COLLOQUIUM MATHEMATICUM

VOL. 74 1997 NO. 2

DEFINABILITY OF PRINCIPAL CONGRUENCES IN EQUIVALENTIAL ALGEBRAS

BY

PAWEŁ M. IDZIAK AND ANDRZEJ WROŃSKI (KRAKÓW)

1. Introduction. We study definability of principal congruences in certain classes of congruence permutable but not congruence distributive groupoids related to Brouwerian semilattices. An example of a congruence permutable variety without DPC generated by a 3-element commutative groupoid is given. This contrasts with a result of A. F. Pixley implying DPC for all arithmetical varieties generated by a set of at most 4-element algebras.

Our example is an equivalential algebra, i.e. a groupoid $\mathbf{A} = (A, \leftrightarrow)$ that can be embedded into a $(\leftrightarrow)$-reduct of a Brouwerian semilattice, where the operation $\leftrightarrow$ is determined by the term $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. This notion was introduced by J. Kabziński and A. Wroński [12]. They showed that the class $\mathcal{E}$ of all equivalential algebras is equationally definable by the following identities:

- $(x \leftrightarrow x) \leftrightarrow y = y$,
- $((x \leftrightarrow y) \leftrightarrow z) \leftrightarrow z = (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow z)$,
- $((x \leftrightarrow y) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z)) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z) = x \leftrightarrow y$.

Although the variety $\mathcal{E}$ is closely related to the variety $\mathcal{BS}$ of all Brouwerian semilattices there are some important differences between them.

First note that the definition of $\mathcal{E}$ shows that $\mathcal{E}$ is locally finite (since $\mathcal{BS}$ is). However, the number of elements of the n-generated free algebra is known only for $n = 0, 1, 2, 3$ when it is 1, 2, 9 and 4415434 respectively (see [18] for the last number).

Note also that the term operation determined by the term $x \leftrightarrow x$ is constant in $\mathcal{BS}$ and in $\mathcal{E}$ as well. The corresponding constant will be denoted by 1. What is more interesting, the congruences are 1-regular, i.e. they are determined by their 1-equivalence classes. This is a well known property of $\mathcal{BS}$, where these equivalence classes are called filters and play an important

1991 *Mathematics Subject Classification*: Primary 08B05; Secondary 03G25.
The first author has been supported by KBN grants 2P03A 031 09.

role. An analogous concept of filter can be defined for equivalential algebras. This concept is well motivated since lattices of congruences and filters of an equivalential algebra are isomorphic under the mapping $\theta \to 1/\theta$. This means that $\mathcal{E}$, like $\mathcal{BS}$, is an ideal determined variety in the sense of H. P. Gumm and A. Ursini [9]. Since all 1-regular varieties are congruence modular and congruence n-permutable for some n (see J. Hagemmann [10]), $\mathcal{E}$ also satisfies these Malcev conditions. Actually, $\mathcal{E}$ is congruence permutable and

$$p(x, y, z) = ((x \leftrightarrow y) \leftrightarrow z) \leftrightarrow (((x \leftrightarrow z) \leftrightarrow z) \leftrightarrow x)$$

serves as a Malcev term.

Since among the identities in the language $(\leftrightarrow)$ the same are satisfied by $\mathcal{E}$ and $\mathcal{BS}$, any Malcev condition which holds in $\mathcal{E}$ also holds in $\mathcal{BS}$. However, the converse fails to hold. The simplest, but important, example is the condition of congruence distributivity. To see this let us only note that among subvarieties of $\mathcal{E}$ there is a smallest non-trivial one, namely the variety $\mathcal{E}_2$ of Boolean groups which is generated by the two-element group and can be axiomatized, relative to $\mathcal{E}$, by adjoining the associativity law. This immediately gives that there is no non-trivial congruence distributive variety of equivalential algebras. From this we see that no non-trivial subvariety of $\mathcal{E}$ has Equationally Definable Principal Congruences (see P. Köhler and D. Pigozzi [14]). However, the variety $\mathcal{E}_2$ does have (first order) Definable Principal Congruences. Moreover, it has the Congruence Extension Property, another condition fulfilled by $\mathcal{BS}$. As we will see later, $\mathcal{E}_2$ is the only non-trivial subvariety of $\mathcal{E}$ which has any of these properties.

To see that CEP fails to hold in $\mathcal{E}$ let us denote by $\mathbf{E}_3$ the $(\leftrightarrow)$-reduct of the 3-element Brouwerian semilattice with elements ordered by $1 > 2 > 3$. Then put $\mathbf{A} = \mathbf{E}_3 \times \mathbf{E}_3$ and observe that $\mathbf{B} = \{(1,1), (1,2), (2,1), (2,2)\}$ is a subuniverse of $\mathbf{A}$. Moreover, we have $((1,2),(2,1)) \in \Theta_{\mathbf{A}}((1,1),(2,2))$ while $((1,2),(2,1)) \notin \Theta_{\mathbf{B}}((1,1),(2,2))$. Thus CEP fails to hold in the variety $\mathcal{E}_3$ generated by $\mathbf{E}_3$. Since for every subvariety $\mathcal{V}$ of $\mathcal{E}$ we have either $\mathcal{V} \subseteq \mathcal{E}_2$ or $\mathcal{E}_3 \subseteq \mathcal{V}$, the trivial subvariety and $\mathcal{E}_2$ are the only ones having CEP.

One of the aims of this paper is to show that DPC is another property which fails in $\mathcal{E}_3$. This means, in particular, that there is a congruence permutable variety generated by a 3-element algebra which does not have DPC.

The notion of definability of principal congruences was introduced by J. T. Baldwin and J. Berman in [2] where they showed that in varieties with DPC certain results of W. Taylor concerning residually small varieties could be sharpened. They also asked if each finitely generated variety has DPC. A negative answer was given by S. Burris [5] who described an example of a 4-element algebra generating a congruence distributive variety without DPC.

Another example, due to W. Taylor [unpublished], is a variety generated by a 3-element commutative semigroup. However, congruences of algebras from Taylor's variety satisfy no non-trivial lattice identity—in particular, they do not permute. The first example of a finite algebra generating a congruence permutable variety without DPC was constructed by S. Burris and J. Lawrence [6]. In the course of their studies of DPC for groups and rings they obtained a 6-element group and a 64-element ring with the above properties. Later, G. Simons [17] obtained an 8-element ring generating a variety without DPC and showed that this is the smallest possible such example. The reader may also wish to consult the paper of K. A. Baker [1], where a complete characterization of finite groups generating varieties with DPC is given.

On the other hand, J. Berman [3] proved that every variety generated by a 2-element algebra does have DPC. Another positive result of this kind is contained in A. F. Pixley's paper [16] where the assumption that all subdirectly irreducible algebras from a variety have linearly ordered congruences (following [11] such varieties will be called *congruence linear*) suffices for a finitely generated arithmetical variety to have DPC. However, congruence linearity is not a necessary condition for a finitely generated arithmetical variety to have DPC, or even EDPC, as examples of some subvarieties of $\mathcal{BS}$ show. On the other hand, it is not true that every finitely generated arithmetical variety has DPC—an 8-element counterexample has been constructed by E. Kiss [13]. We do not know if it is smallest possible, but from Pixley's result it follows that each arithmetical variety $\mathcal{V}$ generated by a set $\mathcal{K}$ of at most 4-element algebras does have DPC.

To see this first note that all subdirectly irreducible algebras from $\mathcal{V}$ are contained in $\mathbf{HS}(\mathcal{K})$ and thus they have at most 4 elements. Therefore their congruence lattices are embeddable into the lattice Π_4 of all partitions of a 4-element set. The only distributive sublattice of Π_4 which has a unique atom μ and is not a chain must consist of exactly 5 elements $0 < \mu < \alpha, \beta < 1$. Moreover, an algebra $\mathbf{A}$ with this congruence lattice must have 4 elements, say a, b, c, d. Without loss of generality we may assume that the monolith μ corresponds to the partition $\{\{a,b\},\{c\},\{d\}\}$. For α and β we have two essentially different situations: either

$$\mathbf{A}/\alpha = \{\{a,b,c\},\{d\}\} \quad \text{and} \quad \mathbf{A}/\beta = \{\{a,b,d\},\{c\}\},$$

or

$$\mathbf{A}/\alpha = \{\{a,b,c\},\{d\}\} \quad \text{and} \quad \mathbf{A}/\beta = \{\{a,b\},\{c,d\}\}.$$

However, in the first case we have

$$c \overset{\alpha}{\equiv} a \overset{\beta}{\equiv} d \ \text{ while } \ (c,d) \notin \beta \circ \alpha,$$

and in the second case

$$a \overset{\alpha}{\equiv} c \overset{\beta}{\equiv} d \quad \text{while} \quad (a,d) \notin \beta \circ \alpha.$$

Consequently, **A** is not congruence permutable. Thus $\mathcal{V}$ is congruence linear and, in view of Pixley's result, it has DPC.

The above description of small algebras generating varieties without DPC leads to the following questions:

QUESTION 1. *How large is a smallest algebra generating an arithmetical variety without DPC*?

QUESTION 2. *Is there a* 3*-element algebra generating a congruence distributive variety without DPC*?

Equivalential algebras are subreducts (i.e. subalgebras of reducts) of Brouwerian semilattices. Actually, for any subvariety $\mathcal{V}$ of $\mathcal{BS}$ all algebras from $\mathcal{V}^{\mathrm{e}}$ are subreducts of those from $\mathcal{V}$, i.e. $\mathcal{V}^{\mathrm{e}} = \mathbf{S}(\mathcal{V}^{\mathrm{r}})$, where $\mathcal{V}^{\mathrm{r}}$ is the class of all $(\leftrightarrow)$-reducts of algebras from $\mathcal{V}$. In particular, we have $\mathcal{E}_2 = \mathcal{BS}_2^{\mathrm{e}} = \mathbf{S}(\mathcal{BS}_2^{\mathrm{r}})$, where $\mathcal{BS}_2$ is the smallest non-trivial subvariety of $\mathcal{BS}$. Actually, we have even more: $\mathcal{E}_2 = \mathcal{BS}_2^{\mathrm{r}}$. Although $\mathcal{E}_3 = \mathcal{BS}_3^{\mathrm{e}}$, where $\mathcal{BS}_3$ is generated by the 3-element Brouwerian semilattice $\mathbf{C}_3$, it is no longer true that $\mathcal{E}_3 = \mathcal{BS}_3^{\mathrm{r}}$. The situation can be even worse since we are unable to prove that $\mathcal{BS}_3^{\mathrm{r}}$ is (first order) axiomatizable. However, any class of the form $\mathcal{V}^{\mathrm{r}}$ is closed under reduced products and thus it is pseudo-elementary (see [7], p. 177). Thus let us state the following

QUESTION 3. *For which varieties $\mathcal{V}$ of Brouwerian semilattices is the class $\mathcal{V}^{\mathrm{r}}$ elementary*?

The main results of the paper are stated as follows:

RESULT 4. *The only non-trivial variety of equivalential algebras with Definable Principal Congruences is the variety $\mathcal{E}_2$ of all Boolean groups.*

RESULT 5. *For a variety $\mathcal{V}$ of Brouwerian semilattices the following conditions are equivalent*:

- *$\mathcal{V}^{\mathrm{r}}$ has Definable Principal Congruences*,
- *$\mathcal{V}$ is generated by a finite chain.*

The last theorem contrasts sharply with the fact that every subvariety of $\mathcal{BS}$ has Equationally Definable Principal Congruences. Actually, as shown by W. Blok, P. Köhler and D. Pigozzi [4], Brouwerian semilattices can serve as a paradigm for varieties with Equationally Definable Principal Congruences.

2. Preliminaries. By an *equivalent algebra* we mean a grupoid $\mathbf{A} = (A, \leftrightarrow)$ satisfying the following three identities (we adopt the convention of associating to the left and ignoring the symbol $\leftrightarrow$ of the equivalence operation):

$$xxy = y, \tag{2.1}$$
$$xyzz = xz(yz), \tag{2.2}$$
$$xy(xzz)(xzz) = xy. \tag{2.3}$$

The variety of equivalential algebras is denoted by $\mathcal{E}$.

For the basic arithmetic of equivalential algebras as well as for the proofs of all facts from this section the reader is referred to [12].

Among the identities fulfilled by $\mathcal{E}$ we have:

$$xy = yx, \tag{2.4}$$
$$xx = yy, \tag{2.5}$$
$$xyy(yxx) = xy, \tag{2.6}$$
$$xyyy = xy, \tag{2.7}$$
$$xyzz = xzz(yzz), \tag{2.8}$$
$$xyyzz = xzzyy, \tag{2.9}$$
$$xyy(yzz)(yzz) = xyy, \tag{2.10}$$
$$xyy(yz)(yz) = xyyzz, \tag{2.11}$$
$$x(yz)y = xyzy, \tag{2.12}$$
$$xyyxzz(xyyx) = xzzxyy(xzzx). \tag{2.13}$$
$$xyy(xzz)(xww) = xyy(xzz(xww)). \tag{2.14}$$

According to (2.1), (2.4) and (2.5) we can define a constant term $1 = xx$ that is a neutral element for the operation given by $\leftrightarrow$.

The identities (2.9) and (2.7) allow us to define an abbreviation $c\&X$ for a finite set X, by putting $c\&\{x_1,\ldots,x_k\} = cx_1x_1\ldots x_kx_k$. (We also put $c\&\emptyset = c$ for convenience.) Moreover, if $\mathtt{X} = \{X_1,\ldots,X_k\}$ is a finite family of finite subsets of an equivalential algebra containing c then, according to (2.4) and (2.14), we can define $c \star \mathtt{X}$ to be $(c\&X_1)\ldots(c\&X_k)$ (or 1 in case $k = 0$, i.e. $\mathtt{X}$ is empty). For a finite subset X of an equivalential algebra we will also use $c \star X$ to denote the element $c \star \{\{x\} : x \in X\}$.

We will need the fact that for every $c \in A$ the subalgebra of $\mathbf{A}$ generated by $\{c\&X : X$ is a finite subset of $A\}$ is associative and has the universe $\{c \star \mathtt{X} : \mathtt{X}$ is a finite family of finite subsets of $A\}$.

By a *filter* of an equivalential algebra $\mathbf{A}$ we mean a non-empty subset F of A satisfying, for all $a, b \in A$,

- if $a, ab \in F$ then $b \in F$,
- if $a \in F$ then $abb \in F$.

As mentioned in the introduction, filters correspond to congruences via the maps

$$F \mapsto \Theta_F = \{(x,y) : xy \in F\}, \qquad \theta \mapsto F_\theta = 1/\theta.$$

For an element c of an equivalential algebra we define the *principal filter* generated by c, denoted by $[c)$, to be the smallest filter containing c.

We will need the following characterization of principal filters proved in [12].

LEMMA 1. *Let a and c belong to an equivalential algebra $\mathbf{A}$. Then $a \in [c)$ iff there is a finite family $\mathtt{X}$ of finite subsets of A such that $acc = c \star \mathtt{X}$.*

Equivalential algebras are subreducts (i.e. subalgebras of reducts) of Brouwerian semilattices. Actually, for any subvariety $\mathcal{V}$ of $\mathcal{BS}$ all algebras from $\mathcal{V}^{\mathrm{e}}$ are subreducts of those from $\mathcal{V}$, i.e. $\mathcal{V}^{\mathrm{e}} = \mathbf{S}(\mathcal{V}^{\mathrm{r}})$, where $\mathcal{V}^{\mathrm{r}}$ is the class of all $(\leftrightarrow)$-reducts of algebras from $\mathcal{V}$. In particular, $\mathcal{E}_2 = \mathcal{BS}_2^{\mathrm{e}} = \mathbf{S}(\mathcal{BS}_2^{\mathrm{r}})$, where $\mathcal{BS}_2$ is the smallest non-trivial subvariety of $\mathcal{BS}$. Actually, $\mathcal{E}_2 = \mathcal{BS}_2^{\mathrm{r}}$. In general, for $n = 1, 2, \ldots, \omega$ we define $\mathcal{E}_n = \mathcal{BS}_n^{\mathrm{e}}$, where $\mathcal{BS}_n$ is the variety of Brouwerian semilattices generated by an n-element chain. Analogously to the case of Brouwerian semilattices all subvarieties of $\mathcal{E}_\omega$ form an infinite chain

$$\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{E}_3 \subseteq \ldots \subseteq \mathcal{E}_\omega.$$

The variety $\mathcal{E}_1$ is trivial, while $\mathcal{E}_2$ is the variety of Boolean groups and it can be axiomatized (relative to $\mathcal{E}$) by either the associativity law or the identity $xyy = x$. Moreover, from [12] we know that the variety $\mathcal{E}_\omega$ can be axiomatized, relative to $\mathcal{E}$, by the identity

$$x(yzz)(yzz)(x(zyy)(zyy)) = x(yz)(yz)x. \tag{1}$$

If a variety $\mathcal{V}$ of equivalential algebras (Brouwerian semilattices) is not contained in $\mathcal{E}_\omega$ (resp. $\mathcal{BS}_\omega$) then the algebra $(\mathbf{2^2}\oplus)^{\mathrm{r}}$ (resp. the Brouwerian semilattice $\mathbf{2^2}\oplus$) belongs to $\mathcal{V}$. Here $\mathbf{2^2}\oplus$ denotes the Brouwerian semilattice obtained from a four-element Boolean algebra by adjoining a new largest element, and $(\mathbf{2^2}\oplus)^{\mathrm{r}}$ its equivalential reduct.

The arithmetic in equivalential algebras that are reducts of Brouwerian semilattices can be slightly simplified. This is due to the fact that the identity

$$xyyzz = x(y \wedge z)(y \wedge z)$$

holds in Brouwerian semilattices. Therefore each element of the form $a\&X$ can be replaced by axx. Consequently, Lemma 1 can be rewritten as

LEMMA 2. *Let a and c belong to a reduct of a Brouwerian semilattice $\mathbf{A}$. Then $a \in [c)$ iff there is a finite subset X of A such that $acc = c \star X$.*

3. Principal congruences in non-Boolean varieties of equivalential algebras are not definable. The aim of this section is to prove the following

THEOREM 3. *The only non-trivial variety of equivalential algebras with definable principal congruences is the variety of Boolean groups.*

In proving the above theorem we will need an auxiliary lemma, an easy proof of which is left to the reader.

LEMMA 4. *Let S be a non-empty $(\leftrightarrow)$-subuniverse of the Boolean group $(\mathcal{P}(X), \leftrightarrow)$ of all subsets of a set X. Then*

$$\mathbf{E}_3[X, S] = \{\alpha \in \mathbf{E}_3^X : \alpha^{-1}(\{1,2\}) \in S\}$$

is a subuniverse of the direct power $\mathbf{E}_3^X$.

Proof of Theorem 3. It is obvious that in any Boolean group the formula

$$\Phi(x, y) \equiv (x = y \text{ or } x = 1)$$

defines the principal filter (i.e. normal subgroup) generated by the element y.

To prove the converse it suffices to show that the variety $\mathcal{E}_3$ does not have definable principal congruences. This will be shown by constructing a family $\{\mathbf{A}_n : n < \omega\}$ of finite algebras from $\mathcal{E}_3$ and two sequences $(\alpha_n : n < \omega)$, $(\gamma_n : n < \omega)$ of elements of $\mathbf{A}_n$ such that $\alpha_n \in [\gamma_n)$ for all $n < \omega$ but $(\alpha_n)/\mathcal{U} \notin [(\gamma_n)/\mathcal{U})$ in the ultraproduct of the $\mathbf{A}_n$'s over an ultrafilter $\mathcal{U}$ containing all cofinite subsets of ω.

Let Π_n be the set of all points of a projective geometry of dimension n over the field $\mathbf{Z}_2$. Then each $t \in \Pi_n$ can be represented as an $(n+1)$-tuple $(t_0, \ldots, t_n) \in \mathbf{Z}_2^{n+1}$ excluding the constant sequence of 0's. By a *hyperplane* we mean any subset of Π_n consisting of all points satisfying an equation (over $\mathbf{Z}_2$) of the form $e_0 t_0 + \ldots + e_n t_n = 0$. The set of all hyperplanes of Π_n is denoted by H_n. It is easy to see that H_n is a $(\leftrightarrow)$-subuniverse of the Boolean algebra $\mathcal{P}(\Pi_n)$, and thus, using Lemma 4, we can form a finite equivalential algebra $\mathbf{A}_n = \mathbf{E}_3[\Pi_n, H_n]$ belonging to $\mathcal{E}_3$.

Now, in each projective space Π_n fix a point p_n and define two elements α_n and γ_n in $\mathbf{A}_n$ by putting

$$\alpha_n(t) = \begin{cases} 2 & \text{if } t = p_n, \\ 1 & \text{otherwise,} \end{cases} \qquad \gamma_n(t) = 2 \text{ for all } t \in \Pi_n.$$

Moreover, for each subset $X \subseteq \Pi_n$ we will need the elements χ_X, δ_X of $\mathbf{E}_3^{\Pi_n}$ defined by

$$\chi_X(t) = \begin{cases} 1 & \text{if } t \in X, \\ 3 & \text{otherwise,} \end{cases} \qquad \delta_X(t) = \begin{cases} 2 & \text{if } t \in X, \\ 1 & \text{otherwise.} \end{cases}$$

Note that $\delta_X \in \mathbf{A}_n$ for each $X \subseteq \Pi_n$ and $\chi_X \in \mathbf{A}_n$ iff $X \in H_n$.

The reader will find no difficulties in checking that

(3.1) $\alpha_n \gamma_n \gamma_n = \alpha_n$,

(3.2) $\gamma_n \& \{\chi_{X_1}, \ldots, \chi_{X_k}\} = \delta_{X_1 \cap \ldots \cap X_k}$ whenever $X_1, \ldots, X_k \subseteq \Pi_n$, (3.3) $(\gamma_n \& \mathcal{X}_1)\ldots(\gamma_n \& \mathcal{X}_s) = \delta_{(\bigcap \mathcal{X}_1)\leftrightarrow\ldots\leftrightarrow(\bigcap \mathcal{X}_s)}$ whenever $\mathcal{X}_1,\ldots,\mathcal{X}_s$ are finite sets of the form $\{\chi_{X_1},\ldots,\chi_{X_k}\}$ with $X_i \subseteq \Pi_n$.

Since the intersection of all hyperplanes containing the point p_n is equal to $\{p_n\}$, by (3.2), we get

$$\gamma_n \& \{\chi_X : p_n \in X \in H_n\} = \alpha_n,$$

which together with (3.1) and Lemma 1 gives $\alpha_n \in [\gamma_n)$ for each $n < \omega$.

Let $\mathcal{U}$ be an ultrafilter on ω containing all cofinite subsets of ω. We are going to show that for $\alpha = (\alpha_n), \gamma = (\gamma_n)$ the element $\alpha/\mathcal{U}$ of the ultraproduct of all $\mathbf{A}_n$'s over $\mathcal{U}$ does not belong to the principal filter generated by $\gamma/\mathcal{U}$. Assume to the contrary that $\alpha/\mathcal{U} \in [\gamma/\mathcal{U})$, which, by Lemma 1, means that there are $s < \omega$ and finite subsets $\mathcal{X}_1,\ldots,\mathcal{X}_s$ of $\prod_{n<\omega} \mathbf{A}_n$ such that

$$[\![\alpha\gamma\gamma = (\gamma \& \mathcal{X}_1)\ldots(\gamma \& \mathcal{X}_s)]\!] \in \mathcal{U}.$$

Note that in any algebra $\mathbf{A}_n$ for each element ξ there is an element ζ such that $\zeta^{-1}(3) = \xi^{-1}(3)$, $\zeta^{-1}(2) = \emptyset$ and therefore $\gamma_n\xi\xi = \gamma\zeta\zeta$. This implies that in (3.4) each $\mathcal{X}_i$ can be assumed to be a subset of $\prod_{n<\omega} \mathbf{A}_n$ such that for each $n < \omega$ the nth coordinate of any element of this subset is of the form $\chi_X \in \mathbf{A}_n$ for some $X \in H_n$. Let $k = |\mathcal{X}_1 \cup \ldots \cup \mathcal{X}_s|$. Denoting by $\mathcal{X}_i^n$ the set of the nth coordinates of elements from $\mathcal{X}_i$, then identifying $\mathcal{X}_i^n$ with $\{X \subseteq \Pi_n : \chi_X \in \mathcal{X}_i^n\}$ and using (3.3) together with $\alpha\gamma\gamma = \alpha$ we get

$$\{n < \omega : \alpha_n = \delta_{(\bigcap \mathcal{X}_1^n)\leftrightarrow\ldots\leftrightarrow(\bigcap \mathcal{X}_s^n)}\} \in \mathcal{U}.$$

This, however, means that there is $k < \omega$ such that for infinitely many $n < \omega$, $\alpha_n = \delta_Z$ for some Boolean combination Z of at most k hyperplanes in Π_n, i.e. that for infinitely many $n < \omega$ the Boolean combination of at most k hyperplanes in Π_n gives a one-element subset of Π_n. This cannot be true, as otherwise representing this Boolean combination in its normal form as a join of intersections of hyperplanes or their complements we find that at least one of those intersections is non-empty. On the other hand, each such intersection consists of solutions of some sound set of at most k linear equations over the field $\mathbf{Z}_2$. However, for sufficiently large n such a set of equations has in Π_n at least $2^{n+1-k} - 1 \geq 3$ solutions and therefore the considered Boolean combination has, for almost all n, at least 3 points. This contradiction finishes the proof of Theorem 3. ■

4. Principal congruences in equivalential reducts of Brouwerian semilattices. In this section we prove that the only varieties of Brouwerian semilattices whose equivalential reducts have definable principal congruences are those generated by a finite chain. This will be done in three steps. First we prove that if the Brouwerian semilattice $\mathbf{2^2}\oplus$ is in the variety $\mathcal{V}$ then $\mathcal{V}^{\mathrm{r}}$ does not have definable principal congruences. To do this we need an auxiliary lemma.

By a *graph* we mean a pair (V, E), where V is a non-empty set and E is a family of non-empty, at most two-element subsets of V. Any graph of the form $(V, K(V))$ with $K(V) = \{\{a, b\} : a, b \in V\}$ is called *complete*.

LEMMA 5. *There is no natural number s such that for each finite graph (V, E) there are $X_1, \ldots, X_t \subseteq V$ with $t \leq s$ and $E = \bigoplus_{i=1}^{t} K(X_i)$, where $\bigoplus$ denotes symmetric difference.*

Proof. Assume that such natural numbers exist and let s be the smallest one. Let p be the number of all subsets of $\{1, \ldots, s\}$ with an odd number of elements.

Now let $V = \{x_0, \ldots, x_p\}$ and E be the family of all singletons of elements from V. By our assumption $E = \bigoplus_{i=1}^{t} K(X_i)$ for some $t \leq s$ and $X_1, \ldots, X_t \subseteq V$. Moreover, put $B_j = \{i \in \{1, \ldots, t\} : x_j \in X_i\}$ for any $j = 0, \ldots, p$. Since $\{x_j\} \in E$, it follows that $|B_j|$ is odd. Because $\{1, \ldots, t\}$ has at most p subsets with odd cardinality, there are $j_1 \neq j_2$ with $B_{j_1} = B_{j_2}$. This gives that $x_{j_1} \in X_i$ iff $x_{j_2} \in X_i$, for all $i = 1, \ldots, t$, and consequently for all i with $x_{j_1} \in X_i$ we have $\{x_{j_1}, x_{j_2}\} \in K(X_i)$. But $|B_{j_1}|$ is odd, which means that $\{x_{j_1}, x_{j_2}\}$ is in an odd number of components of the form $K(X_i)$. Consequently, $\{x_{j_1}, x_{j_2}\} \in E$, which gives $j_1 = j_2$, a contradiction. ■

Now we are ready to prove the following.

THEOREM 6. *If $\mathcal{V}$ is a variety of Brouwerian semilattices containing the algebra $\mathbf{2^2}\oplus$ then the class $\mathcal{V}^{\mathrm{r}}$ of all equivalential reducts of algebras from $\mathcal{V}$ does not have definable principal congruences.*

Proof. With any graph (V, E) we associate the poset $V \cup E$ with the order given by

$$x \leq y \quad \text{iff} \quad x = y \text{ or } x \in y.$$

Now let P_n be the poset associated with the complete n-element graph $K_n = (V_n, K(V_n))$ and $\mathbf{A}_n$ be the Brouwerian semilattice of all upward closed subsets of P_n ordered by reverse inclusion. It is routine to check that $\mathbf{A}_n$ belongs to the variety generated by the Brouwerian semilattice $\mathbf{2^2}\oplus$.

Define two elements α_n, γ_n of $\mathbf{A}_n$ by putting

$$\alpha_n = \{\{a\} : a \in V_n\}, \qquad \gamma_n = \{\{a, b\} : a, b \in V_n\}.$$

Our first claim is that $\alpha_n \in [\gamma_n)$, which means that

$$\alpha_n \gamma_n \gamma_n = (\gamma_n \beta_1 \beta_1) \ldots (\gamma_n \beta_k \beta_k) \tag{2}$$

for some $\beta_1, \ldots, \beta_k \in \mathbf{A}_n$. To see this, observe that for any $\beta_i \in \mathbf{A}_n$ we have $\gamma_n \beta_i \beta_i = \gamma_n(\gamma_n \wedge \beta_i)(\gamma_n \wedge \beta_i)$, and therefore we can consider only β_i's that satisfy $\beta_i \leq \gamma_n$ in $\mathbf{A}_n$, i.e. $\beta_i \supseteq \gamma_n$. This allows us to identify each such β_i with a subset $B_i = \beta_i - \gamma_n$ of V_n. Then we have

$$\gamma_n \beta_i \beta_i = \{\{a, b\} \subseteq V_n : a \notin B_i \text{ and } b \notin B_i\} = K(V_n - B_i).$$

Moreover, we have

$$(\gamma_n\beta_1\beta_1)\dots(\gamma_n\beta_k\beta_k) = \bigoplus_{i=1}^{k} K(V_n - B_i).$$

Now, for $a \in V_n$ put $\beta_a = \gamma_n \cup \{a\}$ and observe that $\gamma_n\beta_a\beta_a = \{\{a\}\} = K(\{a\})$, which together with $\alpha_n\gamma_n\gamma_n = \alpha_n$ gives

$$\alpha_n\gamma_n\gamma_n = \bigoplus_{a\in V_n} K(\{a\}) = \bigoplus_{a\in V_n} \gamma_n\beta_a\beta_a.$$

This means that the β_a's witness the equation (2) and shows that for each $n < \omega$ the element α_n lies in the equivalential filter of $\mathbf{A}_n$ generated by γ_n.

On the other hand, by Lemma 5, there is no uniform (with respect to n) upper bound for k satisfying the equality

$$\alpha_n = \bigoplus_{i=1}^{k} K(V_n - B_i),$$

with $B_1, \dots, B_k \subseteq V_n$. Now the standard ultraproduct argument shows that, in the ultraproduct of the $\mathbf{A}_n$'s over an ultrafilter $\mathcal{U}$ containing all cofinite subsets of ω, the element $(\alpha_n)/\mathcal{U}$ does not belong to the equivalential filter generated by $(\gamma_n)/\mathcal{U}$. ■

In the second step we prove the following.

THEOREM 7. *If $\mathcal{V}$ is a variety of Brouwerian semilattices containing an infinite chain then the class. $\mathcal{V}^{\mathrm{r}}$ of all equivalential reducts of algebras from $\mathcal{V}$ does not have definable principal congruences.*

Proof. Once again we will construct a chain $\{\mathbf{A}_n : n < \omega\}$ of algebras from $\mathcal{V}$ and two chains of elements $\{\alpha^n \in \mathbf{A}_n : n < \omega\}$, $\{\gamma^n \in \mathbf{A}_n : n < \omega\}$ such that for any $n < \omega$, α^n lies in the equivalential filter of $\mathbf{A}_n$ generated by γ^n but $(\alpha^n)/\mathcal{U}$ does not belong to the equivalential filter of the ultraproduct of the $\mathbf{A}_n$'s generated by $(\gamma^n)/\mathcal{U}$.

Let $\mathbf{C}_n$ be the n-element Brouwerian semilattice with elements ordered by $1 > 2 > \dots > n$. It is easy to observe that for each natural number n the set A_n consisting of all elements $\alpha = (\alpha_0, \dots, \alpha_{2n})$ of $\mathbf{C}_{2n+1}^{2n+1}$ satisfying

- if $k < 2n$ and $\alpha_k \neq 1, 2$ then $\alpha_{k+1} = \alpha_k$,
- if $p \in C_{2n+1} - C_2$ then α takes the value p exactly 0 or $p - 1$ times,

is a subuniverse of $\mathbf{C}_{2n+1}^{2n+1}$. Now in the Brouwerian semilattice $\mathbf{A}_n$ put

$$\alpha_i^n = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 2 & \text{if } i \text{ is odd,} \end{cases} \qquad \gamma_i^n = \begin{cases} 2 & \text{if } i \neq 2n, \\ 1 & \text{if } i = 2n. \end{cases}$$

Moreover, for any $p \in C_{2n+1} - \{1\}$, let η^p be the only element of $\mathbf{A}_n$ with the value-set $\{2, p\}$. Then an easy calculation shows that

$$\alpha^n\gamma^n\gamma^n = \alpha^n = (\gamma^n\eta^2\eta^2)(\gamma^n\eta^3\eta^3)\dots(\gamma^n\eta^{2n+1}\eta^{2n+1}),$$

which means that α^n belongs to the equivalential filter generated by γ^n.

On the other hand, each element of the form $\gamma^n\delta\delta$ with $\delta \in \mathbf{A}_n$ belongs to the set $\{\gamma^n\eta^p\eta^p : p = 2, \dots, 2n+1\}$. It is routine to check that $\alpha^n\gamma^n\gamma^n = \alpha^n$ does not belong to any equivalential algebra generated by a proper subset of this set. This actually shows that $(\alpha^n)/\mathcal{U}$ does not belong to the equivalential filter generated by $(\gamma^n)/\mathcal{U}$ in the ultraproduct of the $\mathbf{A}_n$'s over a non-principal ultrafilter $\mathcal{U}$. ■

The last part of this section deals with the proof of the following.

THEOREM 8. *If $\mathcal{V}$ is a variety of Brouwerian semilattices generated by a finite chain then the class $\mathcal{V}^{\mathrm{r}}$ has definable principal congruences.*

Proof. Let A be a Brouwerian semilattice. From Malcev's Lemma we see that if $(a, b) \in \Theta_{\mathbf{A}^{\mathrm{r}}}(c, d)$ then there is a quantifier free (positive) formula $\Phi(x, y, u, v, \overline{z})$ such that

$$A^{\mathrm{r}} \models \exists\overline{z}\ \Phi(a, b, c, d, \overline{z}).$$

Now let A_0 be the Brouwerian subsemilattice of A generated by $\{a, b, c, d, \overline{e}\}$, where $\overline{e} = e_1, \dots, e_k$ are elements of A witnessing the above formula. Then obviously

$$A_0^{\mathrm{r}} \models \exists\overline{z}\ \Phi(a, b, c, d, \overline{z})$$

and consequently $(a, b) \in \Theta_{A_0^{\mathrm{r}}}(c, d)$. Since the variety $\mathcal{BS}$, and therefore each subvariety $\mathcal{V}$ of it, is locally finite, the standard compactness argument (cf. proof of Lemma 2.2 in [15]) gives that the class $\mathcal{V}^{\mathrm{r}}$ has DPC iff its subclass $\mathcal{V}^{\mathrm{r}}_{\mathrm{fin}}$ of all finite members has DPC.

In view of the correspondence between (principal) congruences and (principal) filters in algebras from $\mathcal{E}$ and the above mentioned fact applied for $\mathcal{V} = \mathcal{BS}^{\mathrm{r}}_n$, to prove our theorem it suffices to show that for each $n \in \omega$ the class $(\mathcal{BS}^{\mathrm{r}}_n)_{\mathrm{fin}}$ has definable principal filters.

We claim that the formula

$$\bigvee_{k=0}^{n} \exists x_1 \dots \exists x_k\ acc = (cx_1x_1)\dots(cx_kx_k)$$

defines the relation "$a \in [c)$" in all finite algebras from $\mathcal{BS}^{\mathrm{r}}_n$. In view of Lemma 2 it suffices to show that

(8.1) *If c is an element of a finite $A \in \mathcal{BS}_n$ then for every $X \subseteq A$ there is $Z \subseteq A$ with $|Z| \le n$ and $c \star X = c \star Z$.*

Before proving (8.1) we need some preparations. Through the rest of the proof we assume that A is a finite Brouwerian semilattice and that 0 is its smallest element.

We put $x^{\mathrm{r}} = x00$ and $x^{\mathrm{d}} = x00x$, so that $x = x^{\mathrm{d}}x^{\mathrm{r}}$. An element is said to be *regular* (*dense*) if $x = x^{\mathrm{r}}$ ($x = x^{\mathrm{d}}$). Both these notions are known for (finite) Brouwerian semilattices (see e.g. [8]). Note also that the set A^{d} of all dense elements of A is a subuniverse of A. Moreover, we have $A^{\mathrm{d}} \in \mathcal{BS}_n$ whenever $A \in \mathcal{BS}_{n+1}$.

If x is a regular element then $xuu = x$ for all u. We will use this in the proof of

(8.2) If x is regular then $axx(bx) = abx$.

Indeed, we have

$$
\begin{aligned}
abx &= abxxx && \text{by (2.7)},\\
&= ab(xbb)xx && \text{by regularity of } x,\\
&= a(xb)bbxx && \text{by (2.2)},\\
&= a(xb)(xb)(xb)xx && \text{by (2.11)},\\
&= a(xb)xx && \text{by (2.7)},\\
&= a(bx)xx && \text{by (2.4)},\\
&= axx(bxxx) && \text{by (2.8)},\\
&= axx(bx) && \text{by (2.7)}.
\end{aligned}
$$

Now, with the help of (8.2) and (2.2), straightforward induction gives

$$
(a_1x)\dots(a_kx) = \begin{cases} a_1\dots a_kxx & \text{if } k \text{ is even},\\ a_1\dots a_kx & \text{if } k \text{ is odd}.\end{cases} \tag{8.3}
$$

Our final preparatory claim is the following:

(8.4) If c is a dense element of a finite Brouwerian semilattice A from $\mathcal{BS}_{n+1}$ then for every $X \subseteq A$ there is $Z \subseteq A$ with $|Z| \le n$ and $c \star X = c \star Z$.

We prove the above claim by induction on n. It is trivial for $n = 0$. For $n = 1$ we have $cxx = c$ for all $c, x \in A$, so that $c \star X \in \{c, 1\} = \{c \star \{c\}, c \star \emptyset\}$ for every $X \subseteq A$.

Assume now that $A \in \mathcal{BS}_{n+2}$. Using $x = x^{\mathrm{r}}x^{\mathrm{d}}$ and (1) (see Section 2), we get

$$
\begin{aligned}
cxx &= c(x^{\mathrm{r}}x^{\mathrm{d}})(x^{\mathrm{r}}x^{\mathrm{d}})\\
&= c(c(x^{\mathrm{r}}x^{\mathrm{d}}x^{\mathrm{d}})(x^{\mathrm{r}}x^{\mathrm{d}}x^{\mathrm{d}}))(c(x^{\mathrm{d}}x^{\mathrm{r}}x^{\mathrm{r}})(x^{\mathrm{d}}x^{\mathrm{r}}x^{\mathrm{r}}))\\
&= c(cx^{\mathrm{r}}x^{\mathrm{r}})(cx^{\mathrm{d}}x^{\mathrm{d}})
\end{aligned}
$$

so that for $X = \{x_1, \dots, x_k\}$ we have

$$c \star X = (cx_1^{\mathrm{r}} x_1^{\mathrm{r}}) \dots (cx_k^{\mathrm{r}} x_k^{\mathrm{r}})(cx_1^{\mathrm{d}} x_1^{\mathrm{d}}) \dots (cx_k^{\mathrm{d}} x_k^{\mathrm{d}})c', \tag{3}$$

where $c' \in \{c, 1\}$. On the other hand, for regular elements x, y we get

$$cxx(cyy) = c(xyy)(xyy)(c(yxx)(yxx)) = c(xy)(xy)c,$$

which allows us to shorten (3) to

$$c \star X = (czz)(cx_1^{\mathrm{d}} x_1^{\mathrm{d}}) \dots (cx_k^{\mathrm{d}} x_k^{\mathrm{d}})c'. \tag{4}$$

Now, since all elements $c, x_1^{\mathrm{d}}, \dots, x_k^{\mathrm{d}}, c'$ are dense, they belong to a subalgebra A^{d} of A which itself lies in the variety $\mathcal{BS}_{n+1}$. The induction hypothesis supplies us with $Z \subseteq A^{\mathrm{d}}$ such that $|Z| \le n$ and

$$(cx_1^{\mathrm{d}} x_1^{\mathrm{d}}) \dots (cx_k^{\mathrm{d}} x_k^{\mathrm{d}})c' = c \star Z.$$

Consequently, for $Y = Z \cup \{z\}$ we have $c \star X = c \star Y$ and $|Y| \le n + 1$ as required.

Now to prove (8.1) assume that $X = \{x_1, \dots, x_m\} \subseteq A \in \mathcal{BS}_n$. Then

$$\begin{aligned} c \star X &= (cx_1x_1) \dots (cx_mx_m) \\ &= (c^{\mathrm{d}}c^{\mathrm{r}}x_1x_1) \dots (c^{\mathrm{d}}c^{\mathrm{r}}x_mx_m) \\ &= (c^{\mathrm{d}}x_1x_1(c^{\mathrm{r}}x_1x_1)) \dots (c^{\mathrm{d}}x_mx_m(c^{\mathrm{r}}x_mx_m)) \\ &= (c^{\mathrm{d}}x_1x_1c^{\mathrm{r}}) \dots (c^{\mathrm{d}}x_mx_mc^{\mathrm{r}}), \end{aligned}$$

which, in view of (8.3), is equal either to $(c^{\mathrm{d}}x_1x_1) \dots (c^{\mathrm{d}}x_mx_m)c^{\mathrm{r}}c^{\mathrm{r}}$ or to $(c^{\mathrm{d}}x_1x_1) \dots (c^{\mathrm{d}}x_mx_m)c^{\mathrm{r}}$. Since c^{d} is dense, we can apply (8.4) to get a subset $Z = \{z_1, \dots, z_k\}$ of A with $k \le n - 1$ and

$$(c^{\mathrm{d}}x_1x_1) \dots (c^{\mathrm{d}}x_mx_m) = (c^{\mathrm{d}}z_1z_1) \dots (c^{\mathrm{d}}z_kz_k).$$

On the other hand, $c^{\mathrm{d}}z_iz_i = c00cz_iz_i = c00z_iz_i(cz_iz_i) = c00(cz_iz_i) = c^{\mathrm{r}}(cz_iz_i)$ and consequently $c \star X = c \star Z$ or $c \star X = c \star (Z \cup \{0\})$, as required.

This finishes the proof of (8.1) and therefore that of Theorem 8. ■

REFERENCES

[1] K. A. Baker, *Definable normal closures in locally finite varieties of groups*, Houston J. Math. 7 (1981), 467–471.

[2] J. T. Baldwin and J. Berman, *The number of subdirectly irreducible algebras in a variety*, Algebra Universalis 5 (1975), 379–389.

[3] J. Berman, *A proof of Lyndon's finite basis theorem*, Discrete Math. 29 (1980), 229–233.

[4] W. Blok, P. Köhler and D. Pigozzi, *On the structure of varieties with equationally definable principal congruences II*, Algebra Universalis 18 (1984), 334–379.

[5] S. Burris, *An example concerning definable principal congruences*, ibid. 7 (1977), 403–404.

[6] S. Burris and J. Lawrence, *Definable principal congruences in varieties of rings and groups*, ibid. 9 (1979), 152–164.

[7] C. C. Chang and H. J. Keisler, *Model Theory*, North-Holland, 1973.

[8] A. Eastman and W. Nemitz, *Density and closure in implicative semilattices*, Algebra Universalis 5 (1975), 1–5.

[9] H. P. Gumm and A. Ursini, *Ideals in universal algebras*, ibid. 19 (1984), 45–54.

[10] J. Hagemmann, *On regular and weakly regular congruences*, preprint 75, TH Darmstadt, 1973.

[11] P. M. Idziak, *Varieties with decidable finite algebras I*: *Linearity*, Algebra Universalis 26 (1989), 234–246.

[12] J. K. Kabziński and A. Wroński, *On equivalential algebras*, in: Proc. 1975 Internat. Sympos. on Multiple-Valued Logic (Indiana University, Bloomington, Ind., 1975), IEEE Comput. Soc., Long Beach, Calif., 1975, 419–428.

[13] E. Kiss, *Definable principal congruences in congruence distributive varieties*, Algebra Universalis 21 (1985), 213–224.

[14] P. Köhler and D. Pigozzi, *Varieties with equationally definable principal congruences*, ibid. 11 (1980), 213–219.

[15] R. McKenzie, *Paraprimal varieties*: *A study of finite axiomatizability and definable principal congruences*, ibid. 8 (1978), 336–348.

[16] A. F. Pixley, *Principal congruence formulas in arithmetical varieties*, in: Lecture Notes in Math. 1149, Springer, 1985, 238–254.

[17] G. E. Simons, *Varieties of rings with definable principal congruences*, Proc. Amer. Math. Soc. 87 (1983), 367–402.

[18] A. Wroński, *On the free equivalential algebra with three generators*, Bull. Sec. Logic Polish Acad. Sci. 22 (1993), 37–39.

Paweł M. Idziak
Computer Science Department
Jagiellonian University
Nawojki 11
30-072 Kraków, Poland
E-mail: idziak@ii.uj.edu.pl

Andrzej Wroński
Department of Logic
Jagiellonian University
Grodzka 52
31-044 Kraków, Poland
E-mail: uzwronsk@cyf-kr.edu.pl

Received 28 May 1996;
revised 13 January 1997