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A SEMITOPOLOGICAL ALGEBRA WITHOUT PROPER CLOSED SUBALGEBRAS

BY

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To Czesław Ryll-Nardzewski on his 70th birthday

A *semitopological* (resp. *topological*) algebra is a (Hausdorff) topological vector space provided with an associative separately (resp. jointly) continuous multiplication making it an algebra over the field of scalars ($\mathbb{C}$ or $\mathbb{R}$). It is called locally convex if the underlying topological vector space is locally convex. For detailed information about topological or semitopological algebras the reader is referred to [2], [3], or [4].

In [5] we asked whether every infinite-dimensional Banach algebra A always has a proper (i.e. different from (0) and A) closed subalgebra (for finite dimensions the answer is clearly affirmative unless the dimension is equal to one). We have not been able to answer this question, even if we replace "Banach algebra" by "topological algebra". This problem can be compared with the problem of possessing a proper closed (one-sided) ideal, which is also open for Banach algebras. Aharon Atzmon constructed in [1] a complete locally convex topological algebra which is not a division algebra and in which all proper ideals are dense. Here we construct a locally convex semitopological algebra in which every proper subalgebra is dense. It is not as satisfactory as Atzmon's algebra since it is neither a topological algebra nor a complete algebra; nevertheless, it is a first example of this type.

Our example is a (real or complex) algebra A of all polynomials in one variable t without the constant term (an infinite-dimensional unital algebra always has a closed proper subalgebra: the scalar multiples of the identity element). The topology of A is defined in the following way. Let $(a_i)_{i=1}^{\infty}$ be the sequence of real numbers defined inductively by

$$a_1 = 1 \quad \text{and} \quad a_n = n^n \sum_{s<n} a_s \quad \text{for } n > 1.$$

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The essential property of this sequence, to be used in the sequel, is

$$\lim_n \frac{C^n \sum_{s<n} a_s}{a_n} = 0 \quad \text{for all } C > 0. \tag{1}$$

Define on A a sequence of linear functionals by setting

$$f_k(x) = \sum_i a_{i+k-1}\xi_i \quad \text{for } k = 1, 2, \ldots,$$

where $x = \sum \xi_i t^i \in A$. Clearly the sequence (f_i) is linearly independent. We define on A a topology given by the sequence of seminorms

$$\|x\|_i = |f_i(x)|, \quad i = 1, 2, \ldots$$

It is a locally convex metrizable topology. Later we show that it is Hausdorff. The dual space A' coincides with the vector space of all linear combinations of the functionals f_i. In fact, if f_0 is a continuous linear functional on A, then there is a positive C and a natural n such that

$$|f_0(x)| \le C \max\{\|x\|_1, \ldots, \|x\|_n\} \quad \text{for all } x \in A. \tag{2}$$

We claim that f_0 is a linear combination of $f_1, \ldots, f_n$. If not, then $f_0, \ldots, f_n$ are linearly independent, and we can find $x_0, x_1, \ldots, x_n$ in A with $f_i(x_j) = \delta_{i,j}$. Setting in (2) $x = x_0$ we obtain zero on the right while the left hand side is one. The contradiction proves our assertion. Observe now that the map $x \mapsto tx$ is a continuous linear operator on A. This follows from $\|tx\|_i = \|x\|_{i+1}$ for all i and all x in A. Thus the multiplication by any fixed polynomial in t is a continuous linear operator on A and so A is a semitopological algebra. Later we shall show that the multiplication in A is not jointly continuous so that it is not a topological algebra.

In the sequel we shall need the following well known estimate. Let $x = \sum \xi_i t^i$ be an element in A and put $h(x) = \max_i\{|\xi_i|\}$. Then

$$h(x^n) \le (1+k)^n h(x)^n \tag{3}$$

for all natural n, where k is the degree of x. Our result reads as follows:

THEOREM. *The algebra A is Hausdorff and has no proper closed subalgebras.*

Proof. First we show that each non-zero subalgebra of A is dense. To this end we make the following simple observation. Every non-dense linear subspace of a locally convex space is contained in the kernel of some non-zero continuous linear functional. Hence to prove that each non-zero subalgebra of A is dense we have to show that for every non-zero element x of A and each non-zero element f of A' there is a natural n with $f(x^n) \neq 0$. Write f as $f = \sum c_i f_i$ and denote by i_0 the largest index i for which $c_i \neq 0$. Denote by k the degree of x and denote by c the leading coefficient (the coefficient of t^k) in x. The leading coefficient of x^n is c^n and we can write

$$(4)\quad f(x^n) = c_{i_0} c^n a_{nk+i_0} + \sum_{s<nk+i_0} a_s b_s = a_{nk+i_0}\left(c_{i_0} c^n + \frac{\sum_{s<nk+i_0} a_s b_s}{a_{nk+i_0}} \right),$$

where b_s are suitable linear combinations of coefficients of x^n with coefficients c_i. Setting $\overline{c} = \{|c_1| + \ldots + |c_{i_0}\}|$ and $r = (1+k)h(x)$ we see by (3) that $|b_s| \le \overline{c} r^n$. Formula (1) now implies

$$\left| \frac{\sum_{s<nk+i_0} a_s b_s}{a_{nk+i_0}} \right| \le \overline{c} r^n \frac{\sum_{s<nk+i_0} a_s}{a_{nk+i_0}} \to 0$$

and so, by (4), $f(x^n) \neq 0$ for large n.

It remains to be shown that A is Hausdorff. To this end it is sufficient to show that for each element $x \neq 0$ we have $\lim_n \|x\|_n = \infty$. Suppose that $x = \sum \xi_i t^i$ is of degree k. We have

$$\|x\|_n = a_{n+k} \left| \xi_k + \frac{\sum_{s<n+k} a_s \xi_{s-n}}{a_{n+k}} \right|.$$

The conclusion now follows by reasoning as above.

We now show that the algebra constructed above is not topological. To this end observe first that, as in (2), for any continuous seminorm $\|\cdot\|$ there is a natural n and a positive C such that

$$\|x\| \le C \max\{\|x\|_1, \ldots, \|x\|_n\}$$

for all x in A. Since the intersection of the zero sets of the seminorms $\|\cdot\|_1$, $\ldots, \|\cdot\|_n$ contains some non-zero elements, the above formula implies that there is no continuous norm on A. If A were a topological algebra, then its topology would be given by a sequence $(|\cdot|_i)_{i=1}^{\infty}$ of seminorms satisfying

$$(5)\qquad |x|_i \le |x|_{i+1}$$

for all $x \in A$, $i = 1, 2, \ldots$, and

$$(6)\qquad |xy|_i \le |x|_{i+1} |y|_{i+1}, \qquad x, y \in A,\ i = 1, 2, \ldots$$

(cf. [4]). Since both systems of seminorms are equivalent, (5) implies that there are a natural i and a positive C such that

$$(7)\qquad |f_1(x)| = \|x\|_1 \le C|x|_i$$

for all $x \in A$. Since $|\cdot|_{i+1}$ is not a norm, there is a non-zero element $z \in A$ with $|z|_{i+1} = 0$. Now (6) and (7) with $x = z^{n+1}$ imply

$$|f_1(z^{n+1})| = \|z^{n+1}\|_1 \le C|z^{n+1}|_i \le C|z|_{i+1} |z^n|_{i+1} = 0$$

for all natural n. But this contradicts the fact that $f_1(z^n) \neq 0$ for large n, and consequently A is not a topological algebra.

It would be very interesting to know whether an analogous construction is possible in the class of all topological algebras. So in addition to the above mentioned problem on Banach algebras we formulate a weaker question.

PROBLEM 1. Does there exist an infinite-dimensional topological algebra without proper closed subalgebras?

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