A continuum is a compact connected metric space. A map is a continuous function. For a continuum $X$ with metric $d$, $C(X)$ denotes the hyperspace of subcontinua of $X$ with the Hausdorff metric $H$. Given an onto map $f : X \to Y$ between continua, the induced map $f_1 : C(X) \to C(Y)$ is defined by $f_1(A) = f(A)$ (the image of $A$ under $f$). In a similar way $f_2 : C(C(X)) \to C(C(Y))$ is defined. As is observed in [15, 0.49], $f_1$ is continuous.

Properties of induced maps have been studied by J. J. Charatonik, W. J. Charatonik and H. Hosokawa [2–14]. In [13, Theorem 4.3], H. Hosokawa proved that if $f_1$ is open, then $f$ is open and he gave an example showing that the converse of this implication is not true. In the same paper he asked the following question: Is there an open map $f$ such that $f_1$ is open but $f_2$ is not open?

In this paper we prove the following result.

Theorem. Let $f : X \to Y$ be an onto map. If $Y$ is nondegenerate and $f_2$ is open, then $f$ is a homeomorphism.

As a consequence of this result, we obtain a positive answer to Hosokawa’s question.

Concepts not defined here will be taken as they appear in [15].

Lemma. Let $f : X \to Y$ be a confluent map, let $x_0 \in X$ and let $\beta$ be an order arc in $C(Y)$ such that $f(x_0) \in \bigcap_{B \in \beta} B$. Then there exists an order arc $\alpha$ in $C(X)$ such that $x_0 \in \bigcap_{A \in \alpha} A$ and $f_2(\alpha) = \beta$.

Proof. For each $B \in \beta$, let $A_B$ be the component of $f^{-1}(B)$ such that $x_0 \in A_B$, then $f(A_B) = B$. Define $\alpha_0 = \{A_B : B \in \beta\}$, $B_0 = \bigcap_{B \in \beta} B$ and $B_1 = \bigcup_{B \in \beta} B$. Then $\alpha_0$ has the following properties:

1. If $A \in \alpha_0$, then $A_B \subset A \subset A_{B_1}$, and
2. If $A_1, A_2 \in \alpha_0$, then $A_1 \subset A_2$ or $A_2 \subset A_1$.

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[219]
Proceeding as in Theorem 1.8 in [15], there exists a subset \( \alpha \) of \( C(X) \) such that \( \alpha_0 \subset \alpha \) and \( \alpha \) is maximal with respect to inclusion among all the subsets of \( C(X) \) having properties (1) and (2). Furthermore, as shown in the same theorem, \( \alpha \) is an order arc from \( A_{B_0} \) to \( A_{B_1} \). Let \( \beta_0 = f^2(\alpha) = \{ f_1(A) : A \in \alpha \} \). Notice that \( \beta_0 \) is a subcontinuum of \( C(Y) \) and if \( B_1, B_2 \in \beta_0 \), then \( B_1 \subset B_2 \) or \( B_2 \subset B_1 \). This implies (see [15, Theorem 1.4]) that \( \beta_0 \) is an order arc in \( C(Y) \). Since \( \beta \) is a subarc of \( \beta_0 \) and \( \beta \) contains the end-points \( B_0 \) and \( B_1 \) of \( \beta_0 \), we conclude that \( \beta = \beta_0 \).

**Theorem.** Let \( f : X \to Y \) be an onto map. If \( Y \) is nondegenerate and \( f_2 \) is open, then \( f \) is a homeomorphism.

**Proof.** We only have to prove that \( f \) is one-to-one. Since \( f_2 \) is open, then \( f_1 \) and \( f_2 \) are open ([13, Theorem 4.3]). Thus \( f \) is confluent [1]. For an order arc \( \alpha \) and elements \( A \) and \( B \) in \( \alpha \), we denote by \( \langle A, B \rangle_\alpha \) the subarc of \( \alpha \) which joins \( A \) and \( B \). For each subset \( A \) of \( X \), let \( F_1(A) = \{ \{ p \} : p \in A \} \). For a nonempty closed subset \( A \) of \( X \) and \( \varepsilon > 0 \), define \( N(\varepsilon, A) = \{ x \in X : \text{there exists } a \in A \text{ such that } d(x, a) < \varepsilon \} \). For a nonempty closed subset \( A \) of \( C(X) \) and \( \varepsilon > 0 \), define \( N^1(\varepsilon, A) = \{ B \in C(X) : \text{there exists } A \in A \text{ such that } H(A, B) < \varepsilon \} \). Let \( H^1 \) be the Hausdorff metric in \( C(C(X)) \). We divide the proof into three steps.

**Step 1.** If \( E \subset C(X) \) and \( f(E) \) is nondegenerate, then \( E \) is a component of \( f^{-1}(f(E)) \).

Let \( M = f(E) \). Suppose on the contrary that the component \( C \) of \( f^{-1}(M) \) which contains \( E \) is different from \( E \). Choose points \( p \in C - E \) and \( v \in M - \{ f(p) \} \). Let \( y = f(p) \) and let \( q \in E \) be such that \( f(q) = v \).

Let \( \beta \) and \( \gamma \) be order arcs in \( C(M) \), from \( \{ y \} \) to \( M \) and from \( \{ v \} \) to \( M \), respectively. From the lemma above, there exist order arcs \( \alpha \) and \( \lambda \) in \( C(X) \) such that \( \beta = f_2(\alpha) \), \( \gamma = f_2(\lambda) \), \( p \in \bigcap_{A \in \alpha} A \) and \( q \in \bigcap_{A \in \lambda} A \). Notice that \( \bigcap_{A \in \alpha} A \in \alpha \) (see [15, 1.5, p. 58]) and \( f(\bigcap_{A \in \alpha} A) = \{ y \} \). Taking an order arc from \( \{ p \} \) to \( \bigcap_{A \in \alpha} A \), we can extend \( \alpha \) to an order arc \( \alpha_1 \) in \( C(X) \), from \( \{ p \} \) to \( \bigcup_{A \in \alpha} A \), such that \( \beta = f_2(\alpha_1) \). Similarly, we can extend \( \alpha \) to an order arc from \( \{ p \} \) to \( C \). Thus we may assume that \( \alpha \) is an order arc from \( \{ p \} \) to \( C \). Analogously, we may assume that \( \lambda \) is an order arc from \( \{ q \} \) to \( C \).

Since \( \{ v \} \not\subset \beta \), there exist elements \( G_1 \), \( G_2 \) and \( G_3 \) in \( \gamma - \beta \) such that \( \{ v \} \subsetneq G_1 \subsetneq G_2 \subsetneq G_3 \) and \( \{ v \}, G_3 \} \cap \beta = \emptyset \). Let \( C_1 \), \( C_2 \) and \( C_3 \) in \( \lambda \) be such that \( f_1(G_i) = G_i \), for \( i = 1, 2, 3 \). Then \( \{ q \} \subsetneq C_1 \subsetneq C_2 \subsetneq C_3 \) and \( \{ q \}, C_3 \} \cap \alpha = \emptyset \). Since \( \{ y \} \not\subset \gamma \), there exists an element \( K \) in \( \beta - \{ y \} \) such that \( \{ y \}, K \} \cap \gamma = \emptyset \). Let \( D \) be an element in \( \alpha \) such that \( f(D) = K \). Then \( \{ q \}, D \} \cap \alpha = \emptyset \).

Let \( V \) be an open subset of \( Y \) such that \( y \in V \subset Cl(Y) \subset Y - \{ v \} \). It is easy to check that there exists \( \varepsilon > 0 \) such that:
(a) \(N^1(2\varepsilon, F_1(E) \cup \{q\}, C_1, \lambda) \cap f_1^{-1}((G_2, M, \gamma) \cup (K, M, \beta)) = \emptyset\);
(b) \(N^1(2\varepsilon, \alpha \cup (C_3, C, \lambda) \cap f_1^{-1}(F_1(M - V) \cup \{v\}, G_2, \gamma) = \emptyset\);
(c) \(N^1(2\varepsilon, \lambda) \cap f_1^{-1}(F_1(C \cap (V \cap M)) \cup \{y\}, K, \beta) = \emptyset\); and
(d) \(N^1(2\varepsilon, \alpha) \cap N^1(2\varepsilon, F_1(E)) = \emptyset\).

Let \(A = F_1(E) \cup \alpha \cup \lambda \) and let \(B = f_2(A) = F_1(M) \cup \beta \cup \gamma\). Since \(f_2\) is open, there exists \(\delta > 0\) such that if \(C \in C(C(Y))\) and \(H^1(B, C) < \delta\), then there exists \(D \in C(C(X))\) such that \(H^1(A, D) < \varepsilon\) and \(f_2(D) = C\).

Choose elements \(E_1\) and \(E_2\) in \(\gamma\) such that \(G_1 \subseteq E_1 \subseteq G_2 \subseteq E_2 \subseteq G_3\) and \(\text{diam}(E_1, E_2, \gamma) < \delta\). Define \(C = F_1(M) \cup \beta \cup \{v\}, E_1) \cap (E_2, M, \gamma) \subset B\). Then \(C \in C(C(Y))\) and \(H^1(B, C) < \delta\), so there exists \(D \in C(C(X))\) such that \(H^1(A, D) < \varepsilon\) and \(f_2(D) = C\).

We will show that \(D\) is disconnected; this contradiction will prove Step 1.

Define
\[
D_1 = D \cap \text{Cl}_{C(X)}(N^1(\varepsilon, \alpha \cup (C_1, C, \lambda)) \cap f_1^{-1}((\text{Cl}_{C(Y)}(F_1(V \cap M)) \cup \beta \cup (E_2, M, \gamma))
\]
and
\[
D_2 = D \cap \text{Cl}_{C(X)}(N^1(\varepsilon, F_1(E) \cup \{q\}, C_3, \lambda)) \cap f_1^{-1}(F_1(M) \cup \{y\}, K, \beta) \cup \{v\}, E_1, \gamma).
\]

Then \(D_1\) and \(D_2\) are compact subsets of \(D\).

If there exists an element \(D \in D_1 \cap D_2\), then \(f_1(D) \in \text{Cl}_{C(Y)}(F_1(V \cap M)) \cup \{y\}, K, \beta \cup \{v\}, E_1, \gamma\). This is a contradiction with (c) and (d). Hence \(D_1 \cap D_2 = \emptyset\).

In order to prove that \(D = D_1 \cup D_2\), take \(D \in D\), and let \(A \in A\) be such that \(H(A, D) < \varepsilon\). Since \(f_1(D) \in C\), we have \(f_1(D) \in F_1(\text{Cl}_{C(Y)}(V \cap M)) \cup \beta \cup (E_2, M, \gamma)\), or \(f_1(D) \in F_1(M) \cup \{y\}, K, \beta \cup \{v\}, E_1, \gamma\). In the first case, if \(A \in \alpha \cup (C_1, C, \lambda)\), then \(D \in D_1\). Suppose then that \(A \in F_1(E) \cup \{q\}, C_1, \lambda\). From (a), \(f_1(D) \in C \subset (G_2, M, \gamma) \cup (K, M, \beta)\), so \(f_1(D) \in F_1(M) \cup \{y\}, K, \beta \cup \{v\}, E_1, \gamma\). Therefore \(D \in D_2\).

In the second case, if \(A \in F_1(E) \cup \{q\}, C_3, \lambda\), then \(D \in D_2\). Thus we may assume that \(A \in \alpha \cup (C_3, C, \lambda)\). From (b), \(f_1(D) \in C \subset (F_1(M - V) \cup \{v\}, G_2, \gamma) \subset F_1(V \cap M) \cup \beta \cup (E_2, M, \gamma)\). Therefore \(D \in D_1\). This completes the proof that \(D = D_1 \cup D_2\).

Since \(H^1(A, D) < \varepsilon\) and \(C \in A\), there exists \(D_1 \in D\) such that \(H(C, D_1) < \varepsilon\), and from (b) and (c), \(f_1(D_1) \in C \subset (F_1(M) \cup \{v\}, G_2, \gamma) \cup \{y\}, K, \beta)\), which implies that \(D_1 \in D_1 \cap D_2 \neq \emptyset\). Since \(\{q\} \in A\), there exists \(D_2 \in D\) such that \(H(q, D_2) < \varepsilon\). From (a) and (c), \(f_1(D_2) \in C \subset (\beta \cup (G_2, M, \gamma)\). This implies that \(D_2 \in D_2\). Hence \(D_2 \neq \emptyset\).

Therefore \(D\) is disconnected. This contradiction completes the proof of Step 1.

**Step 2.** \(f\) is light (i.e., fibers of \(f\) are totally disconnected).
Suppose on the contrary that there exists a point $y \in Y$ and a nondegenerate continuum $A$ contained in $f^{-1}(y)$. Choose two points $p \neq q$ in $A$ and let $\varepsilon > 0$ be such that $d(p, q) > 2\varepsilon$. Let $A = f_1(A)$, then $f_2(A) = \{y\}$. Since $f_2$ is open, there exists $\delta > 0$ such that if $C \in C(C(Y))$ and $H^1(\{y\}, C) < \delta$, then there exists $D \in C(C(X))$ such that $H^1(A, D) < \varepsilon$ and $f_2(B) = D$. Since $Y$ is nondegenerate, there exists $D \in C(Y)$ such that $y \in D \neq \{y\}$ and $\text{diam}(D) < \delta$. Then there exists $B \in C(C(X))$ such that $H^1(A, B) < \varepsilon$ and $f_2(B) = \{D\}$. Define $B = \bigcup_{C \subseteq B} C$. Then $B \in C(X)$ (see \cite[Lemma 1.43]{15}) and $f(B) = D$. Since $H^1(A, B) < \varepsilon$, there exist $B_1, B_2 \subseteq B$ such that $H(\{p\}, B_1) < \varepsilon$ and $H(\{q\}, B_2) < \varepsilon$. Then $B_1 \cap B_2 = \emptyset$, so $B_1 \subseteq B$. From Step 1, $B_1$ is a component of $f^{-1}(f(B_1)) = f^{-1}(D)$. This contradicts the fact that $B \subset f^{-1}(\{B\})$ and completes the proof of Step 2.

**Step 3.** $f$ is one-to-one.

Suppose on the contrary that there exist two points $p \neq q$ in $X$ such that $f(p) = f(q)$. Let $y = f(p)$. Let $A$ be a subcontinuum of $X$ such that $A$ is irreducible between $p$ and $q$. Let $B = f(A)$. From Step 2, $B$ is a nondegenerate subcontinuum of $Y$.

We show that $B$ is indecomposable. Suppose on the contrary that there exist proper subcontinua $D$ and $E$ of $B$ such that $B = D \cup E$ and $y \in D$. Let $A_1$ and $A_2$ be the components of $f^{-1}(D)$ such that $p \in A_1$ and $q \in A_2$. Since $f$ is confluent, $f(A_1) = f(A_2) = B$. Then $f(A_1 \cup A_2) = B$ and $A \cup A_1 \cup A_2$ is connected. From Step 1, $A$ is a component of $f^{-1}(B)$, thus $A_1 \cup A_2 \subseteq A$. Irreducibility of $A$ and $f(A_1) \neq f(A)$ imply that $q \not\in A_1$ and $A_1 \cap A_2 = \emptyset$. Let $z$ be a point in $D \cap E$, let $w \in A_1$ be such that $f(w) = z$ and let $B_1$ be a component of $f^{-1}(E)$ such that $w \in B_1$. Step 1 applied to $A$ and to $A_1 \cup B_1$ implies that $A = A_1 \cup B_1$. This implies that $A_2 \subseteq B_1$, so $D \subseteq E$ and $B = E$. This contradiction proves that $B$ is indecomposable.

Let $v$ be a point in $B$ such that $y$ and $v$ are in different components of $B$. Choose a point $u \in A$ such that $f(u) = v$. Let $\beta$ and $\gamma$ be order arcs in $C(B)$, from $\{y\}$ to $B$ and from $\{v\}$ to $B$, respectively. The irreducibility of $B$ between $y$ and $v$ implies that $\beta \cap \gamma = \{B\}$. Since $f(p) = f(q) = y$ and $f(u) = v$, the previous lemma implies that there exist order arcs $\alpha_1, \alpha_2$ and $\lambda$ such that $f_2(\alpha_1) = \beta = f_2(\alpha_2)$, $f_2(\lambda) = \gamma$, $p \in \bigcap_{D \in \alpha_1} D$, $q \in \bigcap_{D \in \alpha_2} D$ and $u \in \bigcap_{D \in \lambda} D$. Since $\{y\} \in \beta$, there exists $D_0 \in \alpha_1$ such that $f(D_0) = \{y\}$. Then $\bigcap_{D \in \alpha_1} D$ is a subcontinuum of $X$ such that $f(\bigcap_{D \in \alpha_1} D) = \{y\}$. From Step 2, we have $\{p\} = \bigcap_{D \in \alpha_1} D$. Since $B \in \beta$, there exists $D_1 \in \alpha_1$ such that $f(D_1) = B$, which implies that $f(\bigcup_{D \in \alpha_1} D) = B$. From Step 1, we obtain $\bigcup_{D \in \alpha_1} D = A$. Hence $\alpha_1$ is an order arc from $\{p\}$ to $A$. Similarly, $\alpha_2$ is an order arc from $\{q\}$ to $A$ and $\lambda$ is an order arc from $\{u\}$ to $A$. The irreducibility of $A$ between $p$ and $q$ implies that $\alpha_1 \cap \alpha_2 = \{A\}$. If
$D \in \alpha \cap \lambda$, $f(D)$ is a subcontinuum of $B$ which contains the points $y$ and $v$, then $f(D) = B$. From Step 1, $D = A$. Thus $\alpha \cap \lambda = \{A\}$ for $i = 1, 2$.

Choose elements $G_1$, $G_2$ and $G_3$ in $\gamma$ such that $\{v\} \subseteq G_1 \subseteq G_2 \subseteq G_3 \subseteq B$ and elements $H_1$, $H_2$ and $H_3$ in $\beta$ such that $\{y\} \subseteq H_1 \subseteq H_2 \subseteq H_3 \subseteq B$. Choose $C_1$, $C_2$ and $C_3$ in $\lambda$ such that $f(C_i) = G_i$, for each $i = 1, 2, 3$. Then $\{u\} \subseteq C_1 \subseteq C_2 \subseteq C_3 \subseteq A$. Choose $A_1 \in \alpha_1$ and $A_2 \in \alpha_2$ such that $f(A_1) = H_2 = f(A_2)$. Then $\{p\} \subseteq A_1 \subseteq A$ and $\{q\} \subseteq A_2 \subseteq A$.

It is easy to verify that there exists $\varepsilon > 0$ such that:

(a) $N^1(2\varepsilon, \{A_1, A\}_\alpha \cup \langle C_3, A \rangle_\lambda) \cap f_1^{-1}(\langle F_1(B) \cup \langle \{y\}, H_1 \rangle_\beta \cup \langle \{v\}, G_2 \rangle_\gamma) = \emptyset$;

(b) $N^1(2\varepsilon, \{A_2\}_\alpha \cup \langle \{w\}, C_1 \rangle_\lambda) \cap f_2^{-1}(\langle H_3, B \rangle_\beta \cup \langle \{w\}, G_2 \rangle_\gamma) = \emptyset$;

(c) $N^1(2\varepsilon, \{A_1 \cup \lambda \} \cap f_1^{-1}(\langle H_1, H_3 \rangle_\beta) = \emptyset$; and

(d) $N^1(2\varepsilon, \{\{q\}, A_2\}_\alpha \cap N^1(2\varepsilon, \{A_1, A\}_\alpha) = \emptyset$.

Define $A = F_1(A) \cup \langle \{q\}, A_2 \rangle_\alpha \cup \langle A_1, A \rangle_\alpha \cup \lambda$, then $A \in C(C(X))$. Define $B = f_2(A) = F_1(B) \cup \beta \cup \gamma$. Since $f_2$ is open, there exists $\delta > 0$ such that if $C \in C(C(Y))$ and $H^1(B, C) < \delta$, then there exists $D \in C(C(X))$ such that $H^1(A, D) < \varepsilon$ and $f_2(D) = C$.

Choose elements $E_1$ and $E_2$ in $\gamma$ such that $G_1 \subseteq E_1 \subseteq G_2 \subseteq E_2 \subseteq G_3$ and $\text{diam}(\{E_1, E_2\}_\gamma) < \delta$. Define $C = F_1(B) \cup \beta \cup \langle \{v\}, E_1 \rangle_\gamma \cup \langle \{E_2, B\}, E_1 \rangle_\gamma$. Then $C \in C(C(Y))$ and $H^1(B, C) < \delta$, so there exists $D \in C(C(X))$ such that $H^1(A, D) < \varepsilon$ and $f_2(D) = C$.

As in the proof of Step 1, the proof of Step 3 will be completed by proving that $D$ is disconnected.

Define $D_1 = D \cap C_{C(X)}(N^1(2\varepsilon, \{A_1, A\}_\alpha \cup \langle C_1, A \rangle_\lambda) \cap f_1^{-1}(\langle H_1, B \rangle_\beta \cup \langle \{E_2, B\}, E_1 \rangle_\gamma)$ and

$$D_2 = D \cap C_{C(X)}(N^1(2\varepsilon, \{A_2\}_\alpha \cup \langle \{u\}, C_3 \rangle_\lambda) \cap f_1^{-1}(\langle F_1(B) \cup \langle \{y\}, H_3 \rangle_\beta \cup \langle \{v\}, E_1 \rangle_\gamma)$$

Then $D_1$ and $D_2$ are closed subsets of $D$.

If there exists an element $D \in D_1 \cap D_2$, then $f_1(D) \in \langle H_1, H_3 \rangle_\beta$. From (c), $D \notin N^1(2\varepsilon, F_1(A) \cup \lambda)$. Since $D \in D_1 \cap D_2$, we have $D \in N^1(2\varepsilon, \{\{q\}, A_2\}_\alpha) \cap N^1(2\varepsilon, \{A_1, A\}_\alpha)$, which contradicts (d). Thus $D_1 \cap D_2 = \emptyset$.

We prove that $D = D_1 \cup D_2$. Let $D \in D$ and let $E \in A$ be such that $H(E, D) < \varepsilon$. Then $f_1(D) \in F_1(B) \cup \langle \{y\}, H_1 \rangle_\beta \cup \langle \{v\}, E_1 \rangle_\gamma$ or $f_1(D) \in \langle H_3, B \rangle_\beta \cup \langle \{v\}, E_1 \rangle_\gamma$ or $f_1(D) \in \langle H_1, H_3 \rangle_\beta$. In the first case, from (a), $E \in A - \langle \{A_1, A\}_\alpha \cup \langle C_3, A \rangle_\lambda \rangle$, which is impossible. In the second case, from (b), $E \in A - \langle F_1(A) \cup \langle \{q\}, A_2 \rangle_\alpha \cup \langle \{u\}, C_3 \rangle_\lambda \rangle$. This implies that $D \in D_2$. In the second case, from (c), $E \in A - \langle F_1(A) \cup \lambda \rangle$, which is impossible. Therefore, $D_1 \cup D_2$.
Since $A \in \mathcal{A}$, there exists $D_1 \in \mathcal{D}$ such that $H(A, D_1) < \varepsilon$. From (a), $f_1(D_1) \in C - (F_1(B) \cup \langle \{y\}, H_1 \rangle \cup \langle \{v\}, G_2 \rangle)$. Thus $D_1 \in \mathcal{D}_1$ and $\mathcal{D}_1 \neq \emptyset$. Since $\{u\} \in \mathcal{A}$, there exists $D_2 \in \mathcal{D}$ such that $H(\{u\}, D_2) < \varepsilon$. From (b), $f_1(D_2) \in C - (\langle H_3, B \rangle \cup \langle G_2, B \rangle)$. Thus $D_2 \in \mathcal{D}_2$ and $\mathcal{D}_2 \neq \emptyset$.

Therefore $\mathcal{D}$ is disconnected. This contradiction proves Step 3 and completes the proof of the theorem.

**Corollary.** Let $f : X \to Y$ be an onto map. If $Y$ is nondegenerate then $f_2$ is open if and only if $f$ is a homeomorphism.

**Example.** Let $X$ be the square $[0, 1] \times [0, 1]$, $Y = [0, 1]$ and let $f : X \to Y$ be the natural projection onto the first coordinate. It is easy to check that $f$ is open and $f_1$ is also open. From the theorem above, $f_2$ is not open. This example answers Hosokawa’s question.

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