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THE OPENNESS OF INDUCED MAPS ON HYPERSPACES

BY

ALEJANDRO ILLANES (MÉXICO)

A *continuum* is a compact connected metric space. A *map* is a continuous function. For a continuum X with metric d, $C(X)$ denotes the hyperspace of subcontinua of X with the Hausdorff metric H. Given an onto map $f: X \to Y$ between continua, the induced map $f_1 : C(X) \to C(Y)$ is defined by $f_1(A) = f(A)$ (the image of A under f). In a similar way $f_2 : C(C(X)) \to C(C(Y))$ is defined. As is observed in [15, 0.49], f_1 is continuous.

Properties of induced maps have been studied by J. J. Charatonik, W. J. Charatonik and H. Hosokawa [2–14].

In [13, Theorem 4.3], H. Hosokawa proved that if f_1 is open, then f is open and he gave an example showing that the converse of this implication is not true. In the same paper he asked the following question: Is there an open map f such that f_1 is open but f_2 is not open?

In this paper we prove the following result.

THEOREM. *Let $f : X \to Y$ be an onto map. If Y is nondegenerate and f_2 is open, then f is a homeomorphism.*

As a consequence of this result, we obtain a positive answer to Hosokawa's question.

Concepts not defined here will be taken as they appear in [15].

LEMMA. *Let $f : X \to Y$ be a confluent map, let $x_0 \in X$ and let β be an order arc in $C(Y)$ such that $f(x_0) \in \bigcap_{B\in\beta} B$. Then there exists an order arc α in $C(X)$ such that $x_0 \in \bigcap_{A\in\alpha} A$ and $f_2(\alpha) = \beta$.*

Proof. For each $B \in \beta$, let A_B be the component of $f^{-1}(B)$ such that $x_0 \in A_B$, then $f(A_B) = B$. Define $\alpha_0 = \{A_B : B \in \beta\}$, $B_0 = \bigcap_{B\in\beta} B$ and $B_1 = \bigcup_{B\in\beta} B$. Then α_0 has the following properties:

(1) If $A \in \alpha_0$, then $A_{B_0} \subset A \subset A_{B_1}$ and

(2) If $A_1, A_2 \in \alpha_0$, then $A_1 \subset A_2$ or $A_2 \subset A_1$.

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Proceeding as in Theorem 1.8 in [15], there exists a subset α of $C(X)$ such that $\alpha_0 \subset \alpha$ and α is maximal with respect to inclusion among all the subsets of $C(X)$ having properties (1) and (2). Furthermore, as shown in the same theorem, α is an order arc from A_{B_0} to A_{B_1}. Let $\beta_0 = f_2(\alpha) = \{f_1(A) : A \in \alpha\}$. Notice that β_0 is a subcontinuum of $C(Y)$ and if $B_1, B_2 \in \beta_0$, then $B_1 \subset B_2$ or $B_2 \subset B_1$. This implies (see [15, Theorem 1.4]) that β_0 is an order arc in $C(Y)$. Since β is a subarc of β_0 and β contains the end-points B_0 and B_1 of β_0, we conclude that $\beta = \beta_0$.

THEOREM. *Let $f : X \to Y$ be an onto map. If Y is nondegenerate and f_2 is open, then f is a homeomorphism.*

Proof. We only have to prove that f is one-to-one. Since f_2 is open, then f_1 and f are open ([13, Theorem 4.3]). Thus f is confluent [1]. For an order arc α and elements A and B in α, we denote by $\langle A, B\rangle_\alpha$ the subarc of α which joins A and B. For each subset A of X, let $F_1(A) = \{\{p\} : p \in A\}$. For a nonempty closed subset A of X and $\varepsilon > 0$, define $N(\varepsilon, A) = \{x \in X :$ there exists $a \in A$ such that $d(x, a) < \varepsilon\}$. For a nonempty closed subset $\mathcal{A}$ of $C(X)$ and $\varepsilon > 0$, define $N^1(\varepsilon, \mathcal{A}) = \{B \in C(X) :$ there exists $A \in \mathcal{A}$ such that $H(A, B) < \varepsilon\}$. Let H^1 be the Hausdorff metric in $C(C(X))$. We divide the proof into three steps.

STEP 1. *If $E \in C(X)$ and $f(E)$ is nondegenerate, then E is a component of $f^{-1}(f(E))$.*

Let $M = f(E)$. Suppose on the contrary that the component C of $f^{-1}(M)$ which contains E is different from E. Choose points $p \in C - E$ and $v \in M - \{f(p)\}$. Let $y = f(p)$ and let $q \in E$ be such that $f(q) = v$.

Let β and γ be order arcs in $C(M)$, from $\{y\}$ to M and from $\{v\}$ to M, respectively. From the lemma above, there exist order arcs α and λ in $C(X)$ such that $\beta = f_2(\alpha)$, $\gamma = f_2(\lambda)$, $p \in \bigcap_{A\in\alpha} A$ and $q \in \bigcap_{A\in\lambda} A$. Notice that $\bigcap_{A\in\alpha} A \in \alpha$ (see [15, 1.5, p. 58]) and $f(\bigcap_{A\in\alpha} A) = \{y\}$. Taking an order arc from $\{p\}$ to $\bigcap_{A\in\alpha} A$, we can extend α to an order arc α_1 in $C(X)$, from $\{p\}$ to $\bigcup_{A\in\alpha} A$, such that $\beta = f_2(\alpha_1)$. Similarly, we can extend α to an order arc from $\{p\}$ to C. Thus we may assume that α is an order arc from $\{p\}$ to C. Analogously, we may assume that λ is an order arc from $\{q\}$ to C.

Since $\{v\} \notin \beta$, there exist elements G_1, G_2 and G_3 in $\gamma - \beta$ such that $\{v\} \subsetneq G_1 \subsetneq G_2 \subsetneq G_3$ and $\langle \{v\}, G_3\rangle_\gamma \cap \beta = \emptyset$. Let C_1, C_2 and C_3 in λ be such that $f_1(C_i) = G_i$, for $i = 1, 2, 3$. Then $\{q\} \subsetneq C_1 \subsetneq C_2 \subsetneq C_3$ and $\langle \{q\}, C_3\rangle_\lambda \cap \alpha = \emptyset$. Since $\{y\} \notin \gamma$, there exists an element K in $\beta - \{y\}$ such that $\langle \{y\}, K\rangle_\beta \cap \gamma = \emptyset$. Let D be an element in α such that $f(D) = K$. Then $\langle \{p\}, D\rangle_\alpha \cap \lambda = \emptyset$.

Let V be an open subset of Y such that $y \in V \subset \mathrm{Cl}_Y(V) \subset Y - \{v\}$. It is easy to check that there exists $\varepsilon > 0$ such that:

(a) $N^1(2\varepsilon, F_1(E) \cup \langle\{q\}, C_1\rangle_\lambda) \cap f_1^{-1}(\langle G_2, M\rangle_\gamma \cup \langle K, M\rangle_\beta) = \emptyset$;
(b) $N^1(2\varepsilon, \alpha \cup \langle C_3, C\rangle_\lambda) \cap f_1^{-1}(F_1(M - V) \cup \langle\{v\}, G_2\rangle_\gamma) = \emptyset$;
(c) $N^1(2\varepsilon, \lambda) \cap f_1^{-1}(F_1(\mathrm{Cl}_Y(V \cap M)) \cup \langle\{y\}, K\rangle_\beta) = \emptyset$; and
(d) $N^1(2\varepsilon, \alpha) \cap N^1(2\varepsilon, F_1(E)) = \emptyset$.

Let $\mathcal{A} = F_1(E) \cup \alpha \cup \lambda$ and let $\mathcal{B} = f_2(\mathcal{A}) = F_1(M) \cup \beta \cup \gamma$. Since f_2 is open, there exists $\delta > 0$ such that if $\mathcal{C} \in C(C(Y))$ and $H^1(\mathcal{B}, \mathcal{C}) < \delta$, then there exists $\mathcal{D} \in C(C(X))$ such that $H^1(\mathcal{A}, \mathcal{D}) < \varepsilon$ and $f_2(\mathcal{D}) = \mathcal{C}$.

Choose elements E_1 and E_2 in γ such that $G_1 \subsetneq E_1 \subsetneq G_2 \subsetneq E_2 \subsetneq G_3$ and $\mathrm{diam}(\langle E_1, E_2\rangle_\gamma) < \delta$. Define $\mathcal{C} = F_1(M) \cup \beta \cup \langle\{v\}, E_1\rangle_\gamma \cup \langle E_2, M\rangle_\gamma \subset \mathcal{B}$. Then $\mathcal{C} \in C(C(Y))$ and $H^1(\mathcal{B}, \mathcal{C}) < \delta$, so there exists $\mathcal{D} \in C(C(X))$ such that $H^1(\mathcal{A}, \mathcal{D}) < \varepsilon$ and $f_2(\mathcal{D}) = \mathcal{C}$.

We will show that $\mathcal{D}$ is disconnected; this contradiction will prove Step 1.

Define

$$\mathcal{D}_1 = \mathcal{D} \cap \mathrm{Cl}_{C(X)}(N^1(\varepsilon, \alpha \cup \langle C_1, C\rangle_\lambda)) \cap f_1^{-1}(\mathrm{Cl}_{C(Y)}(F_1(V \cap M)) \cup \beta \cup \langle E_2, M\rangle_\gamma)$$

and

$$\mathcal{D}_2 = \mathcal{D} \cap \mathrm{Cl}_{C(X)}(N^1(\varepsilon, F_1(E) \cup \langle\{q\}, C_3\rangle_\lambda)) \cap f_1^{-1}(F_1(M) \cup \langle\{y\}, K\rangle_\beta \cup \langle\{v\}, E_1\rangle_\gamma).$$

Then $\mathcal{D}_1$ and $\mathcal{D}_2$ are compact subsets of $\mathcal{D}$.

If there exists an element $D \in \mathcal{D}_1 \cap \mathcal{D}_2$, then $f_1(D) \in \mathrm{Cl}_{C(Y)}(F_1(V \cap M)) \cup \langle\{y\}, K\rangle_\beta$ and $D \in N^1(2\varepsilon, \alpha \cup \langle C_1, C\rangle_\lambda) \cap N^1(2\varepsilon, (F_1(E) \cup \langle\{q\}, C_3\rangle_\lambda))$. This is a contradiction with (c) and (d). Hence $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$.

In order to prove that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, take $D \in \mathcal{D}$, and let $A \in \mathcal{A}$ be such that $H(A, D) < \varepsilon$. Since $f_1(D) \in \mathcal{C}$, we have $f_1(D) \in F_1(\mathrm{Cl}_{C(Y)}(V \cap M)) \cup \beta \cup \langle E_2, M\rangle_\gamma$ or $f_1(D) \in F_1(M) \cup \langle\{y\}, K\rangle_\beta \cup \langle\{v\}, E_1\rangle_\gamma$. In the first case, if $A \in \alpha \cup \langle C_1, C\rangle_\lambda$, then $D \in \mathcal{D}_1$. Suppose then that $A \in F_1(E) \cup \langle\{q\}, C_1\rangle_\lambda$. From (a), $f_1(D) \in \mathcal{C} - (\langle G_2, M\rangle_\gamma \cup \langle K, M\rangle_\beta)$, so $f_1(D) \in F_1(M) \cup \langle\{y\}, K\rangle_\beta \cup \langle\{v\}, E_1\rangle_\gamma$. Therefore $D \in \mathcal{D}_2$. In the second case, if $A \in F_1(E) \cup \langle\{q\}, C_3\rangle_\lambda$, then $D \in \mathcal{D}_2$. Thus we may assume that $A \in \alpha \cup \langle C_3, C\rangle_\lambda$. From (b), $f_1(D) \in \mathcal{C} - (F_1(M - V) \cup \langle\{v\}, G_2\rangle_\gamma) \subset F_1(V \cap M) \cup \beta \cup \langle E_2, M\rangle_\gamma$. Therefore $D \in \mathcal{D}_1$. This completes the proof that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$.

Since $H^1(\mathcal{A}, \mathcal{D}) < \varepsilon$ and $C \in \mathcal{A}$, there exists $D_1 \in \mathcal{D}$ such that $H(C, D_1) < \varepsilon$, and from (b) and (c), $f_1(D_1) \in \mathcal{C} - (F_1(M) \cup \langle\{v\}, G_2\rangle_\gamma \cup \langle\{y\}, K\rangle_\beta)$, which implies that $D_1 \in \mathcal{D}_1$ and $\mathcal{D}_1 \neq \emptyset$. Since $\{q\} \in \mathcal{A}$, there exists $D_2 \in \mathcal{D}$ such that $H(\{q\}, D_2) < \varepsilon$. From (a) and (c), $f_1(D_2) \in \mathcal{C} - (\beta \cup \langle G_2, M\rangle_\gamma)$. This implies that $D_2 \in \mathcal{D}_2$. Hence $D_2 \neq \emptyset$.

Therefore $\mathcal{D}$ is disconnected. This contradiction completes the proof of Step 1.

STEP 2. *f is light (i.e., fibers of f are totally disconnected).*

Suppose on the contrary that there exists a point $y \in Y$ and a nondegenerate continuum A contained in $f^{-1}(y)$. Choose two points $p \neq q$ in A and let $\varepsilon > 0$ be such that $d(p,q) > 2\varepsilon$. Let $\mathcal{A} = F_1(A)$, then $f_2(\mathcal{A}) = \{\{y\}\}$. Since f_2 is open, there exists $\delta > 0$ such that if $\mathcal{C} \in C(C(Y))$ and $H^1(\{\{y\}\}, \mathcal{C}) < \delta$, then there exists $\mathcal{D} \in C(C(X))$ such that $H^1(\mathcal{A}, \mathcal{D}) < \varepsilon$ and $f_2(\mathcal{D}) = \mathcal{C}$. Since Y is nondegenerate, there exists $D \in C(Y)$ such that $y \in D \neq \{y\}$ and $\operatorname{diam}(D) < \delta$. Then there exists $\mathcal{B} \in C(C(X))$ such that $H^1(\mathcal{A}, \mathcal{B}) < \varepsilon$ and $f_2(\mathcal{B}) = \{D\}$. Define $B = \bigcup_{C \in \mathcal{B}} C$. Then $B \in C(X)$ (see [15, Lemma 1.43]) and $f(B) = D$. Since $H^1(\mathcal{A}, \mathcal{B}) < \varepsilon$, there exist $B_1, B_2 \in \mathcal{B}$ such that $H(\{p\}, B_1) < \varepsilon$ and $H(\{q\}, B_2) < \varepsilon$. Then $B_1 \cap B_2 = \emptyset$, so $B_1 \subsetneq B$. From Step 1, B_1 is a component of $f^{-1}(f(B_1)) = f^{-1}(D)$. This contradicts the fact that $B \subset f^{-1}(D)$ and completes the proof of Step 2.

STEP 3. *f is one-to-one.*

Suppose on the contrary that there exist two points $p \neq q$ in X such that $f(p) = f(q)$. Let $y = f(p)$. Let A be a subcontinuum of X such that A is irreducible between p and q. Let $B = f(A)$. From Step 2, B is a nondegenerate subcontinuum of Y.

We show that B is indecomposable. Suppose on the contrary that there exist proper subcontinua D and E of B such that $B = D \cup E$ and $y \in D$. Let A_1 and A_2 be the components of $f^{-1}(D)$ such that $p \in A_1$ and $q \in A_2$. Since f is confluent, $f(A_1) = D = f(A_2)$. Then $f(A \cup A_1 \cup A_2) = B$ and $A \cup A_1 \cup A_2$ is connected. From Step 1, A is a component of $f^{-1}(B)$, thus $A_1 \cup A_2 \subset A$. Irreducibility of A and $f(A_1) \neq f(A)$ imply that $q \notin A_1$ and $A_1 \cap A_2 = \emptyset$. Let z be a point in $D \cap E$, let $w \in A_1$ be such that $f(w) = z$ and let B_1 be the component of $f^{-1}(E)$ such that $w \in B_1$. Step 1 applied to A and to $A_1 \cup B_1$ implies that $A = A_1 \cup B_1$. This implies that $A_2 \subset B_1$, so $D \subset E$ and $B = E$. This contradiction proves that B is indecomposable.

Let v be a point in B such that y and v are in different composants of B. Choose a point $u \in A$ such that $f(u) = v$. Let β and γ be order arcs in $C(B)$, from $\{y\}$ to B and from $\{v\}$ to B, respectively. The irreducibility of B between y and v implies that $\beta \cap \gamma = \{B\}$. Since $f(p) = f(q) = y$ and $f(u) = v$, the previous lemma implies that there exist order arcs α_1, α_2 and λ such that $f_2(\alpha_1) = \beta = f_2(\alpha_2)$, $f_2(\lambda) = \gamma$, $p \in \bigcap_{D \in \alpha_1} D$, $q \in \bigcap_{D \in \alpha_2} D$ and $u \in \bigcap_{D \in \lambda} D$. Since $\{y\} \in \beta$, there exists $D_0 \in \alpha_1$ such that $f(D_0) = \{y\}$. Then $\bigcap_{D \in \alpha_1} D$ is a subcontinuum of X such that $f(\bigcap_{D \in \alpha_1} D) = \{y\}$. From Step 2, we have $\{p\} = \bigcap_{D \in \alpha_1} D$. Since $B \in \beta$, there exists $D_1 \in \alpha_1$ such that $f(D_1) = B$, which implies that $f(\bigcup_{D \in \alpha_1} D) = B$. From Step 1, we obtain $\bigcup_{D \in \alpha_1} D = A$. Hence α_1 is an order arc from $\{p\}$ to A. Similarly, α_2 is an order arc from $\{q\}$ to A and λ is an order arc from $\{u\}$ to A. The irreducibility of A between p and q implies that $\alpha_1 \cap \alpha_2 = \{A\}$. If

$D \in \alpha_i \cap \lambda$, $f(D)$ is a subcontinuum of B which contains the points y and v, then $f(D) = B$. From Step 1, $D = A$. Thus $\alpha_i \cap \lambda = \{A\}$ for $i = 1, 2$.

Choose elements G_1, G_2 and G_3 in γ such that $\{v\} \subsetneq G_1 \subsetneq G_2 \subsetneq G_3 \subsetneq B$ and elements H_1, H_2 and H_3 in β such that $\{y\} \subsetneq H_1 \subsetneq H_2 \subsetneq H_3 \subsetneq B$. Choose C_1, C_2 and C_3 in λ such that $f(C_i) = G_i$, for each $i = 1, 2, 3$. Then $\{u\} \subsetneq C_1 \subsetneq C_2 \subsetneq C_3 \subsetneq A$. Choose $A_1 \in \alpha_1$ and $A_2 \in \alpha_2$ such that $f(A_1) = H_2 = f(A_2)$. Then $\{p\} \subsetneq A_1 \subsetneq A$ and $\{q\} \subsetneq A_2 \subsetneq A$.

It is easy to verify that there exists $\varepsilon > 0$ such that:

(a) $N^1(2\varepsilon, \langle A_1, A\rangle_{\alpha_1} \cup \langle C_3, A\rangle_\lambda) \cap f_1^{-1}(F_1(B) \cup \langle \{y\}, H_1\rangle_\beta \cup \langle \{v\}, G_2\rangle_\gamma) = \emptyset$;
(b) $N^1(2\varepsilon, F_1(A) \cup \langle \{q\}, A_2\rangle_{\alpha_2} \cup \langle \{u\}, C_1\rangle_\lambda) \cap f_1^{-1}(\langle H_3, B\rangle_\beta \cup \langle G_2, B\rangle_\gamma) = \emptyset$;
(c) $N^1(2\varepsilon, F_1(A) \cup \lambda) \cap f_1^{-1}(\langle H_1, H_3\rangle_\beta) = \emptyset$; and
(d) $N^1(2\varepsilon, \langle \{q\}, A_2\rangle_{\alpha_2}) \cap N^1(2\varepsilon, \langle A_1, A\rangle_{\alpha_1}) = \emptyset$.

Define $\mathcal{A} = F_1(A) \cup \langle \{q\}, A_2\rangle_{\alpha_2} \cup \langle A_1, A\rangle_{\alpha_1} \cup \lambda$, then $\mathcal{A} \in C(C(X))$. Define $\mathcal{B} = f_2(\mathcal{A}) = F_1(B) \cup \beta \cup \gamma$. Since f_2 is open, there exists $\delta > 0$ such that if $\mathcal{C} \in C(C(Y))$ and $H^1(\mathcal{B}, \mathcal{C}) < \delta$, then there exists $\mathcal{D} \in C(C(X))$ such that $H^1(\mathcal{A}, \mathcal{D}) < \varepsilon$ and $f_2(\mathcal{D}) = \mathcal{C}$.

Choose elements E_1 and E_2 in γ such that $G_1 \subsetneq E_1 \subsetneq G_2 \subsetneq E_2 \subsetneq G_3$ and $\operatorname{diam}(\langle E_1, E_2\rangle_\gamma) < \delta$. Define $\mathcal{C} = F_1(B) \cup \beta \cup \langle \{v\}, E_1\rangle_\gamma \cup \langle E_2, B\rangle_\gamma$. Then $\mathcal{C} \in C(C(Y))$ and $H^1(\mathcal{B}, \mathcal{C}) < \delta$, so there exists $\mathcal{D} \in C(C(X))$ such that $H^1(\mathcal{A}, \mathcal{D}) < \varepsilon$ and $f_2(\mathcal{D}) = \mathcal{C}$.

As in the proof of Step 1, the proof of Step 3 will be completed by proving that $\mathcal{D}$ is disconnected.

Define

$$\mathcal{D}_1 = \mathcal{D} \cap \mathrm{Cl}_{C(X)}(N^1(\varepsilon, \langle A_1, A\rangle_{\alpha_1} \cup \langle C_1, A\rangle_\lambda)) \cap f_1^{-1}(\langle H_1, B\rangle_\beta \cup \langle E_2, B\rangle_\gamma)$$

and

$$\mathcal{D}_2 = \mathcal{D} \cap \mathrm{Cl}_{C(X)}(N^1(\varepsilon, F_1(A) \cup \langle \{q\}, A_2\rangle_{\alpha_2} \cup \langle \{u\}, C_3\rangle_\lambda)) \cap f_1^{-1}(F_1(B) \cup \langle \{y\}, H_3\rangle_\beta \cup \langle \{v\}, E_1\rangle_\gamma).$$

Then $\mathcal{D}_1$ and $\mathcal{D}_2$ are closed subsets of $\mathcal{D}$.

If there exists an element $D \in \mathcal{D}_1 \cap \mathcal{D}_2$, then $f_1(D) \in \langle H_1, H_3\rangle_\beta$. From (c), $D \notin N^1(2\varepsilon, F_1(A) \cup \lambda)$. Since $D \in \mathcal{D}_1 \cap \mathcal{D}_2$, we have $D \in N^1(2\varepsilon, \langle \{q\}, A_2\rangle_{\alpha_2}) \cap N^1(2\varepsilon, \langle A_1, A\rangle_{\alpha_1})$, which contradicts (d). Thus $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$.

We prove that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. Let $D \in \mathcal{D}$ and let $E \in \mathcal{A}$ be such that $H(E, D) < \varepsilon$. Then $f_1(D) \in F_1(B) \cup \langle \{y\}, H_1\rangle_\beta \cup \langle \{v\}, E_1\rangle_\gamma$ or $f_1(D) \in \langle H_3, B\rangle_\beta \cup \langle E_2, B\rangle_\gamma$ or $f(D) \in \langle H_1, H_3\rangle_\beta$. In the first case, from (a), $E \in \mathcal{A} - (\langle A_1, A\rangle_{\alpha_1} \cup \langle C_3, A\rangle_\lambda)$. So $E \in F_1(A) \cup \langle \{q\}, A_2\rangle_{\alpha_2} \cup \langle \{u\}, C_3\rangle$. This implies that $D \in \mathcal{D}_2$. In the second case, from (b), $E \in \mathcal{A} - (F_1(A) \cup \langle \{q\}, A_2\rangle_{\alpha_2} \cup \langle \{u\}, C_1\rangle_\lambda)$, so $D \in \mathcal{D}_1$. Finally, in the third case, from (c), $E \in \mathcal{A} - (F_1(A) \cup \lambda)$, so $E \in \langle A_1, A\rangle_{\alpha_1} \cup \langle \{q\}, A_2\rangle_{\alpha_2}$. This implies that $D \in \mathcal{D}_1 \cup \mathcal{D}_2$. Therefore $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$.

Since $A \in \mathcal{A}$, there exists $D_1 \in \mathcal{D}$ such that $H(A, D_1) < \varepsilon$. From (a), $f_1(D_1) \in \mathcal{C} - (F_1(B) \cup \langle \{y\}, H_1 \rangle_\beta \cup \langle \{v\}, G_2 \rangle_\gamma)$. Thus $D_1 \in \mathcal{D}_1$ and $\mathcal{D}_1 \neq \emptyset$. Since $\{u\} \in \mathcal{A}$, there exists $D_2 \in \mathcal{D}$ such that $H(\{u\}, D_2) < \varepsilon$. From (b), $f_1(D_2) \in \mathcal{C} - (\langle H_3, B \rangle_\beta \cup \langle G_2, B \rangle_\gamma)$. Thus $D_2 \in \mathcal{D}_2$ and $\mathcal{D}_2 \neq \emptyset$.

Therefore $\mathcal{D}$ is disconnected. This contradiction proves Step 3 and completes the proof of the theorem.

Corollary. *Let $f : X \to Y$ be an onto map. If Y is nondegenerate then f_2 is open if and only if f is a homeomorphism.*

Example. Let X be the square $[0,1] \times [0,1]$, $Y = [0,1]$ and let $f : X \to Y$ be the natural projection onto the first coordinate. It is easy to check that f is open and f_1 is also open. From the theorem above, f_2 is not open. This example answers Hosokawa's question.

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Instituto de Matemáticas
Circuito Exterior
Cd. Universitaria
México, D.F. 04510, Mexico
E-mail: illanes@gauss.matem.unam.mx

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