the complete algebra $k[[X]]$ is of this form.

PROPOSITION 1.1. *If* (H,Δ,S,ε) *is a complete Hopf algebra*, *then*

(1) $S(gh) = S(h)S(g)$ *for all* $h, g \in H$,

(2) $S(1) = 1$,

(3) $\varepsilon S = \varepsilon$,

(4) $T(S \mathbin{\widehat{\otimes}} S)\Delta = \Delta S$, *where* $T : H \mathbin{\widehat{\otimes}} H \to H \mathbin{\widehat{\otimes}} H$ *is the twist map* $h \mathbin{\widehat{\otimes}} g \mapsto g \mathbin{\widehat{\otimes}} h$.

Proof. Apply the arguments used in the proof of Proposition 4.0.1 of [5]. ∎

2. Complete Hopf modules and complete comodule algebras. By analogy with definitions of comodule algebras and Hopf modules (and their morphisms) over an ordinary Hopf algebra (see [5]) we define also the concept of a complete comodule algebra and a complete Hopf module (and their morphisms) over a complete Hopf algebra. These two concepts are of special interest for us, so we give precise definitions. For that purpose assume that (H,Δ,S,ε) is a complete Hopf algebra. If V, W are complete right H-modules, via $t : V \mathbin{\widehat{\otimes}} H \to V$ and $t' : W \mathbin{\widehat{\otimes}} H \to W$, respectively, then $V \mathbin{\widehat{\otimes}} W$ is also a right H-module, via the composed morphism

$$V \mathbin{\widehat{\otimes}} W \mathbin{\widehat{\otimes}} H \xrightarrow{1\widehat{\otimes}1\widehat{\otimes}\Delta} V \mathbin{\widehat{\otimes}} W \mathbin{\widehat{\otimes}} H \mathbin{\widehat{\otimes}} H \xrightarrow{1\widehat{\otimes}T\widehat{\otimes}1} V \mathbin{\widehat{\otimes}} H \mathbin{\widehat{\otimes}} W \mathbin{\widehat{\otimes}} H \xrightarrow{t\widehat{\otimes}t'} V \mathbin{\widehat{\otimes}} W,$$

where T is the twist map.

DEFINITION 2.1. A *complete H-comodule algebra* is a complete algebra A together with a morphism of algebras $D : A \to A \mathbin{\widehat{\otimes}} H$, which makes A a complete H-comodule. If A and A' are complete H-comodule algebras, then a *morphism* $A \to A'$ is a morphism of complete algebras, which is also a morphism of complete H-comodules.

If A is a complete algebra, then $A \mathbin{\widehat{\otimes}} H$ is a complete H-comodule algebra, via $1 \mathbin{\widehat{\otimes}} \Delta$.

DEFINITION 2.2. A (*right*) *complete Hopf module* over H is a complete module M satisfying the following conditions.

(1) M is a complete right H-module,

(2) M is a complete right H-comodule, via $\varrho : M \to M \mathbin{\widehat{\otimes}} H$,

(3) ϱ is a morphism of complete H-modules, where H acts on itself by right multiplication. A morphism of complete Hopf modules M, M' is a morphism of complete modules $M \to M'$ which is a morphism of complete H-modules and complete H-comodules.

Let H be a complete Hopf algebra. If M is a complete module, then $M \mathbin{\widehat{\otimes}} H$ is a complete Hopf module, via $1 \mathbin{\widehat{\otimes}} m_H : M \mathbin{\widehat{\otimes}} H \mathbin{\widehat{\otimes}} H \to M \mathbin{\widehat{\otimes}} H$ and $1 \mathbin{\widehat{\otimes}} \Delta : M \mathbin{\widehat{\otimes}} H \to M \mathbin{\widehat{\otimes}} H \mathbin{\widehat{\otimes}} H$, where m_H is the multiplication in H. Similarly to [5], for each complete H-comodule $\varrho : M \to M \mathbin{\widehat{\otimes}} H$, we define the module of (co)invariants $M^H = \{m \in M : \varrho(m) = m \mathbin{\widehat{\otimes}} 1\}$. One easily checks that M^H (with the induced topology) is a complete submodule of M. If A is a complete H-comodule algebra, then clearly A^H is a complete subalgebra of A. Our first result is a generalization of [5, Theorem 4.1.1].

THEOREM 2.3. *If $(H, \Delta, S, \varepsilon)$ is a complete Hopf algebra and M is a complete Hopf module over H, then the map $\alpha : M^H \mathbin{\widehat{\otimes}} H \to M$, $\alpha(m \otimes h) = m.h$, is an isomorphism of complete Hopf modules.*

Proof. Denote by Φ_M and Φ_H the families of submodules defining the topologies in M and H, respectively. Let $P : M \to M$ be the composition

$$M \xrightarrow{\varrho} M \mathbin{\widehat{\otimes}} H \xrightarrow{1 \widehat{\otimes} S} M \mathbin{\widehat{\otimes}} H \xrightarrow{t} M,$$

where t makes M a complete H-module. We are going to show that $P(M) \subseteq M^H$. It suffices to show that for each $m \in M$ and each $M_1 \in \Phi_M$, $H_1 \in \Phi_H$, $\varrho(P(m)) = P(m) \mathbin{\widehat{\otimes}} 1$ modulo $(M_1 \otimes H + M \otimes H_1)^\wedge$. Fix then $m \in M$ and $M_1 \in \Phi_M, H_1 \in \Phi_H$. Since $M \mathbin{\widehat{\otimes}} H/(M' \otimes H + M \otimes H')^\wedge = M/M' \otimes H/H'$ for all $M' \in \Phi_M$, $H' \in \Phi_H$, and we deal with continuous morphisms, there exists a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\bar{\varrho}"] \arrow[d, "\bar{\varrho}"'] & M/M_7 \otimes H/H_7 \arrow[d, "R_1"] \\
M/M_6 \otimes H/H_6 \arrow[r, "\bar{\varrho} \otimes \bar{\Delta}"] \arrow[d, "1 \otimes \bar{S}"'] & M/M_5 \otimes H/H_5 \otimes H/H_5 \otimes H/H_5 \arrow[d, "R_2"] \\
M/M_4 \otimes H/H_4 \arrow[r, "\bar{\varrho} \otimes \bar{\Delta}"] \arrow[d, "\bar{t}"'] & M/M_3 \otimes H/H_3 \otimes H/H_3 \otimes H/H_3 \arrow[d, "R_3"] \\
M/M_2 \arrow[r, "\bar{\varrho}"] & M/M_1 \otimes M/M_1,
\end{tikzcd}$$

where $R_1 = 1 \otimes (\overline{\Delta} \otimes 1)\overline{\Delta}$, $R_2 = 1 \otimes 1 \otimes T(\overline{S} \otimes \overline{S})$, $R_3 = (\overline{t} \otimes \overline{m}_H)(1 \otimes T \otimes 1)$, $M_i \in \Phi_M, H_i \in \Phi_H$, $i = 1, \ldots, 7$, m_H is the multiplication in H, and $\overline{\varrho}, \overline{\Delta}, \overline{S}, \overline{t}, \overline{m}_H$ denote the maps induced by the maps $\varrho, \Delta, S, t, m_H$, respectively (commutativity follows from the corresponding definitions and Proposition 1). Now, using the sigma notation from [5], we proceed in exactly the same way as in [5, Proof of Theorem 4.1.1]. Let $\overline{\varrho}(m) = \sum m_{(0)} \otimes m_{(1)} \in M/M_6 \otimes H/H_6$ and, as above, write $t(a \otimes h) = a.h$. Then we get the following equalities modulo $(M_1 \otimes H + M \otimes H_1)^\wedge$:

$$\begin{aligned}
\varrho(P(m)) &= \sum \overline{\varrho}(m_{(0)}.\overline{S}(m_{(1)})) = \sum m_{(0)}.\overline{S}(m_{(3)}) \otimes m_{(1)}\overline{S}(m_{(2)}) \\
&= \sum m_{(0)}.\overline{S}(m_{(2)}) \otimes \overline{\varepsilon}(m_{(1)}) \\
&= \sum m_{(0)}.\overline{S}(m_{(1)}) \otimes 1 = P(m) \mathbin{\widehat{\otimes}} 1.
\end{aligned}$$

Thus we have shown that $P(M) \subseteq M^H$. Define the map $\beta : M \to M^H \mathbin{\widehat{\otimes}} H$ by $\beta = (P \mathbin{\widehat{\otimes}} 1)\varrho$. We show that $\alpha\beta = 1_M$ and $\beta\alpha = 1_{M^H \widehat{\otimes} H}$. The first equality is a consequence of the commutativity of the diagram

$$\begin{tikzcd}
M \mathbin{\widehat{\otimes}} H \arrow[r, "\varrho \widehat{\otimes} 1"] & M \mathbin{\widehat{\otimes}} H \mathbin{\widehat{\otimes}} H \arrow[r, "1 \widehat{\otimes} S \widehat{\otimes} 1"] & M \mathbin{\widehat{\otimes}} H \mathbin{\widehat{\otimes}} H \arrow[r, "t \widehat{\otimes} 1"] \arrow[d, "1 \widehat{\otimes} m_H"] & M \mathbin{\widehat{\otimes}} H \arrow[d, "t"] \\
M \arrow[u, "\varrho"] \arrow[r, "\varrho"] & M \mathbin{\widehat{\otimes}} H \arrow[u, "1 \widehat{\otimes} \Delta"] \arrow[r, "1 \widehat{\otimes} \varepsilon\eta"] & M \mathbin{\widehat{\otimes}} H \arrow[r, "t"] & M
\end{tikzcd}$$

To prove the second one, fix $m' \in M^H$, $h \in H$ and $M_1 \in \Phi_M, H_1 \in \Phi_H$, and observe that there exists a commutative diagram

$$\begin{tikzcd}
M^H \otimes H \arrow[rr, "\bar{t}"] \arrow[d, "\varrho \widehat{\otimes} 1"] & & M \arrow[d, "\bar{\varrho}"] \\
M/M_6 \otimes (H/H_6)^{\otimes^3} \arrow[rr, "(\bar{t} \otimes \bar{m}_H)(1 \otimes T \otimes 1)"] \arrow[d, "\bar{\varrho} \otimes 1 \otimes \bar{\Delta} \otimes 1"] & & M/M_4 \otimes M/M_4 \arrow[d, "\bar{\varrho} \otimes 1"] \\
M/M_5 \otimes (H/H_5)^{\otimes^5} \arrow[rr, "(\bar{t} \otimes \bar{m}_H \otimes \bar{m}_H)(1 \otimes T')"] & & M/M_3 \otimes H/H_3 \otimes H/H_3 \arrow[d, "1 \otimes \bar{S} \otimes 1"] \\
& & M/M_2 \otimes H/H_2 \otimes H/H_2 \arrow[d, "\bar{t} \otimes 1"] \\
& & M/M_1 \otimes H/H_1
\end{tikzcd}$$

where as above all M_i's are in Φ_M, all H_i's are in Φ_H, and $T' : (H/H_5)^{\otimes^5} \to (H/H_5)^{\otimes^5}$ is given by $T'(x_1 \otimes x_2 \otimes \ldots \otimes x_5) = x_3 \otimes x_4 \otimes x_1 \otimes x_2 \otimes x_5$ (notation and arguments as above). Now as in [5, the proof of Theorem 4.1.1] we get the following equalities modulo $(M_1 \otimes H + M \otimes H_1)^\wedge$:

$$\begin{aligned}\beta\alpha(m'\otimes h) &= \sum \overline{P}(m'.h_{(1)})\otimes h_{(2)} = \sum (m'.h_{(1)}).\bar{S}(h_{(2)})\otimes h_{(3)}\\ &= \sum m'.(h_{(1)}\bar{S}(h_{(2)})\otimes h_{(3)} = \sum m'\bar{\varepsilon}(h_{(1)})\otimes h_{(2)} = m'\otimes h.\end{aligned}$$

Hence $\beta\alpha = 1$. It remains to verify that α is a morphism of complete Hopf modules. α is trivially a morphism of complete H-modules. Since $\varrho : M \to M \mathbin{\widehat{\otimes}} H$ is a morphism of H-modules, α is also a morphism of complete comodules. ■

THEOREM 2.4. *Suppose that A is a complete H-comodule algebra, via $\varrho : A \to A \mathbin{\widehat{\otimes}} H$, and that there exists a morphism of complete H-comodule algebras $f : H \to A$. Then*

(a) *the map $t : A \mathbin{\widehat{\otimes}} H \to A$, $t(a\otimes h) = af(h)$, defines a complete H-Hopf module structure on (A,ϱ),*

(b) *the map $\alpha : A^H \mathbin{\widehat{\otimes}} H \to A$, $\alpha(a\otimes h) = af(h)$, is an isomorphism of complete Hopf modules and left A^H-modules. Moreover, $\beta = \alpha^{-1} : A \to A^H \mathbin{\widehat{\otimes}} H$ is given as in the proof of Theorem* 2.3,

(c) *if $f(h)y = yf(h)$ for $h \in H$ and $y \in A^H$, then α is an isomorphism of complete algebras.*

Proof. Part (a) is a simple calculation. Parts (b) and (c) are immediate consequences of (a) and Theorem 2.3.

Remark 2.5. If A and H are discrete as topological modules, then parts (b) and (c) of the above theorem can be easily deduced from [4, Proposition 7.2.3], because every morphism of algebras $H \to A$ is invertible in the convolution algebra $(\mathrm{Hom}_k(H,A),*)$, see [5, Chap. IV] or [4, Def. 1.4.1].

Remark 2.6. If k is an algebraically closed field and $G\times Y \to Y$ is an (algebraic) action of an algebraic group G over k on an algebraic variety Y admitting a G-morphism $f : Y \to G$, then the geometric quotient Y/G exists, and the G-varieties Y and $Y/G \times G$ are isomorphic. Theorem 2.4(c) can be viewed as a Hopf-theoretic counterpart of this fact.

3. Applications. In this section we give some consequences of Theorem 2.4.

If A is a complete algebra and A' is a subalgebra of A, then we consider the set

$$C_A(A') = \{a\in A : \forall_{y\in A'}\, ay = ya\}.$$

Let $\mathbb{N}$ be the set of all non-negative rational integers and let n be a positive rational integer. If $\gamma = (\gamma_1,\ldots,\gamma_n)$, $\eta = (\eta_1,\ldots,\eta_n)$ are in $\mathbb{N}^n$, we set

$$(\gamma,\eta) = \binom{\gamma_1+\eta_1}{\eta_1}\cdots\binom{\gamma_n+\eta_n}{\eta_n}.$$

Recall that an *n-dimensional differentiation* of an algebra A is a morphism of algebras $D : A \to A[[X]], X = \{X_1, \ldots, X_n\}$, such that $D(a) = a$ (mod X). Given such a differentiation D, A^D will denote the subalgebra of its constants, i.e., $A^D = \{a \in A : D(a) = a\}$. Recall also that an n-dimensional differentiation D is said to be *locally nilpotent* if $D(A) \subseteq A[X]$, and D is said to be *iterative* if $D_\gamma D_\eta = (\gamma, \eta) D_{\gamma+\eta}$, where $D_\mu : A \to A$, $\mu \in \mathbb{N}^n$, are the maps determined by the equality $D(a) = \sum D_\mu(a) X^\mu$ ($X^\mu = X_1^{\mu_1} \ldots X_n^{\mu_n}$). Finally, if A is an algebra, $U(A)$ will denote the group of invertible elements of A.

The following corollary is well-known (for $n = 1$ and commutative A (see [3, Lemma 1.4]).

COROLLARY 3.1. *Suppose that $n \geq 1$ is an integer and $D : A \to A[[X]]$ is an n-dimensional locally nilpotent iterative differentiation of an algebra A such that there are elements $a_1, \ldots, a_n \in A$ with $a_i a_j = a_j a_i$ and $D(a_i) = a_i + X_i$ for $i, j = 1, \ldots, n$. If $a_1, \ldots, a_n \in C_A(A^D)$, then the map $A^D \otimes k[X] \to A$, $a \otimes g(X) \mapsto a g(a_1, \ldots, a_n)$ is an isomorphism of A^D-algebras.*

Proof. Let H denote the Hopf algebra $(k[X], \Delta, S, \varepsilon)$, where $\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i$, $S(X_i) = -X_i$, $\varepsilon(X_i) = 0$. Then an H-comodule algebra structure on A is nothing else than an n-dimensional, locally nilpotent, iterative differentiation $D : A \to A[[X]]$, and, given such a D, a morphism of H-comodule algebras $H \to A$ is simply a sequence $(a_1, \ldots, a_n)$ of commuting elements from A such that $D(a_i) = a_i + X_i$ for all i. So, the corollary follows from Theorem 2.4(c), because $A^H = A^D$. ■

COROLLARY 3.2. *Let G be an (abstract) group and let $A = \bigoplus_{g \in G} A_g$ be a G-graded algebra such that there exists a group homomorphism $t : G \to U(A)$ with $t(g) \in A_g$ for $g \in G$. If $t(G) \subseteq C_A(A_1)$, then the map $\alpha : A_1 \otimes kG \to A$, $\alpha(a \otimes g) = at(g)$, is an A_1-linear isomorphism of G-graded algebras, where kG denotes the group algebra of G over k.*

Proof. Denote by H the Hopf algebra kG with $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$, $\varepsilon(g) = 1$ for $g \in G$. Then an H-comodule algebra structure on A is simply a G-grading $A = \bigoplus_{g \in G} A_g$, A^H is then equal to A_1, and a morphism of H-comodule algebras $H \to A$ is a homomorphism of groups $t : G \to U(A)$ such that $t(g) \in A_g$ for all $g \in G$. So, we are done, again by Theorem 2.4(c). ■

Recall that a derivation $d : A \to A$ of an algebra A is called *locally nilpotent* if for each $a \in A$ there is an $s \in \mathbb{N}$ with $d^s(a) = 0$. Given such a derivation d, $A^d = \operatorname{Ker} d$ is the algebra of its constants.

By P we denote the divided power algebra $\bigoplus_{i=0}^{\infty} ky_i$ with $y_i y_j = (i,j)y_{i+j}$ for $i, j \geq 0$.

COROLLARY 3.3. *Let $d : A \to A$ be a locally nilpotent derivation of an algebra A such that there exists a sequence $1 = a_0, a_1, \ldots, a_i \in A$, with $a_i a_j = (i,j)a_{i+j}$ and $d(a_{i+1}) = a_i$ for $i \geq 0$. If $a_i \in C_A(A^d)$ for all i, then the map $\alpha : A^d \otimes P \to A$, $\alpha(a \otimes y_i) = aa_i$, $i \geq 0$, is an isomorphism of A^d-algebras.*

Proof. Apply Theorem 2.4(c) to the Hopf algebra $H = (P, \Delta, S, \varepsilon)$, where $\Delta(y_n) = \sum_{i+j=n} y_i \otimes y_j$, $\varepsilon(y_i) = \delta_{0i}$. The existence of an antipode S is an easy exercise. ■

By applying Theorem 2.4(c) to complete Hopf algebras of the form $H(F)$, where F is a formal group (see Section 1), and to complete algebras with topologies defined by powers of some ideal we get the following result.

COROLLARY 3.4. *Let $F(X,Y) = (F_1(X,Y), \ldots, F_n(X,Y))$ be an n-dimensional formal group, and let A be a complete algebra with topology defined by powers of a two-sided ideal J. Moreover, let $D : A \to A[[X]]$ be an n-dimensional differentiation satisfying the conditions*:

(i) $\sum_{\gamma,\mu} D_\gamma D_\mu(a) X^\gamma Y^\mu = \sum_\eta D_\eta(a) F(X,Y)^\eta$.

(ii) *There exists a sequence $a = (a_1, \ldots, a_n)$ with $a_i \in J$, $a_i a_j = a_j a_i$, and $D(a_i) = F_i(a, X)$, $i, j = 1, \ldots, n$.*

If $a_1, \ldots, a_n \in C_A(A^D)$, then the map $\alpha : A^D \mathbin{\widehat{\otimes}} k[[X]] \to A$, $\alpha(a \otimes g(X)) = ag(a_1, \ldots, a_n)$, is an isomorphism of A^D-algebras. In particular, the induced topology in A^D is equivalent to the $J \cap A^D$-adic topology in A^D.

Remark 3.5. (a) In the situation of the above corollary, if $F = X+Y$, then condition (i) says that D is an iterative differentiation of A, and in this case the corollary is well known.

(b) If $n = 1$ and $F = X + Y + XY$, then condition (i) says that $D_i D_j = \sum_k \binom{k}{i}\binom{i}{i+j-k}$ $\left(\binom{r}{s} = 0 \text{ when } r < s\right)$ for all i, j, and condition (ii) says that $D(a) = a + (1+a)X$ for some $a \in J$.

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