HEAT KERNELS AND RIESZ TRANSFORMS
ON NILPOTENT LIE GROUPS

BY

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We consider pure mth order subcoercive operators with complex coefficients acting on a connected nilpotent Lie group. We derive Gaussian bounds with the correct small time singularity and the optimal large time asymptotic behaviour on the heat kernel and all its derivatives, both right and left. Further we prove that the Riesz transforms of all orders are bounded on the L_p-spaces with $p \in (1, \infty)$. Finally, for second-order operators with real coefficients we derive matching Gaussian lower bounds and deduce Harnack inequalities valid for all times.

1. Introduction. The heat kernel plays a significant role in classical harmonic analysis since it encapsulates the most important analytic information. It is consequently crucial in the study of analytic properties of Lie groups to have efficient estimates on the semigroup kernels associated with elliptic or subelliptic operators. There are three distinct characteristics of these estimates: the Gaussian decay on the group, the short time singularity and the long time decay. The first two features are of universal nature and are well understood (see, for example, [Rob], Chapter III, or [EIR6]) but the asymptotic behaviour with time is a more specific feature. If the group volume grows polynomially, the asymptotic decrease of the heat kernel is expected to be dictated by the available volume. In this paper we demonstrate that this expectation is realized for the heat kernels of pure mth order complex subelliptic operators on a general connected nilpotent group. Our estimates, which are valid for the kernel and all its derivatives, then allow us to analyze various aspects which are sensitive to global growth. In particular, we are able to define and analyze the Riesz transforms of all orders.

Let G be a connected nilpotent Lie group with (bi-invariant) Haar measure dg and Lie algebra $\mathfrak{g}$. The exponential map is surjective by [Var], The-

orem 3.6.1. One can associate a subelliptic distance $(g, h) \mapsto d'(g\,;h)$ with each fixed algebraic basis $a_1, \ldots, a_{d'}$ of $\mathfrak{g}$. This distance has the characterization

$$d'(g\,;h) = \sup \left\{ |\psi(g) - \psi(h)| : \psi \in C_{\mathrm{b}}^{\infty}(G),\ \sum_{i=1}^{d'} |(A_i\psi)|^2 \leq 1,\ \psi\ \text{real} \right\}$$

where we emphasize that the ψ are real-valued ([Rob], Lemma IV.2.3, or [EIR5], Lemma 4.2). Let $g \mapsto |g|' = d'(g\,;e)$, where e is the identity element of G, denote the corresponding modulus. Then the Haar measure $|B'(g\,;\varrho)|$ of the subelliptic ball $B'(g\,;\varrho) = \{h \in G : |gh^{-1}|' < \varrho\}$ is independent of g. Set $V(\varrho) = |B'(g\,;\varrho)|$. Next, for all $i \in \{1, \ldots, d'\}$ let $A_i = dL(a_i)$ and $B_i = dR(a_i)$ denote the generators of left L, and right R, translations acting on the classical function spaces in the direction a_i, respectively. Multiple derivatives are denoted with multi-index notation, e.g., if $\alpha = (i_1, \ldots, i_n)$ with $i_j \in \{1, \ldots, d'\}$ then $A^\alpha = A_{i_1} \ldots A_{i_n}$ and $|\alpha| = n$. (In general, we adopt the notation of [Rob].)

We consider right-invariant subelliptic operators of all orders. Since the notion of subellipticity for operators of order greater than two is slightly indirect we initially summarize our main results for the second-order case. The general case is covered in the body of the paper. Consider the homogeneous second-order operators

$$(1) \qquad\qquad H = -\sum_{i,j=1}^{d'} c_{ij} A_i A_j$$

acting on the L_p-spaces, $L_p(G\,;dg)$, with $c_{ij} \in \mathbb{C}$ satisfying the ellipticity condition

$$(2) \qquad\qquad \mathrm{Re} \sum_{i,j=1}^{d'} c_{ij} \overline{\xi}_i \xi_j \geq \mu |\xi|^2$$

for some $\mu > 0$ and all $\xi \in \mathbb{C}^{d'}$. Then H is closed on $L_2(G\,;dg)$ and generates a holomorphic contraction semigroup S with a C^∞-kernel K with Gaussian decay (see [EIR4]). In particular, S extends to a continuous semigroup, which we also denote by S, on each of the L_p-spaces, $L_p(G\,;dg)$, or on related Banach spaces such as $C_{\mathrm{b}}(G)$.

The first result of this paper is the following set of optimal kernel bounds.

THEOREM 1.1. *There exists a $b > 0$ and for all multi-indices α, β an $a_{\alpha,\beta} > 0$ such that*

$$|(A^\alpha B^\beta K_t)(g)| \leq a_{\alpha,\beta} t^{-(|\alpha|+|\beta|)/2} V(t^{1/2})^{-1} e^{-b(|g|')^2 t^{-1}}$$

for all $t > 0$ and all $g \in G$.

Although we have only stated this result for second-order operators, an analogous statement is derived in Section 3 for homogeneous operators of higher order. Moreover, the kernel bounds for real t readily extend to complex t, in a suitable sector, by rotation. There is a $\theta \in (0, \pi/2)$ such that $e^{i\phi}H$ is a subelliptic operator of the type under consideration for all $\phi \in (0, \theta)$. Then the holomorphy sector of the semigroup S automatically contains the sector $\Lambda(\theta) = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$ and the kernel bounds extend to $z \mapsto K_z$ for $z \in \Lambda(\phi)$, with t replaced by $|z|$, for all $\phi < \theta$.

Various special cases of this theorem are already known. If the coefficients c_{ij} of H are real-valued and the matrix $C = (c_{ij})$ is symmetric then bounds of the type stated in the theorem are known for K_t and its left derivatives $A_i K_t$ for all groups of polynomial growth [Sal] (see also [Rob], Corollary IV.4.19). In addition, in the real case, one can obtain good estimates on the exponent b of the Gaussian. But the techniques used to obtain the large time estimates do not extend to complex coefficients, or to more general derivatives of the kernel. Alternatively, if the group is stratified, and the operator H is homogeneous with respect to the dilations on the group, then bounds analogous to those of the theorem, but slightly weaker, have been given in [EIR3]. The derivation of these bounds relies heavily on the dilation properties of the group. The current results are much stronger as they are valid for all connected nilpotent Lie groups and no dilation structure is necessary. They are derived by transference from a related homogeneous group $\widetilde{G}$ with d' generators and the same rank as G. This group is defined in Section 2 together with a version of the transference result of [CoW2] adapted to the current situation.

The transference procedure which we use differs conceptually from the method developed in [CoW2]. The latter reference examines two different representations of a fixed group and transfers estimates on integral operators from one representation to the other. In our analysis we examine two different groups but one fixed representation, the left regular representation, and transfer estimates from one group to the other. More significantly, the standard transference procedures are restricted to L_p-estimates but we develop a technique to transfer pointwise estimates.

The transfer of L_p-estimates is, however, relevant to the discussion of Riesz transforms. The natural Lie group analogues of the classical Riesz transforms are the operators $R_\alpha = A^\alpha H^{-|\alpha|/2}$ but it is initially unclear whether these operators have a useful domain of definition on the L_p-spaces. The transforms are products of unbounded operators and viewed as integral operators they are highly singular. These problems are discussed in detail in Section 4 where we prove a precise version of the following statement.

THEOREM 1.2. *The Riesz transforms R_α extend to bounded operators on each of the spaces $L_p(G\,;dg)$, $p \in (1,\infty)$.*

In fact, we bound the norms of the Riesz transforms by multiples of the norms of the analogous transforms on the auxiliary homogeneous group. In addition we deduce that the R_α are of weak type $(1,1)$.

The simplest Riesz transforms $A_i H^{-1/2}$, with $H = -\sum_{i=1}^{d'} A_i^2$ the sublaplacian, have been shown to be bounded on $L_p(G\,;dg)$, $p \in (1,\infty)$, for various types of Lie groups. The result was established by Folland [Fol] for stratified groups, by Lohoué and Varopoulos [LoV] for nilpotent groups and by Alexopoulos [Ale] for groups of polynomial growth (see also [Sal]). But the properties of the higher-order transforms are less well understood. If the group is stratified then these transforms are bounded [Fol], or if the group is compact [BER]. But Alexopoulos [Ale] has given an example of a group of polynomial growth for which $A_i^2 H^{-1}$ can be unbounded (see also [GQS] for an example with a group of exponential growth). In the light of such examples Theorem 1.2 is possibly unexpected. In fact, we prove boundedness of all operators $A^{\alpha_1} H^{-n_1/2} A^{\alpha_2} H^{-n_2/2} \dots A^{\alpha_k} H^{-n_k/2}$ with $|\alpha_1| + \dots + |\alpha_k| = n_1 + \dots + n_k$ and the analogues for higher-order operators.

Finally, the kernel estimates can be applied in various ways. We discuss applications to Lipschitz spaces and holomorphic functional calculus in Section 5. Moreover, for second-order operators with real coefficients we apply our techniques to the derivation of Gaussian lower bounds and Harnack inequalities valid for all $t > 0$.

2. Free groups and transference. In this section we examine convolution operators acting on the L_p-spaces over the connected nilpotent Lie group G formed with respect to the Haar measure dg. We estimate bounds on these operators by transference from a homogeneous group $\widetilde{G}$ constructed from G and an algebraic basis $a_1, \dots, a_{d'}$ of the Lie algebra $\mathfrak{g}$ of G.

Let r be the rank of the nilpotent Lie algebra $\mathfrak{g}$. Then the rank of the basis $a_1, \dots, a_{d'}$ is at most r. Next let $\mathfrak{g}(d', r)$ denote the nilpotent Lie algebra with d' generators which is free of step r. Thus $\mathfrak{g}(d', r)$ is the quotient of the free Lie algebra with d' generators by the ideal generated by the commutators of order at least $r + 1$. Further let $G(d', r)$ be the connected simply connected Lie group with Lie algebra $\mathfrak{g}(d', r)$. It is automatically a non-compact group. We call $G(d', r)$ the nilpotent Lie group on d' generators free of step r and use the notation $\widetilde{\mathfrak{g}} = \mathfrak{g}(d', r)$, $\widetilde{G} = G(d', r)$ for brevity. Generally, we add a tilde to distinguish between quantities associated with $\widetilde{G}$ and those associated with G. For example, we denote the generators of $\widetilde{\mathfrak{g}}$ by $\widetilde{a}_1, \dots, \widetilde{a}_{d'}$. We also set $L_p = L_p(G\,;dg)$ and $L_{\widetilde{p}} = L_p(\widetilde{G}\,;d\widetilde{g})$ and denote the corresponding

norms by $\|\cdot\|_p$ and $\|\cdot\|_{\widetilde{p}}$. Then the norm of an operator X on L_p is denoted by $\|X\|_{p\to p}$ and the norm of an operator $\widetilde{X}$ on $L_{\widetilde{p}}$ by $\|\widetilde{X}\|_{\widetilde{p}\to\widetilde{p}}$. One simple example of this construction is for the Abelian nilpotent group $G = \mathbb{T}^n$. Then $\widetilde{G} = \mathbb{R}^n$.

Now we compare G and $\widetilde{G}$. There exists a unique Lie algebra homomorphism $\Lambda : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ such that $\Lambda(\widetilde{a}_i) = a_i$ for all $i \in \{1, \dots, d'\}$ and this lifts to a surjective homomorphism $\pi : \widetilde{G} \to G$ by the exponential map. Explicitly,

$$\pi = \exp \circ \Lambda \circ \widetilde{\exp}^{-1}$$

where $\widetilde{\exp} : \widetilde{\mathfrak{g}} \to \widetilde{G}$ and $\exp : \mathfrak{g} \to G$. For any function $\varphi : G \to \mathbb{C}$ define $\pi^*\varphi : \widetilde{G} \to \mathbb{C}$ by $\pi^*\varphi = \varphi \circ \pi$. The map π^* is contractive,

$$\|\pi^*\varphi\|_{\widetilde{\infty}} = \sup_{\widetilde{g}\in\widetilde{G}} |\varphi(\pi(\widetilde{g}))| \leq \sup_{g\in G} |\varphi(g)| = \|\varphi\|_\infty$$

for all $\varphi \in L_\infty$. Next, for any finite measure $\widetilde{\mu}$ on $\widetilde{G}$ let $\pi_*(\widetilde{\mu})$ denote the image measure on G. Then

$$\int\limits_G d\pi_*(\widetilde{\mu})(g)\,\varphi(g) = \int\limits_{\widetilde{G}} d\widetilde{\mu}(\widetilde{g})\,(\pi^*\varphi)(\widetilde{g})$$

for all $\varphi \in L_1(G\,;\pi_*(\widetilde{\mu}))$. Note that the image measure is again a finite measure. If $\widetilde{\mu}$ is a complex measure on $\widetilde{G}$ then we also use the notation $\pi_*(\widetilde{\mu})$ to denote the complex image measure on G. If $M(G)$ and $M(\widetilde{G})$ denote the Banach spaces of all complex measures on G and $\widetilde{G}$, respectively, then

$$\|\pi_*(\widetilde{\mu})\|_{M(G)} = \sup_{\varphi\in C_c(G),\,\|\varphi\|_\infty\leq 1} \left| \int\limits_G d\pi_*(\widetilde{\mu})(g)\,\varphi(g) \right|$$

$$\leq \sup_{\varphi\in C_c(G),\,\|\varphi\|_\infty\leq 1} \int\limits_{\widetilde{G}} d|\widetilde{\mu}|(\widetilde{g})\,|(\pi^*\varphi)(\widetilde{g})| \leq \|\widetilde{\mu}\|_{M(\widetilde{G})}.$$

Thus the map $\pi_* : M(\widetilde{G}) \to M(G)$ is also contractive.

The space $L_{\widetilde{1}}$ is naturally isomorphic to the space of all absolutely continuous measures on $\widetilde{G}$. So for each $\widetilde{\psi} \in L_{\widetilde{1}}$ there exists a complex measure $\pi_*(\widetilde{\psi}) \in M(G)$ such that

$$\int\limits_G d\pi_*(\widetilde{\psi})(g)\,\varphi(g) = \int\limits_{\widetilde{G}} d\widetilde{g}\,\widetilde{\psi}(\widetilde{g})\,(\pi^*\varphi)(\widetilde{g})$$

for all $\varphi \in L_\infty(G)$. Then $\|\pi_*(\widetilde{\psi})\|_{M(G)} \leq \|\widetilde{\psi}\|_{\widetilde{1}}$.

Standard results involve transference of norm bounds on convolution operators from one representation of a group to another. Define the isometric representation L_π of $\widetilde{G}$ on L_p, with $p \in [1, \infty]$, by

$$L_\pi(\widetilde{g})\varphi = L(\pi(\widetilde{g}))\varphi.$$

Let q be the dual exponent to p. Then for all $\widetilde{\psi} \in L_{\widetilde{1}}$, $\varphi \in L_p$ and $\tau \in L_q$ one has

$$(\tau, L_\pi(\widetilde{\psi})\varphi) = \int\limits_{\widetilde{G}} d\widetilde{g}\, \widetilde{\psi}(\widetilde{g})(\tau, L_\pi(\widetilde{g})\varphi) = \int\limits_{\widetilde{G}} d\widetilde{g}\, \widetilde{\psi}(\widetilde{g})(\tau, L(\pi(\widetilde{g}))\varphi)$$

$$= \int\limits_{G} d\pi_*(\widetilde{\psi})(g)\,(\tau, L(g)\varphi) = (\tau, L(\pi_*(\widetilde{\psi}))\varphi).$$

So $L_\pi(\widetilde{\psi}) = L(\pi_*(\widetilde{\psi}))$ as operators on L_p.

The principal transference theorem of [CoW2], when applied to the representation L_π of $\widetilde{G}$ on L_p and the left regular representation $\widetilde{L}$ of $\widetilde{G}$ on $L_{\widetilde{p}}$, gives a relationship between the norms of operators $L(\pi_*(\widetilde{\psi}))$ on $L_p(G\,;dg)$ and $\widetilde{L}(\widetilde{\psi})$ on $L_p(\widetilde{G}\,;d\widetilde{g})$. This result is significant because, in the context of subelliptic semigroups, $\pi_*(\widetilde{\psi})$ and $\widetilde{\psi}$ correspond to the kernel of the semigroup generated by the operator with the same coefficients but on G and $\widetilde{G}$, respectively (see Lemma 3.2 below).

THEOREM 2.1. *If* $\widetilde{\psi} \in L_1(\widetilde{G}\,;d\widetilde{g})$ *then*

$$\|L(\pi_*(\widetilde{\psi}))\|_{p\to p} = \|L_\pi(\widetilde{\psi})\|_{p\to p} \leq \|\widetilde{L}(\widetilde{\psi})\|_{\widetilde{p}\to\widetilde{p}}$$

for all $p \in [1,\infty]$.

Proof. The group $\widetilde{G}$ has polynomial growth, hence is amenable by [Pat], Proposition 0.13. Therefore if $p < \infty$ then the theorem is precisely Theorem 2.4 of [CoW2] with $L(\pi_*(\widetilde{\psi}))$ replaced by $L_\pi(\widetilde{\psi})$.

Finally, on L_∞ one has

$$\|L(\pi_*\widetilde{\psi})\varphi\|_\infty = \|L_\pi(\widetilde{\psi})\varphi\|_\infty \leq \|\pi_*\widetilde{\psi}\|_{M(G)}\|\varphi\|_\infty$$

$$\leq \|\widetilde{\psi}\|_{\widetilde{1}}\|\varphi\|_\infty = \|\widetilde{L}(\widetilde{\psi})\|_{\widetilde{\infty}\to\widetilde{\infty}}\|\varphi\|_\infty$$

for all $\varphi \in L_\infty$. This establishes the L_∞-estimate of the theorem. ∎

This theorem will be applied in Section 4 to transfer knowledge about Riesz transforms on $\widetilde{G}$ to knowledge about the comparable transforms on G acting on the L_p-spaces with $p \in (1,\infty)$. There is a similar transference result, based on Theorem 2.6 of [CoW2], which deals with estimates of weak type $(1,1)$. But this will not play a role in the sequel.

3. Heat kernels of subcoercive operators. In the introduction we discussed second-order subelliptic operators but our results are valid for subelliptic operators of all orders. We begin this section by describing the notion of subcoercivity, or subellipticity, for the higher-order situation.

Let m be an even positive integer and for every multi-index α with $|\alpha| = m$ let $c_\alpha \in \mathbb{C}$. We consider the homogeneous mth order operator

$$H = \sum_{|\alpha|=m} c_\alpha A^\alpha$$

on L_p with domain $D(H) = L'_{p;m}$ and compare it with its analogue

$$\widetilde{H} = \sum_{|\alpha|=m} c_\alpha \widetilde{A}^\alpha$$

on $L_{\widetilde{p}}$. We assume the operator H is subcoercive of step r, where r is the rank of the nilpotent Lie algebra $\mathfrak{g}$, in the sense of [E1R3]. This means that the comparison operator $\widetilde{H}$ satisfies a Gårding inequality on $L_{\widetilde{2}}$, i.e., there exists $\mu > 0$ such that

$$\mathrm{Re}(\widetilde{\varphi}, \widetilde{H}\widetilde{\varphi}) \geq \mu \sum_{|\alpha|=m/2} \|\widetilde{A}^\alpha \widetilde{\varphi}\|_{\widetilde{2}}$$

uniformly for all $\widetilde{\varphi} \in C_c^\infty(\widetilde{G})$. For second-order operators there is a simpler description, at least for $r \geq 2$.

PROPOSITION 3.1. *Define the $d' \times d'$ matrix C by $C_{ij} = -c_{(i,j)}$. If $r \geq 2$ then the second-order operator H is subcoercive of step r if, and only if, the matrix $\Re C = 2^{-1}(C + C^*)$ is strictly positive, i.e.,*

$$\mathrm{Re} \sum_{i,j=1}^{d'} C_{ij}\overline{\xi}_i\xi_j \geq \mu|\xi|^2$$

for some $\mu > 0$ and all $\xi \in \mathbb{C}^{d'}$.

P r o o f. See [EIR4], Proposition 3.7. ∎

If $r = 1$ then the "only if" part of Proposition 3.1 fails, but the "if" part is still valid ([EIR4], Corollary 3.6). So for any $r \in \mathbb{N}$, operators of the form (1) which satisfy (2) are subcoercive of step r.

The main result of [EIR3] and [EIR4] is that H generates a holomorphic semigroup S_t on the L_p-spaces with a C^∞-kernel K_t satisfying Gaussian bounds and $\widetilde{H}$ generates a similar semigroup $\widetilde{S}_t$ on the $L_{\widetilde{p}}$-spaces with a kernel $\widetilde{K}_t$.

Now if $\varphi \in C_b(G)$ then $\pi^*\varphi = \varphi \circ \pi \in C_b(\widetilde{G})$ and

$$\widetilde{A}_i \pi^*\varphi = \pi^*(A_i\varphi).$$

Consequently,

$$\widetilde{H}\pi^*\varphi = \pi^*(H\varphi).$$

Therefore

$$(\lambda I + \widetilde{H})\pi^*\varphi = \pi^*((\lambda I + H)\varphi) \quad \text{and} \quad (\lambda I + \widetilde{H})^{-1}\pi^*\varphi = \pi^*((\lambda I + H)^{-1}\varphi)$$

for all large $\lambda > 0$. Hence, by the usual semigroup algorithms,

$$\widetilde{S}_t \pi^* \varphi = \pi^* (S_t \varphi)$$

for all $t > 0$. This allows one to relate the kernels of the two semigroups.

LEMMA 3.2. *The identities*

$$\int_G dg\, \varphi(g) (A^\alpha B^\beta K_t)(g) = \int_{\widetilde{G}} d\widetilde{g}\, (\pi^* \varphi)(\widetilde{g}) (\widetilde{A}^\alpha \widetilde{B}^\beta \widetilde{K}_t)(\widetilde{g})$$

are valid for all $\varphi \in C_b(G)$, all $t > 0$ and all multi-indices α, β. Hence

$$A^\alpha B^\beta K_t = \pi_* (\widetilde{A}^\alpha \widetilde{B}^\beta \widetilde{K}_t)$$

for all $t > 0$ and all multi-indices α, β, where we identify $L_1(G\,; dg)$ with the space of all absolutely continuous measures on G.

P r o o f. Consider the case $|\alpha| = 0 = |\beta|$. Introduce $\check{\varphi}$ by setting $\check{\varphi}(g) = \varphi(g^{-1})$. Then

$$(\widetilde{S}_t \pi^* \check{\varphi})(\widetilde{e}) = (\widetilde{S}_t (\pi^* \varphi)\check{\,})(\widetilde{e}) = \int_{\widetilde{G}} dg\, (\pi^* \varphi)(g) \widetilde{K}_t(g)$$

because π is a homomorphism. But

$$(\widetilde{S}_t \pi^* \check{\varphi})(\widetilde{e}) = (\pi^*(S_t \check{\varphi}))(\widetilde{e}) = (S_t \check{\varphi})(e) = \int_G dg\, \varphi(g) K_t(g)$$

since $\pi(\widetilde{e}) = e$. Combining these equations gives the required identities for $|\alpha| = 0 = |\beta|$. The general case follows similarly. ∎

The identification $K_t = \pi_*(\widetilde{K}_t)$ is the key to obtaining good Gaussian bounds on the kernel K_t by transference. But first one needs optimal bounds on the kernel $\widetilde{K}_t$. These can be obtained by exploiting the scaling properties of the group $\widetilde{G}$ and the homogeneity of $\widetilde{H}$ (see [EIR3]).

The group $\widetilde{G}$ is homogeneous with respect to a semigroup of dilations $(\gamma_u)_{u>0}$. These dilations are initially defined as the Lie algebra isomorphism on $\widetilde{\mathfrak{g}}$ satisfying $\gamma_u(\widetilde{a}_i) = u \widetilde{a}_i$. The dilations of $\widetilde{\mathfrak{g}}$ then induce dilations of $\widetilde{G}$ via the exponential map. It follows automatically that $|\gamma_u(\widetilde{g})|' = u|\widetilde{g}|'$ for all $\widetilde{g} \in \widetilde{G}$ where $|\cdot|'$ now denotes the subelliptic distance on $\widetilde{G}$ associated with the generators $\widetilde{a}_1, \ldots, \widetilde{a}_{d'}$. Moreover, there is an integer $\widetilde{D}$, called the *homogeneous dimension* of $\widetilde{G}$ with respect to the basis $\widetilde{a}_1, \ldots, \widetilde{a}_{d'}$, such that

$$\widetilde{V}(\varrho) = |\widetilde{B}'(\widetilde{g}\,; \varrho)| = \varrho^{\widetilde{D}} |\widetilde{B}'(\widetilde{g}\,; 1)| = \varrho^{\widetilde{D}} \widetilde{V}(1)$$

for all $\varrho \in (0, \infty)$. Since the subcoercive operator $\widetilde{H}$ is a pure mth order operator it follows that one has the scaling property

$$(\widetilde{H}(\varphi \circ \gamma_u)) \circ \gamma_{u^{-1}} = u^m (\widetilde{H} \varphi)$$

for all $\varphi \in D(\widetilde{H})$. Therefore the associated kernel satisfies

$$u^{-\widetilde{D}}\widetilde{K}_t(\gamma_{u^{-1}}(\widetilde{g})) = \widetilde{K}_{u^m t}(\widetilde{g})$$

for all $t, u > 0$ and all $\widetilde{g} \in \widetilde{G}$. More generally,

$$u^{-\widetilde{D}-|\alpha|-|\beta|}(\widetilde{A}^\alpha \widetilde{B}^\beta \widetilde{K}_t)(\gamma_{u^{-1}}(\widetilde{g})) = (\widetilde{A}^\alpha \widetilde{B}^\beta \widetilde{K}_{u^m t})(\widetilde{g}).$$

These relations allow one to deduce large t, or small t, bounds on the kernel $\widetilde{K}$ and its derivatives from bounds at $t = 1$.

The utility of the scaling relations in combination with the kernel identification of Lemma 3.2 is illustrated by the following pair of estimates. First, one has

$$(3) \qquad \|S_t\|_{p \to p} = \|L(\pi_*(\widetilde{K}_t))\|_{p \to p} \leq \|\pi_*(\widetilde{K}_t)\|_1 \leq \|\widetilde{K}_t\|_{\widetilde{1}} = \|\widetilde{K}_1\|_{\widetilde{1}}$$

where the last identification uses the scaling property of $\widetilde{G}$. These bounds are uniform for $p \in [1, \infty]$ and $t > 0$. Since the operator H has complex coefficients, this simple proof of uniformity is somewhat surprising. (Note that $\|\widetilde{K}_t\|_{\widetilde{1}} \geq 1$ with equality if, and only if, $\widetilde{K}$ is positive or, equivalently, H is a second-order operator and the coefficients of H are real. The first equivalence follows because $\widetilde{S}_t I = I$ and hence $\int \widetilde{K}_t = 1$. The second equivalence is established in [Rob], Section III.5.) Secondly, one has the holomorphy estimates

$$(4) \qquad \|HS_t\|_{p \to p} = \|L(\pi_*(\partial \widetilde{K}_t))\|_{p \to p}$$
$$\leq \|\pi_*(\partial \widetilde{K}_t)\|_1 \leq \|\partial \widetilde{K}_t\|_{\widetilde{1}} = t^{-1}\|\partial \widetilde{K}_1\|_{\widetilde{1}}$$

where ∂ denotes the partial derivative with respect to t. Again these bounds are uniform for $p \in [1, \infty]$ and are valid for all $t > 0$.

Now we return to the discussion of pointwise bounds.

LEMMA 3.3. *There exist $a, b > 0$ such that*

$$|\widetilde{K}_t(\widetilde{g})| \leq at^{-\widetilde{D}/m}e^{-b((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}$$

for all $t > 0$ and all $\widetilde{g} \in \widetilde{G}$. Moreover, for each $\varepsilon > 0$ and all multi-indices α, β there exists an $a_{\alpha,\beta} > 0$ such that

$$|(\widetilde{A}^\alpha \widetilde{B}^\beta \widetilde{K}_t)(\widetilde{g})| \leq a_{\alpha,\beta} t^{-(\widetilde{D}+|\alpha|+|\beta|)/m}e^{-(b-\varepsilon)((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}$$

uniformly for all $t > 0$ and all $\widetilde{g} \in \widetilde{G}$.

P r o o f. The bounds on $\widetilde{K}$ are given in [EIR3], Corollary 4.10. This reference also gives bounds on the left derivatives of the kernel but with the value of the Gaussian exponent dependent on α. The proof is based on the identification of the $\widetilde{A}^\alpha \widetilde{K}_t$ with the kernels of the bounded operators $\widetilde{A}^\alpha \widetilde{S}_t$. But one calculates straightforwardly that $\widetilde{B}^\beta \widetilde{K}_t$ is the kernel of the

bounded closure of the densely defined operator $(-1)^{|\beta|}\widetilde{S}_t\widetilde{A}^{\beta_*}$, where β_* is the multi-index obtained from β by reversing its order. Specifically,

$$
\begin{aligned}
(\widetilde{S}_t\widetilde{L}(\exp(-t\widetilde{a}))\widetilde{\varphi})(\widetilde{g}) &= \int\limits_{\widetilde{G}} d\widetilde{h}\, \widetilde{K}_t(\widetilde{h})\widetilde{\varphi}(\exp(t\widetilde{a})\widetilde{h}^{-1}\widetilde{g}) \\
&= \int\limits_{\widetilde{G}} d\widetilde{h}\, \widetilde{K}_t(\widetilde{h}\exp(-t\widetilde{a}))\widetilde{\varphi}(\widetilde{h}^{-1}\widetilde{g}).
\end{aligned}
$$

Therefore $\widetilde{A}^\alpha\widetilde{B}^\beta\widetilde{K}_t$ is the kernel of $(-1)^{|\beta|}\widetilde{A}^\alpha\widetilde{S}_t\widetilde{A}^{\beta_*}$ and the bounds on the mixed derivatives follow from the bounds on the left derivatives derived in [EIR3] by duality. It remains, however, to prove that one may choose the Gaussian exponent independent of α and β and arbitrarily close to b.

Let $L_{\widetilde{p}}^\varrho = L_p(\widetilde{G}; e^{\varrho|\widetilde{g}|'}d\widetilde{g})$ with norm $\|\cdot\|_{\widetilde{p}}^\varrho$ for all $\varrho \in \mathbb{R}$. Then for each $\varepsilon \in (0, 2^{-1})$ one has

$$
\begin{aligned}
e^{-\varrho|\widetilde{g}|'}|(\widetilde{A}^\alpha\widetilde{B}^\beta\widetilde{K}_t)(\widetilde{g})| &= \sup_{\|\varphi\|_{\widetilde{1}}^\varrho \leq 1} |(\widetilde{A}^\alpha\widetilde{S}_t\widetilde{A}^{\beta_*}\varphi)(e)| \leq \|\widetilde{A}^\alpha\widetilde{S}_t\widetilde{A}^{\beta_*}\|_{\widetilde{1}\to\widetilde{\infty}}^\varrho \\
&\leq \|\widetilde{S}_{\varepsilon t}\widetilde{A}^{\beta_*}\|_{\widetilde{1}\to\widetilde{2}}^\varrho \|\widetilde{S}_{(1-2\varepsilon)t}\|_{\widetilde{2}\to\widetilde{2}}^\varrho \|\widetilde{A}^\alpha\widetilde{S}_{\varepsilon t}\|_{\widetilde{2}\to\widetilde{\infty}}^\varrho \\
&= \|\widetilde{A}^\beta\widetilde{S}_{\varepsilon t}^*\|_{\widetilde{2}\to\widetilde{\infty}}^{-\varrho} \|\widetilde{S}_{(1-2\varepsilon)t}\|_{\widetilde{2}\to\widetilde{2}}^\varrho \|\widetilde{A}^\alpha\widetilde{S}_{\varepsilon t}\|_{\widetilde{2}\to\widetilde{\infty}}^\varrho
\end{aligned}
$$

where the crossnorms are now between the weighted spaces. But it follows from the Gaussian bounds on the left derivatives of $\widetilde{K}$ that

$$
\begin{aligned}
\|\widetilde{A}^\alpha\widetilde{S}_t\|_{\widetilde{2}\to\widetilde{\infty}}^\varrho &\leq \sup_{\widetilde{h}\in\widetilde{G}} \left(\int\limits_{\widetilde{G}} d\widetilde{g}\, (|(\widetilde{A}^\alpha\widetilde{K}_t)(\widetilde{h}\widetilde{g}^{-1})|e^{|\varrho|\cdot|\widetilde{h}\widetilde{g}^{-1}|'})^2 \right)^{1/2} \\
&\leq a_\alpha t^{-\widetilde{D}/4}t^{-|\alpha|/m}e^{\omega_\alpha \varrho^m t}
\end{aligned}
$$

by a quadrature estimate. Similarly,

$$
\|\widetilde{A}^\beta\widetilde{S}_t^*\|_{\widetilde{2}\to\widetilde{\infty}}^{-\varrho} \leq a_\beta' t^{-\widetilde{D}/4}t^{-|\beta|/m}e^{\omega_\beta' \varrho^m t}.
$$

Moreover, $\|\widetilde{L}(\widetilde{g})\|_{\widetilde{2}\to\widetilde{2}}^\varrho \leq e^{|\varrho|\cdot|\widetilde{g}|'}$. Therefore

$$
\begin{aligned}
\|\widetilde{S}_t\|_{\widetilde{2}\to\widetilde{2}}^\varrho \leq \|\widetilde{K}_t\|_{\widetilde{1}}^{|\varrho|} &\leq \int\limits_{\widetilde{G}} d\widetilde{g}\, a t^{-\widetilde{D}/m}e^{-b((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}e^{|\varrho|\cdot|\widetilde{g}|'} \\
&\leq \sup_{\widetilde{h}\in\widetilde{G}} e^{-(b-\varepsilon)((|\widetilde{h}|')^m t^{-1})^{1/(m-1)}} \\
&\quad \times e^{|\varrho|\cdot|\widetilde{h}|'} \int\limits_{\widetilde{G}} d\widetilde{g}\, a t^{-\widetilde{D}/m}e^{-\varepsilon((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}} \leq a_\varepsilon' e^{\omega_{b-\varepsilon}\varrho^m t},
\end{aligned}
$$

where $\omega_{b-\varepsilon} = (b-\varepsilon)^{-(m-1)}(m-1)^{m-1}m^{-m}$. Combination of these estimates gives

$$
e^{-\varrho|\widetilde{g}|'}|(\widetilde{A}^\alpha\widetilde{B}^\beta\widetilde{K}_t)(\widetilde{g})| \leq a_{\alpha,\beta,\varepsilon}t^{-(\widetilde{D}+|\alpha|+|\beta|)/m}e^{((1-2\varepsilon)\omega_{b-\varepsilon}+\varepsilon(\omega_\alpha+\omega_\beta'))\varrho^m t}.
$$

Hence for all sufficiently small $\varepsilon > 0$ one obtains bounds

$$e^{-\varrho|\widetilde{g}|'}|(\widetilde{A}^\alpha \widetilde{B}^\beta \widetilde{K}_t)(\widetilde{g})| \le a'_{\alpha,\beta,\varepsilon} t^{-(\widetilde{D}+|\alpha|+|\beta|)/m} e^{\omega_{b-\varepsilon} \varrho^m t}.$$

Minimizing over ϱ then gives

$$|(\widetilde{A}^\alpha \widetilde{B}^\beta \widetilde{K}_t)(\widetilde{g})| \le a''_{\alpha,\beta,\varepsilon} t^{-(\widetilde{D}+|\alpha|+|\beta|)/m} e^{-(b-\varepsilon)((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}$$

as desired. ∎

Lemma 3.2 provides the mechanism for transferring the Gaussian bounds of Lemma 3.3 from the semigroup kernel $\widetilde{K}$ to the kernel K and the next lemma provides the transference channel.

LEMMA 3.4. *Let G be a group with polynomial growth, $a_1, \ldots, a_{d'}$ an algebraic basis of the Lie algebra of G and K^Δ the kernel associated with the sublaplacian $\Delta = -\sum_{i=1}^{d'} A_i^2$. Then there exist $a, b, a', b' > 0$ such that*

$$a'V(t^{1/2})^{-1} e^{-b'(|g|')^2 t^{-1}} \le K_t^\Delta(g) \le aV(t^{1/2})^{-1} e^{-b(|g|')^2 t^{-1}}$$

for all $t > 0$ and all $g \in G$.

P r o o f. This result can be pieced together from [Rob], Theorem IV.4.16, Proposition IV.4.19 and Proposition IV.4.21. ∎

Note that since each nilpotent group has polynomial growth, the lemma applies to both K^Δ on G and the corresponding kernel $\widetilde{K}^{\widetilde{\Delta}}$ on the homogeneous group $\widetilde{G}$.

At this point we can readily derive the estimates for the kernel K stated in the introduction for second-order operators. Let $\varphi \in C_{\mathrm{b}}(G)$ be positive. Then $\pi^*\varphi \ge 0$ and

$$\left| \int_G dg\, \varphi(g)(A^\alpha B^\beta K_t)(g) \right| \le \int_{\widetilde{G}} d\widetilde{g}\, (\pi^*\varphi)(\widetilde{g})|(\widetilde{A}^\alpha \widetilde{B}^\beta \widetilde{K}_t)(\widetilde{g})|$$

$$\le a_{\alpha,\beta} \int_{\widetilde{G}} d\widetilde{g}\, (\pi^*\varphi)(\widetilde{g}) t^{-(\widetilde{D}+|\alpha|+|\beta|)/2} e^{-b(|\widetilde{g}|')^2 t^{-1}}$$

$$\le a'_{\alpha,\beta} t^{-(|\alpha|+|\beta|)/2} \int_{\widetilde{G}} d\widetilde{g}\, (\pi^*\varphi)(\widetilde{g}) \widetilde{K}^{\widetilde{\Delta}}_{\omega t}(\widetilde{g})$$

$$= a'_{\alpha,\beta} t^{-(|\alpha|+|\beta|)/2} \int_G dg\, \varphi(g) K^\Delta_{\omega t}(g),$$

by application of Lemmas 3.2–3.4 to both the kernels K_t^Δ and $\widetilde{K}_t^{\widetilde{\Delta}}$. It then follows by the Lebesgue theorem that

$$|(A^\alpha B^\beta K_t)(g)| \le a'_{\alpha,\beta} t^{-(|\alpha|+|\beta|)/2} K^\Delta_{\omega t}(g)$$

for all $t > 0$ and $g \in G$. Since K_t^Δ satisfies Gaussian bounds with the correct

asymptotic behaviour, by Lemma 3.4 applied to G and Δ, one obtains the desired bounds.

The comparable result for higher-order operators is derived by similar reasoning supplemented by some more detailed Gaussian estimates. It is a consequence of the next theorem and Lemma 3.3.

THEOREM 3.5. *Let* $a, b > 0$ *be such that*

$$|\widetilde{K}_t(\widetilde{g})| \leq a t^{-\widetilde{D}/m} e^{-b((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}$$

for all $t > 0$ *and all* $\widetilde{g} \in \widetilde{G}$. *Then for all* $\varepsilon > 0$ *and all multi-indices* α, β *there exists an* $a_{\alpha,\beta} > 0$ *such that*

$$|(A^\alpha B^\beta K_t)(g)| \leq a_{\alpha,\beta} t^{-(|\alpha|+|\beta|)/m} V(t^{1/m})^{-1} e^{-(b-\varepsilon)((|g|')^m t^{-1})^{1/(m-1)}}$$

for all $t > 0$ *and all* $g \in G$.

Proof. Let $\varrho \geq 0$ and $\varphi \in C_c(G)$. Then for all $\varepsilon > 0$ one deduces that

$$\left| \int_G dg\, \varphi(g) e^{\varrho|g|'} (A^\alpha B^\beta K_t)(g) \right|$$

$$\leq \int_{\widetilde{G}} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| e^{\varrho|\pi(\widetilde{g})|'} |(\widetilde{A}^\alpha \widetilde{B}^\beta \widetilde{K}_t)(\widetilde{g})|$$

$$\leq a_{\alpha,\beta} \int_{\widetilde{G}} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| e^{\varrho|\widetilde{g}|'} t^{-(\widetilde{D}+|\alpha|+|\beta|)/m} e^{-(b-\varepsilon)((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}$$

$$\leq a_{\alpha,\beta} t^{-(|\alpha|+|\beta|)/m} \sup_{\widetilde{h} \in \widetilde{G}} \left(e^{|\varrho| \cdot |\widetilde{h}|'} e^{-(b-2\varepsilon)((|\widetilde{h}|')^m t^{-1})^{1/(m-1)}} \right)$$

$$\times \int_{\widetilde{G}} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| t^{-\widetilde{D}/m} e^{-\varepsilon((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}$$

for suitable $a_{\alpha,\beta} > 0$ by Lemmas 3.3 and 3.2 since $|\pi(\widetilde{g})|' \leq |\widetilde{g}|'$. Moreover,

$$e^{\varrho|\widetilde{h}|'} e^{-(b-2\varepsilon)((|\widetilde{h}|')^m t^{-1})^{1/(m-1)}} \leq e^{\omega_{b-2\varepsilon} \varrho^m t}$$

for all $\widetilde{h} \in \widetilde{G}$, where $\omega_{b-2\varepsilon} = (b-2\varepsilon)^{-(m-1)}(m-1)^{m-1} m^{-m}$.

Next we estimate

$$\int_{\widetilde{G}} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| t^{-\widetilde{D}/m} e^{-\varepsilon((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}$$

by a detour through second-order operators. Again let K_t^Δ and $\widetilde{K}_t^{\widetilde{\Delta}}$ denote the semigroup kernels associated with the Laplacians $\Delta = -\sum_{i=1}^{d'} A_i^2$ and $\widetilde{\Delta} = -\sum_{i=1}^{d'} \widetilde{A}_i^2$, respectively. These kernels satisfy upper and lower Gaussian bounds with the correct asymptotic behaviour by Lemma 3.4. Then one has

$$s^{-\widetilde{D}/2} e^{-(|\widetilde{g}|')^2 s^{-1}} \leq a \widetilde{K}_{\omega s}^{\widetilde{\Delta}}(\widetilde{g})$$

for suitable $a, \omega > 0$ and all $s > 0$ and $\widetilde{g} \in \widetilde{G}$ by Lemma 3.4 applied to $\widetilde{G}$ and $\widetilde{\Delta}$. Therefore

$$
\begin{aligned}
\int_{\widetilde{G}} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| s^{-\widetilde{D}/2} e^{-(|\widetilde{g}|')^2 s^{-1}} &\leq a \int_{\widetilde{G}} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| \widetilde{K}^{\widetilde{\Delta}}_{\omega s}(\widetilde{g}) \\
&= a \int_{G} dg\, |\varphi(g)| K^{\Delta}_{\omega s}(g) \\
&\leq a\|K^{\Delta}_{\omega s}\|_\infty \|\varphi\|_1 \leq a' V(s^{1/2})^{-1} \|\varphi\|_1,
\end{aligned}
$$

by Lemma 3.2 and application of Lemma 3.4 to the kernels K^{Δ}_t.

Next consider the decomposition

$$
\int_{\widetilde{G}} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| t^{-\widetilde{D}/m} e^{-\varepsilon((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}
$$

$$
= \sum_{n=0}^{\infty} \int_{\Omega_n} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| t^{-\widetilde{D}/m} e^{-\varepsilon((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}},
$$

where $\Omega_n = \{\widetilde{g} \in \widetilde{G} : n \leq (|\widetilde{g}|')^m t^{-1} \leq n+1\}$. For all $n \in \mathbb{N}_0$ one has

$$
\int_{\Omega_n} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| t^{-\widetilde{D}/m} e^{-\varepsilon((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}
$$

$$
\leq t^{-\widetilde{D}/m} e^{-\varepsilon n^{1/(m-1)}} \int_{\Omega_n} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})|
$$

$$
\leq t^{-\widetilde{D}/m} e^{-\varepsilon n^{1/(m-1)}} s^{\widetilde{D}/2} e^{((n+1)t)^{2/m} s^{-1}} \int_{\Omega_n} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| s^{-\widetilde{D}/2} e^{-(|\widetilde{g}|')^2 s^{-1}}
$$

$$
\leq t^{-\widetilde{D}/m} e^{-\varepsilon n^{1/(m-1)}} s^{\widetilde{D}/2} e^{((n+1)t)^{2/m} s^{-1}} \int_{\widetilde{G}} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| s^{-\widetilde{D}/2} e^{-(|\widetilde{g}|')^2 s^{-1}}
$$

$$
\leq t^{-\widetilde{D}/m} e^{-\varepsilon n^{1/(m-1)}} s^{\widetilde{D}/2} e^{((n+1)t)^{2/m} s^{-1}} a' V(s^{1/2})^{-1} \|\varphi\|_1
$$

for all $s > 0$. Now set $s = (n+1) t^{2/m}$. Then

$$
\int_{\Omega_n} d\widetilde{g}\, |(\pi^*\varphi)(\widetilde{g})| t^{-\widetilde{D}/m} e^{-\varepsilon((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}}
$$

$$
\leq a' V((n+1)^{1/2} t^{1/m})^{-1} (n+1)^{\widetilde{D}/2} e^{-\varepsilon n^{1/(m-1)} + (n+1)^{-1+2/m}} \|\varphi\|_1
$$

$$
\leq a' e V(t^{1/m})^{-1} (n+1)^{\widetilde{D}/2} e^{-\varepsilon n^{1/(m-1)}} \|\varphi\|_1.
$$

Since

$$
\sum_{n=0}^{\infty} e^{-\varepsilon n^{1/(m-1)}} (n+1)^{\widetilde{D}/2} < \infty
$$

it follows that

$$\int\limits_{\widetilde{G}} d\widetilde{g}\,|(\pi^*\varphi)(\widetilde{g})|t^{-\widetilde{D}/m}e^{-\varepsilon((|\widetilde{g}|')^m t^{-1})^{1/(m-1)}} \leq a''V(t^{1/m})^{-1}\|\varphi\|_1.$$

Hence

$$\left|\int\limits_G dg\,\varphi(g)e^{\varrho|g|'}(A^\alpha B^\beta K_t)(g)\right| \leq a''a_{\alpha,\beta}e^{\omega_{b-2\varepsilon}\varrho^m t}t^{-(|\alpha|+|\beta|)/m}V(t^{1/m})^{-1}\|\varphi\|_1$$

for all $t > 0$ and $\varphi \in C_c(G)$. Therefore

$$e^{\varrho|g|'}|(A^\alpha B^\beta K_t)(g)| \leq a''a_{\alpha,\beta}e^{\omega_{b-2\varepsilon}\varrho^m t}t^{-(|\alpha|+|\beta|)/m}V(t^{1/m})^{-1}$$

for all $t > 0$ and $g \in G$. Finally, minimizing over ϱ one obtains the bounds

$$|(A^\alpha B^\beta K_t)(g)| \leq a''a_{\alpha,\beta}t^{-(|\alpha|+|\beta|)/m}V(t^{1/m})^{-1}e^{-(b-2\varepsilon)((|g|')^m t^{-1})^{1/(m-1)}}$$

for all $t > 0$ and $g \in G$. ∎

The exponent $b-\varepsilon$ in the Gaussian bounds for the kernel on G is smaller than the corresponding exponent b for the homogeneous group but can be chosen arbitrarily close to b at the risk of increasing the value of $a_{\alpha,\beta}$.

4. Riesz transforms. In this section we examine the Lie group analogues $R_\alpha = A^\alpha H^{-|\alpha|/m}$ of the classical Riesz transforms on the L_p-spaces with $p \in (1,\infty)$. If G is not compact, define $k_\alpha : G \setminus \{e\} \to \mathbb{C}$ by

$$(5) \qquad k_\alpha(g) = \Gamma(|\alpha|/m)^{-1}\int\limits_0^\infty dt\, t^{-1+|\alpha|/m}(A^\alpha K_t)(g).$$

Then, formally, $R_\alpha = L(k_\alpha)$ and the bounds on the derivatives of K given by Theorem 3.5 lead to estimates

$$|k_\alpha(g)| \leq aV(|g|')^{-1}$$

if G is not compact (see, for example, the appendix of [EIR2]). These estimates reflect the expected singularities of the kernel. They demonstrate that its integral is logarithmically divergent both at the identity and at infinity. The kernel k_α defined by (5) is also differentiable away from the identity and its left derivatives satisfy bounds

$$(6) \qquad |(A^\gamma k_\alpha)(g)| \leq a(|g|')^{-|\gamma|}V(|g|')^{-1}$$

for all multi-indices γ. These kernel bounds indicate the viability of singular integration techniques (see, for example, [Ste], Chapter I, or [CoW1], Chapter III) to bound the R_α. But the difficulty with this approach is that it requires *a priori* knowledge of the boundedness of the transforms on L_2, or on one of the other L_p-spaces with $p \in (1,\infty)$. It is, however, unclear whether the R_α are even densely defined on these spaces. In the classical setting, $G = \mathbb{R}^d$, this question is readily resolved by Fourier theory but in

the general Lie group setting it is not straightforward except in special cases such as second-order self-adjoint H on L_2 and $|\alpha| = 1$. In the latter case one has $D(H^{1/2}) = \bigcap_{i=1}^{d'} D(A_i)$ and

$$\mu \sum_{i=1}^{d'} \|A_i\varphi\|_2^2 \leq \|H^{1/2}\varphi\|_2^2 \leq \|C\| \sum_{i=1}^{d'} \|A_i\varphi\|_2^2$$

for all $\varphi \in D(H^{1/2})$ with μ the ellipticity constant defined by (2) and $\|C\|$ the norm of the matrix of coefficients. It follows that ψ is orthogonal to the range of $H^{1/2}$ if, and only if, $\psi \in D(H^{1/2})$ and $H^{1/2}\psi = 0$, i.e., if, and only if, $A_i\psi = 0$ for all $i \in \{1, \ldots, d'\}$ or, equivalently, ψ is constant. But L_2 contains non-zero constant functions if, and only if, G is compact and as G is nilpotent this occurs if, and only if, $G = \mathbb{T}^{d'}$ (see [HeR], Corollary 29.44). One concludes that $R(H^{1/2})$ is dense in L_2 and $H^{-1/2}$ is densely defined except in the special case $G = \mathbb{T}^{d'}$. Moreover, the foregoing estimates then establish that the Riesz transforms, $R_{(i)}$, are norm-densely defined and have bounded closures except in the special case of the tori. In the latter situation the Riesz transforms are norm-densely defined on the subspace of the functions with mean value zero. Then one can define $R_{(i)} = 0$ on the constant functions and with this convention these lowest order Riesz transforms are again bounded on L_2.

If $|\alpha| > 1$, or if H is of order $m > 2$, or if H fails to be self-adjoint, then similar simple observations no longer yield boundedness of the Riesz transforms on L_2. Therefore it is difficult to apply directly the techniques of singular integration. Nevertheless, one can make an indirect application of these techniques combined with regularization, following [BER], and then gain extra uniformity by transference arguments. This is the approach we take.

Let H again denote a homogeneous subcoercive operator of order m acting on the L_p-spaces. Then $H^{-N/m}$ is defined on the range of $H^{N/m}$ and although it is not evident that this subspace is dense, it does follow from the uniform boundedness (3) of the S_t that the range of $(\nu I + H)^{N/m}$ is dense for each $\nu > 0$. Therefore our tactic is to analyze the operators $R_{\alpha;\nu} = A^\alpha(\nu I + H)^{-|\alpha|/m}$ and then recuperate information about the R_α by taking the limit $\nu \downarrow 0$. Again it is not evident that the $R_{\alpha;\nu}$ are densely defined because this requires $D(H^{|\alpha|/m}) \subseteq D(A^\alpha)$. But this follows from [BER], at least on L_p with $p \in (1, \infty)$, and, moreover, the $R_{\alpha;\nu}$ have bounded closures for sufficiently large values of ν by Theorem 2.3 of [BER]. These results hold for a general Lie group. If, however, the group is homogeneous, the norms are independent of ν, by scaling, and one could hope to transfer the bounds of the transforms on the homogeneous group to the transforms on the nilpotent group G by Theorem 2.1. But this is again not straightforward since the kernels of the $R_{\alpha;\nu}$ are not integrable.

For any nilpotent Lie group G and all $\nu > 0$ define $k_{\alpha;\nu} : G \setminus \{e\} \to \mathbb{C}$ by

$$(7) \qquad k_{\alpha;\nu}(g) = \Gamma(|\alpha|/m)^{-1} \int_0^\infty dt\, t^{-1+|\alpha|/m} e^{-\nu t} (A^\alpha K_t)(g).$$

Then for all $\varphi, \psi \in C_c^\infty(G)$ with disjoint supports one has

$$\begin{aligned}
(\psi, R_{\alpha;\nu}\varphi) &= (-1)^{|\alpha|} (A^{\alpha*}\psi, (\nu I + H)^{-|\alpha|/m}\varphi) \\
&= (-1)^{|\alpha|} \Gamma(|\alpha|/m)^{-1} \Big(A^{\alpha*}\psi, \int_0^\infty dt\, t^{-1+|\alpha|/m} e^{-\nu t} K_t * \varphi \Big) \\
&= \Gamma(|\alpha|/m)^{-1} \Big(\psi, \int_0^\infty dt\, t^{-1+|\alpha|/m} e^{-\nu t} (A^\alpha K_t) * \varphi \Big) \\
&= (\psi, k_{\alpha;\nu} * \varphi).
\end{aligned}$$

So by density,

$$(8) \qquad\qquad (\psi, R_{\alpha;\nu}\varphi) = (\psi, k_{\alpha;\nu} * \varphi)$$

for all $\varphi \in L_p$ and $\psi \in L_q$ both with compact, but disjoint, supports. Moreover, the bounds of Theorem 3.5 now lead to estimates

$$(9) \qquad\qquad |k_{\alpha;\nu}(g)| \leq a V(|g|')^{-1} e^{-b\nu^{1/m}|g|'},$$

if G is not compact, with a and b independent of ν and g. The exponential factor ensures integrability of the kernel at infinity but the integral of the kernel is still logarithmically divergent at the identity. Therefore the transference theorem, Theorem 2.1, is not directly applicable to the $R_{\alpha;\nu}$. In order to circumvent this difficulty we again follow the ideas of [BER] and consider the regularized transforms

$$R_{\alpha;\nu,\varepsilon} = A^\alpha (\nu I + H)^{-|\alpha|/m} (I + \varepsilon H)^{-n}$$

with $\varepsilon > 0$ and n a positive integer. The kernels $k_{\alpha;\nu,\varepsilon}$ of these operators are less singular since the extra $(I + \varepsilon H)^{-n}$ introduces a factor $(|g|')^{nm}$. Therefore if n is sufficiently large the kernels are integrable although the norms $\|k_{\alpha;\nu,\varepsilon}\|_1$ diverge as $\nu \downarrow 0$ or $\varepsilon \downarrow 0$. (For detailed estimates see [BER], Lemma 2.4.) But using these regularizations and singular integration theory it is established in [BER] that

$$\|A^\alpha (\nu I + H)^{-|\alpha|/m}\|_{p \to p} < \infty$$

for all $p \in (1, \infty)$, all large $\nu > 0$ and all multi-indices α. These estimates are the starting point for our analysis of the Riesz transforms. We begin by applying them to the homogeneous group $\widetilde{G}$.

LEMMA 4.1. *The regularized transforms $\widetilde{R}_{\alpha;\nu,\varepsilon}$ defined on $\widetilde{G}$ satisfy bounds*

$$\|\widetilde{R}_{\alpha;\nu,\varepsilon}\|_{\widetilde{p}\to\widetilde{p}} \leq \|\widetilde{S}_1\|_{\widetilde{p}\to\widetilde{p}}\|\widetilde{A}^\alpha(I+\widetilde{H})^{-|\alpha|/m}\|_{\widetilde{p}\to\widetilde{p}}$$

for all $p \in (1,\infty)$, $\varepsilon > 0$ and $\nu > 0$.

P r o o f. It follows from the definition of $\widetilde{R}_{\alpha;\nu,\varepsilon}$ that

$$\|\widetilde{R}_{\alpha;\nu,\varepsilon}\|_{\widetilde{p}\to\widetilde{p}} \leq \|\widetilde{A}^\alpha(\nu I+\widetilde{H})^{-|\alpha|/m}\|_{\widetilde{p}\to\widetilde{p}}\|(I+\varepsilon\widetilde{H})^{-n}\|_{\widetilde{p}\to\widetilde{p}}.$$

But

$$\|(I+\varepsilon\widetilde{H})^{-n}\|_{\widetilde{p}\to\widetilde{p}} \leq \Gamma(n)^{-1}\int\limits_0^\infty dt\, e^{-t}t^{-1+n}\|\widetilde{S}_{\varepsilon t}\|_{\widetilde{p}\to\widetilde{p}} \leq \|\widetilde{S}_1\|_{\widetilde{p}\to\widetilde{p}}$$

where the last estimate uses scaling. Moreover,

$$\|\widetilde{A}^\alpha(\nu I+\widetilde{H})^{-|\alpha|/m}\|_{\widetilde{p}\to\widetilde{p}} = \|\widetilde{A}^\alpha(I+\widetilde{H})^{-|\alpha|/m}\|_{\widetilde{p}\to\widetilde{p}}$$

by another application of scaling. The statement of the lemma follows from combination of these estimates. ∎

Since $k_{\alpha;\nu,\varepsilon} = \pi_*(\widetilde{k}_{\alpha;\nu,\varepsilon})$, where $\widetilde{k}_{\alpha;\nu,\varepsilon}$ is defined in the obvious way, by the argument used to prove Lemma 3.2, one can apply the transference theorem, Theorem 2.1, to the regularized transforms to obtain

$$(10) \qquad \|R_{\alpha;\nu,\varepsilon}\|_{p\to p} = \|L(k_{\alpha;\nu,\varepsilon})\|_{p\to p} \leq \|\widetilde{L}(\widetilde{k}_{\alpha;\nu,\varepsilon})\|_{\widetilde{p}\to\widetilde{p}} = \|\widetilde{R}_{\alpha;\nu,\varepsilon}\|_{\widetilde{p}\to\widetilde{p}}.$$

Combination of these bounds with Lemma 4.1 immediately yields the key estimates on the Riesz transforms.

LEMMA 4.2. *For each multi-index α one has $D(H^{|\alpha|/m}) \subseteq D(A^\alpha)$ and*

$$(11) \qquad \|A^\alpha\varphi\|_p \leq s_{\alpha,p}\|H^{|\alpha|/m}\varphi\|_p$$

for all $\varphi \in D(H^{|\alpha|/m})$ and all $p \in (1,\infty)$, where

$$s_{\alpha,p} = \limsup_{\varepsilon\downarrow 0,\,\nu\downarrow 0}\|\widetilde{R}_{\alpha;\nu,\varepsilon}\|_{\widetilde{p}\to\widetilde{p}} \leq \|\widetilde{S}_1\|_{\widetilde{p}\to\widetilde{p}}\|\widetilde{A}^\alpha(I+\widetilde{H})^{-|\alpha|/m}\|_{\widetilde{p}\to\widetilde{p}}.$$

P r o o f. Let $N \in \mathbb{N}$. It follows from the foregoing estimate that

$$\|R_{\alpha;\nu,\varepsilon}\psi\|_p \leq \|\widetilde{R}_{\alpha;\nu,\varepsilon}\|_{\widetilde{p}\to\widetilde{p}}\|\psi\|_p$$

for all $\psi \in L_p$, $p \in (1,\infty)$, $\varepsilon,\nu > 0$ and α with $|\alpha| = N$. Therefore, setting $\psi = (\nu I + H)^{|\alpha|/m}(I+\varepsilon H)^n\varphi$ with $\varphi \in D_\infty(H) = \bigcap_{m\geq 1}D(H^m) = L_{p;\infty}$ one has

$$\|A^\alpha\varphi\|_p \leq \|\widetilde{R}_{\alpha;\nu,\varepsilon}\|_{\widetilde{p}\to\widetilde{p}}\|(\nu I+H)^{|\alpha|/m}(I+\varepsilon H)^n\varphi\|_p.$$

Taking the limits $\nu \downarrow 0$ and $\varepsilon \downarrow 0$ gives the bounds (11) for $\varphi \in D_\infty(H) = L_{p;\infty}$. (The upper bound on $s_{\alpha,p}$ follows from Lemma 4.1.) Here we have used

$$\lim_{\nu\downarrow0}\|(\nu I + H)^\gamma\psi - H^\gamma\psi\|_p = 0$$

for all $\gamma > 0$ and all $\psi \in D_\infty(H)$. This is evident for integer values of γ and follows from Lemma II.3.2 of [Rob] if $\gamma \in (0, 1)$. The general result then follows by combination of these special cases.

In particular, setting $\alpha_i = (i, \dots, i)$ with $i \in \{1, \dots, d'\}$, one obtains $\|A_i^N\varphi\|_p \le s_{\alpha_i,p}\|H^{N/m}\varphi\|_p$ for all $\varphi \in D_\infty(H)$. Since $D_\infty(H)$ is a core for $H^{N/m}$ and A_i^N is closed, it then follows that $D(H^{N/m}) \subset D(A_i^N)$. So $D(H^{N/m}) \subseteq \bigcap_{i=1}^{d'} D(A_i^N) = L'_{p;N}$, by [BER], Corollary 2.6. Finally, if $\varphi \in D(H^{N/m})$ and $\varphi_1, \varphi_2, \dots \in D_\infty(H)$ are such that $\lim \varphi_n = \varphi$ and $\lim H^{N/m}\varphi_n = H^{N/m}\varphi$ in L_p, then $\lim H^{j/m}\varphi_n = H^{j/m}\varphi$ in L_p for all $j \in \{0, 1, \dots, N\}$. Hence by induction on $|\alpha|$ it follows from the estimates (11) and the closedness of the A_i that $\lim A^\alpha\varphi_n = A^\alpha\varphi$ for all α with $|\alpha| \le N$. Then (11) is established for all $\varphi \in D(H^{N/m})$ by limiting. ∎

COROLLARY 4.3. *For all $N \in \mathbb{N}$ and $p \in (1, \infty)$ one has $D(H^{N/m}) = L'_{p;N}$ and there exists a $c > 0$ such that*

$$\|H^{N/2}\varphi\|_p \le c \max_{|\alpha|=N}\|A^\alpha\varphi\|_p$$

for all $\varphi \in L'_{p;N}$. Therefore the seminorms $\varphi \mapsto \|H^{N/m}\varphi\|_p$ and $\varphi \mapsto \max_{|\alpha|=N}\|A^\alpha\varphi\|_p$ are equivalent.

Proof. We first consider the cases $N \in \{1, \dots, m-1\}$. For every multi-index α with $|\alpha| = m$ let α' and α'' be multi-indices such that $\alpha = \langle\alpha', \alpha''\rangle$ and $|\alpha'| = m - N$ and $|\alpha''| = N$. Since the adjoint H^* of H is a subelliptic operator of the same kind as H, the foregoing conclusions apply equally well to the H^* on the dual L_q of L_p. One then has

$$
\begin{aligned}
|(\psi, \varphi)| &= |(\psi, (\nu I + H)^{-(m-N)/m}(\nu I + H)(\nu I + H)^{-N/m}\varphi)| \\
&\le \sum_{|\alpha|=m} |c_\alpha|\,\|A^{\alpha'*}(\nu I + H^*)^{-(m-N)/m}\psi\|_q\|A^{\alpha''}(\nu I + H)^{-N/m}\varphi\|_p \\
&\quad + \nu\|\psi\|_q\|(\nu I + H)^{-1}\varphi\|_p \\
&\le \sum_{i,j=1}^{d'} |c_\alpha|\,\|\widetilde{S}_1^*\|_{\tilde{q}\to\tilde{q}}\|\widetilde{A}^{\alpha'*}(I + \widetilde{H}^*)^{-(m-N)/m}\|_{\tilde{q}\to\tilde{q}} \\
&\quad \times \|\psi\|_q\|A^{\alpha''}(\nu I + H)^{-N/m}\varphi\|_p + \nu\|\psi\|_q\|(\nu I + H)^{-1}\varphi\|_p
\end{aligned}
$$

for all $\varphi \in L'_{p;\infty}$, $\psi \in L_q$ and $\nu > 0$, by Lemma 4.1 and the bounds (10). Hence replacing φ by $(\nu I + H)^{N/m}\varphi$ one gets

$$\|(\nu I + H)^{N/m}\varphi\|_p$$

$$\leq \sum_{|\alpha|=m} |c_\alpha| \,\|\widetilde{S}_1^*\|_{\widetilde{q}\to\widetilde{q}} \|\widetilde{A}^{\alpha'*}(I+\widetilde{H}^*)^{-(m-N)/m}\|_{\widetilde{q}\to\widetilde{q}} \|A^{\alpha''}\varphi\|_p$$

$$+ \nu\|(\nu I+H)^{-(m-N)/m}\varphi\|_p$$

for all $\varphi \in L'_{p;\infty}$. Taking the limit $\nu \downarrow 0$ it then follows that

$$\|H^{N/m}\varphi\|_p \leq \sum_{|\alpha|=m} |c_\alpha| \,\|\widetilde{S}_1^*\|_{\widetilde{q}\to\widetilde{q}} \|\widetilde{A}^{\alpha'*}(I+\widetilde{H}^*)^{-(m-N)/m}\|_{\widetilde{q}\to\widetilde{q}} \|A^{\alpha''}\varphi\|_p$$

for all $\varphi \in L'_{p;\infty}$, and so, by density, for all $\varphi \in L'_{p;N}$. This proves the cases $N < m$.

The general case follows easily by induction. ∎

At this stage it is straightforward to argue that the Riesz transforms extend to bounded operators. First, the foregoing conclusions apply to the adjoint H^* of H on the dual L_q of L_p. Therefore $(H^*)^{N/m}\varphi = 0$ implies $A^\beta\varphi = 0$ for all β with $|\beta| = N$ and as a consequence $A_i\varphi = 0$ for all $i \in \{1, \ldots, d'\}$. Since $a_1, \ldots, a_{d'}$ is an algebraic basis this implies that φ is constant. But $q \neq \infty$ and hence L_q contains non-zero constant functions if, and only if, G is compact, and as G is nilpotent this occurs if, and only if, $G = \mathbb{T}^{d'}$ (see [HeR], Corollary 29.44). One concludes that $R(H^{N/m})$ is dense in L_p except in the special case $G = \mathbb{T}^{d'}$. Moreover, the foregoing estimates then establish that the Riesz transforms R_α are norm-densely defined and have bounded closures except in the special case of the tori. Alternatively, if G is compact one can replace the space L_p by the closed subspace of functions with mean value zero, i.e., one considers the space L_p modulo the constant functions. Then left translations act on this subspace and the Riesz transforms are defined and bounded. Moreover, on L_p, define $R_\alpha = 0$ on the constant functions and with this convention the Riesz transforms are again bounded. In the sequel we assume that one or the other of these modifications has been made in the compact case.

THEOREM 4.4. *The Riesz transforms R_α extend to bounded operators on each of the L_p-spaces, $p \in (1, \infty)$, and*

$$\|R_\alpha\|_{p\to p} \leq a_p b_p^k \|\widetilde{R}_\alpha\|_{\widetilde{p}\to\widetilde{p}}$$

where

$$a_p = \sup_{t>0} \|\widetilde{S}_1 - \widetilde{S}_{1+t}\|_{\widetilde{p}\to\widetilde{p}}, \qquad b_p = \|I - \widetilde{S}_1\|_{\widetilde{p}\to\widetilde{p}}$$

and k is the integer part of $|\alpha|/m$.

Proof. Using Lemma 4.2 it remains to show that

$$\limsup_{\varepsilon\downarrow 0,\,\nu\downarrow 0} \|\widetilde{R}_{\alpha;\nu,\varepsilon}\|_{\widetilde{p}\to\widetilde{p}} \leq a_p b_p^k \|\widetilde{R}_\alpha\|_{\widetilde{p}\to\widetilde{p}}.$$

But
$$\limsup_{\varepsilon\downarrow 0,\,\nu\downarrow 0} \|\widetilde{R}_{\alpha;\nu,\varepsilon}\|_{\widetilde{p}\to\widetilde{p}}$$

$$\leq \|\widetilde{R}_\alpha\|_{\widetilde{p}\to\widetilde{p}} \limsup_{\varepsilon\downarrow 0,\,\nu\downarrow 0} \|\widetilde{H}^{|\alpha|/m}(\nu I + \widetilde{H})^{-|\alpha|/m}(I + \varepsilon\widetilde{H})^{-n}\|_{\widetilde{p}\to\widetilde{p}}$$

$$= \|\widetilde{R}_\alpha\|_{\widetilde{p}\to\widetilde{p}} \limsup_{\delta\downarrow 0} \|\widetilde{H}^{|\alpha|/m}(I + \widetilde{H})^{-|\alpha|/m}(I + \delta\widetilde{H})^{-n}\|_{\widetilde{p}\to\widetilde{p}}$$

where the last identification uses scaling on $\widetilde{G}$. Now $|\alpha|/m = k + \gamma$ with $\gamma \in [0,1)$ and
$$\|\widetilde{H}^k(I + \widetilde{H})^{-k}\|_{\widetilde{p}\to\widetilde{p}} \leq (\|\widetilde{H}(I + \widetilde{H})^{-1}\|_{\widetilde{p}\to\widetilde{p}})^k.$$
But
$$\|\widetilde{H}(I + \widetilde{H})^{-1}\|_{\widetilde{p}\to\widetilde{p}} = \left\| \int_0^\infty dt\, e^{-t}(I - \widetilde{S}_t) \right\|_{\widetilde{p}\to\widetilde{p}} \leq b_p.$$

Next, if J is the generator of a uniformly bounded semigroup then
$$J^\gamma = n_\gamma^{-1} \int_0^\infty d\lambda\, \lambda^{-1+\gamma} J(\lambda I + J)^{-1}$$

for $\gamma \in (0,1)$ with $n_\gamma = \int_0^\infty d\lambda\, \lambda^{-1+\gamma}(\lambda + 1)^{-1}$. Since $\widetilde{H}$ generates a uniformly bounded semigroup it follows that $J = \widetilde{H}(I + \widetilde{H})^{-1}$ also generates a uniformly bounded semigroup. Hence applying the foregoing fractional power formula we get
$$\widetilde{H}^\gamma(I + \widetilde{H})^{-\gamma} = (\widetilde{H}(I + \widetilde{H})^{-1})^\gamma$$

$$= n_\gamma^{-1} \int_0^\infty d\lambda\, \lambda^{-1+\gamma}(1 + \lambda)^{-1}\widetilde{H}(\lambda(1 + \lambda)^{-1}I + \widetilde{H})^{-1}.$$

Therefore
$$\|\widetilde{H}^\gamma(I + \widetilde{H})^{-\gamma}(I + \delta\widetilde{H})^{-n}\|_{\widetilde{p}\to\widetilde{p}} \leq \sup_{0\leq\varepsilon\leq 1} \|\widetilde{H}(\varepsilon I + \widetilde{H})^{-1}(I + \delta\widetilde{H})^{-n}\|_{\widetilde{p}\to\widetilde{p}}$$

$$\leq \sup_{0\leq\varepsilon\leq 1} \|\widetilde{H}(I + \widetilde{H})^{-1}(I + \delta\varepsilon\widetilde{H})^{-n}\|_{\widetilde{p}\to\widetilde{p}}$$

where the last step uses scaling. Now
$$\widetilde{H}(I + \widetilde{H})^{-1}(I + \nu\widetilde{H})^{-n} = \Gamma(n)^{-1} \int_0^\infty ds \int_0^\infty dt\, e^{-(s+t)} s^{-1+n} \widetilde{S}_{\nu s}(I - \widetilde{S}_t)$$

and combination of these observations gives
$$\limsup_{\delta\downarrow 0} \|\widetilde{H}^\gamma(I + \widetilde{H})^{-\gamma}(I + \delta\widetilde{H})^{-n}\|_{\widetilde{p}\to\widetilde{p}}$$

$$\leq \limsup_{\nu\downarrow 0} \|\widetilde{H}(I + \widetilde{H})^{-1}(I + \nu\widetilde{H})^{-n}\|_{\widetilde{p}\to\widetilde{p}} \leq a_p.$$

The statement of the proposition follows immediately. ∎

Note that if $p = 2$ and H is self-adjoint then $a_2 = b_2 = 1$ and hence $\|R_\alpha\|_{2\to 2} \leq \|\widetilde{R}_\alpha\|_{\widetilde{2}\to\widetilde{2}}$. Alternatively, if $m = 2$ and S_t is a Markovian semigroup then $a_p, b_p \leq 2$ and $a_2 = b_2 = 1$. Thus by interpolation $a_p, b_p \leq 2^{|1-2/p|}$.

In the non-commutative context of Lie groups there are various operators other than the R_α which are analogues of the Riesz transforms. But these can be bounded by additional arguments such as duality.

COROLLARY 4.5. *The transforms* $R_{\alpha,\beta} = A^\alpha H^{-(|\alpha|+|\beta|)/m} A^\beta$ *extend to bounded operators on each of the spaces* $L_p(G\,;dg)$ *with* $p \in (1,\infty)$.

P r o o f. One has $R_{\alpha,\beta}(H) = R_\alpha(H)(R_{\beta_*}(H^*))^*$ where we have temporarily modified the notation to indicate the dependence of the transforms on H, or H^*. Therefore the $R_{\alpha,\beta}(H)$ extend to bounded operators on the L_p-spaces as the product of the bounded $R_\alpha(H)$ and the bounded adjoints $R_{\beta_*}(H^*)$. ∎

Alternatively, one can examine multiple products of derivatives and inverse powers of H.

COROLLARY 4.6. *The transforms* $A^{\alpha_1} H^{-n_1/m} \ldots A^{\alpha_k} H^{-n_k/m}$ *on* $L_{p;\infty}$ *extend to bounded operators on each of the spaces* $L_p(G\,;dg)$ *with* $p \in (1,\infty)$ *for all* $k \in \mathbb{N}$, *all multi-indices* $\alpha_1, \ldots, \alpha_k$ *and all* $n_1, \ldots, n_k \in \mathbb{N}_0$ *such that* $|\alpha_1| + \ldots + |\alpha_k| = n_i + \ldots + n_k$.

P r o o f. The proof follows from the observation that for all $N \in \mathbb{Z}$, $i \in \{1, \ldots, d'\}$ and $p \in (1,\infty)$ there exists a $c > 0$ such that

$$\|H^{(N-1)/m} A_i H^{-N/m} \varphi\|_p \leq c\|\varphi\|_p$$

uniformly for $\varphi \in L_{p;\infty}$.

The case $N = 0$ has already been established. Therefore suppose $N \geq 1$. Then

$$\|H^{(N-1)/m} A_i H^{-N/m} \varphi\|_p \leq c \max_{|\beta|=N-1} \|A^\beta A_i H^{-N/m} \varphi\|_p$$

$$\leq c \max_{|\alpha|=N} \|A^\alpha H^{-N/m} \varphi\|_p \leq c'\|\varphi\|_p$$

by Lemma 4.2 and Corollary 4.3. Finally, the case $N < 0$ follows by duality from the case $N \geq 0$. ∎

Although we have established boundedness of the Riesz transforms on the L_p-spaces with $p \in (1,\infty)$, our arguments do not establish directly that the R_α are weak type $(1,1)$. One deduces from [BER] that the regularized operators $R_{\alpha;\nu,\varepsilon}$ are weak type $(1,1)$ but it appears difficult to conclude by limiting arguments that this property extends to the R_α. Nevertheless, since we now know that the R_α are bounded on L_2 we are in a position to

deduce the weak type $(1, 1)$ property by the standard arguments of singular integration theory.

PROPOSITION 4.7. *The Riesz transforms R_α extend to operators of weak type* $(1, 1)$.

P r o o f. We will verify the assumptions of Theorem 3 of Chapter I of [Ste] for the operators R_α. Then the proposition is a corollary. Note that although the discussion of Chapter I of [Ste] is restricted to the Euclidean space $\mathbb{R}^d$ equipped with the Lebesgue measure dx, the analogue of Theorem 3 is valid on a homogeneous space X with a measure μ satisfying the doubling property. (Singular integration theory is described in this latter setting in [CoW1] but the main result, Theorem III.2.4, contains an extraneous assumption of square integrability of the operator kernel.)

First suppose G is not compact. Fix $p \in (1, \infty)$. Then R_α is bounded on L_p by Theorem 4.4. Moreover, $H^{|\alpha|/m}(\nu I + H)^{-|\alpha|/m}$ is bounded for each $\nu > 0$. But it is readily verified that

$$\lim_{\nu \downarrow 0} \|H^{|\alpha|/m}(\nu I + H)^{-|\alpha|/m}\varphi - \varphi\|_p = 0$$

for all $\varphi \in L_p$. Therefore

$$(\psi, R_\alpha\varphi) = \lim_{\nu \downarrow 0}(\psi, R_\alpha H^{|\alpha|/m}(\nu I + H)^{-|\alpha|/m}\varphi)$$

$$= \lim_{\nu \downarrow 0}(\psi, A^\alpha(\nu I + H)^{-|\alpha|/m}\varphi)$$

for all $\varphi \in L_p$ and $\psi \in L_q$, the dual of L_p.

But $(\psi, A^\alpha(\nu I + H)^{-|\alpha|/m}\varphi) = (\psi, k_{\alpha;\nu} * \varphi)$ if $\varphi \in L_p$ and $\psi \in L_q$, with compact and disjoint supports, by (8). Moreover, $k_{\alpha,\nu}$ satisfies the bounds (9), uniformly for all $\nu > 0$. Therefore one deduces from the Lebesgue dominated convergence theorem that

$$(\varphi, R_\alpha\psi) = \lim_{\nu \downarrow 0}(\varphi, k_{\alpha,\nu} * \psi) = (\varphi, k_\alpha * \psi)$$

for φ, ψ with disjoint supports and with k_α defined by (8). Thus R_α is determined by the singular kernel k_α in this weak sense. But this weak relation between the operator and the singular kernel, together with the bounds (9), is sufficient for R_α to be of weak type $(1, 1)$ (see [Ste], Section I.5, for a discussion of this point in the case $G = \mathbb{R}^n$).

Finally, if G is compact, let $P\varphi = \int_G dg \, L(g)\varphi$ denote the projection of φ on the space of constant functions. Then on the subspace $(I - P)L_1$ of L_1 the restriction H_r of the operator H has a bounded inverse as a direct consequence of spectral properties (see [Rob], Proposition I.7.1). Therefore the operator $(\nu I + H_\mathrm{r})H_\mathrm{r}^{-1}$ is bounded for all $\nu > 0$. Alternatively, the operator $A^\alpha(\nu I + H)^{-1}$ and hence the operator $A^\alpha(\nu I + H_\mathrm{r})^{-1}$ is of weak

type $(1,1)$, by a direct application of singular integration theory to the operator $R_{\alpha;\nu}$ (cf. [Ste], Section I.5). Therefore

$$R_\alpha = (A^\alpha (\nu I + H_{\mathrm{r}})^{-1})((\nu I + H_{\mathrm{r}})H_{\mathrm{r}}^{-1})(I - P)$$

is an operator of weak type $(1,1)$. ∎

5. Concluding remarks. The heat kernel bounds of Section 3 and the bounds on the Riesz transforms of Section 4 can be used to investigate various other aspects of the analytic structure of the nilpotent Lie group G. For example, one can extend the analysis of subelliptic Lipschitz spaces given in [EIR1] (see also [Rob], Chapter II) in a manner which takes into account both the local smoothness and the asymptotic decay. Secondly, we briefly discuss a holomorphic functional calculus, and finally, for second-order operators with real coefficients we consider lower bounds for large t and Harnack inequalities.

5.1. *Lipschitz spaces.* It was established in Corollary 4.3 that the seminorms $\varphi \mapsto \max_{|\alpha|=n} \|A^\alpha \varphi\|_p$ and $\varphi \mapsto \|H^{n/m} \varphi\|_p$ are equivalent on the $L'_{p;n}$-spaces with $p \in (1,\infty)$. But these seminorms are in fact norms unless G is compact, i.e., unless $G = \mathbb{T}^{d'}$, and in this latter case they are norms on the subspace of functions with mean value zero. In the following we assume that G is not compact, although the results are also valid for compact G if one reads seminorms instead of norms. Note, however, the spaces $L'_{p;n}$ equipped with either of these norms are not Banach spaces. Now one can develop a theory of intermediate spaces for the C^n-subspaces $L'_{p;n}$ equipped with these norms (see, for example, [BeL] for background information). Since the A_i are group generators and H generates a bounded holomorphic semigroup S, one then has alternative characterizations in terms of Lipschitz norms, etc. Fix $\gamma > 0$ and $n \in \mathbb{N}$ with $n > \gamma$ and $q \in [1,\infty)$.

Define $\| \cdot \|_{p;\gamma,q}^{(n)} : L_p \to [0,\infty]$ by

$$\|\varphi\|_{p;\gamma,q}^{(n)} = \Big(\int\limits_0^\infty dt\, t^{-1} (t^{-\gamma} \kappa_\varphi^{(n)}(t))^q \Big)^{1/q},$$

where

$$\kappa_\varphi^{(n)}(t) = \inf\{\|\varphi_0\|_p + t^n \max_{|\alpha|=n} \|A^\alpha \varphi\|_p : \varphi = \varphi_0 + \varphi_n,\ \varphi_n \in L'_{p;n}\}$$

and we use the C^n-(semi)norm $\varphi \mapsto \max_{|\alpha|=n} \|A^\alpha \varphi\|_p$. Then the spaces

$$L_{p;\gamma,q}^{(n)} = (L_p, L'_{p;n})_{\gamma,q} = \{\varphi \in L_p : \|\varphi\|_{p;\gamma,q}^{(n)} < \infty\}$$

correspond to the real interpolation spaces for the C^n-structure of the left regular representation L of G on $L_p(G\,;dg)$. They are normed spaces with

respect to the norms $\|\cdot\|_{p;\gamma,q}^{(n)}$. Then $L'_{p;n}$ is dense in $L_{p;\gamma,q}^{(n)}$ (see [BeL], Theorem 3.4.2). Alternatively, they can be equipped with an equivalent norm defined as above but with $\varphi \mapsto \max_{|\alpha|=n} \|A^\alpha\varphi\|$ replaced by $\varphi \mapsto \|H^{n/m}\varphi\|_p$. Then the spaces correspond to the intermediate spaces of the fractional powers of H. But H generates the holomorphic semigroup S and the general theory of interpolation spaces for holomorphic semigroups establishes that these spaces can be characterized as the spaces $L_{p;\gamma/m,q}^{(n,S)}$ of all $\varphi \in L_p$ for which

$$\|\varphi\|_{p;\gamma/m,q}^{(n,S)} = \Big(\int\limits_0^\infty dt\, t^{-1}(t^{-\gamma/m}\|(I-S_t)^n\varphi\|_p)^q \Big)^{1/q}$$

is finite, or the spaces $L_{p;\gamma/m,q}^{(n,H)}$ for which

$$\|\varphi\|_{p;\gamma/m,q}^{(n,H)} = \Big(\int\limits_0^\infty dt\, t^{-1}(t^{n-\gamma/m}\|H^n S_t\varphi\|_p)^q \Big)^{1/q}$$

is finite. The equivalence of the last two norms is a version of a result of Peetre (see, for example, [BuB], Chapter III). It follows whenever the semigroup S and its generator H satisfy bounds $\|S_t\| \leq M$ and $\|HS_t\| \leq Mt^{-1}$ uniformly for all $t > 0$. But in the current context these latter bounds are given by (3) and (4).

Once one has the foregoing equivalences it follows by a slight variation of the arguments used in Steps 1, 2 and 3 of the proof of Theorem 3.2 in [EIR1] that $L_{p;\gamma,q}^{(n)}$ is equal to the space $L_{p;\gamma,q}^{(n,L)}$ of all $\varphi \in L_p$ for which

$$\|\varphi\|_{p;\gamma,q}^{(n,L)} = \Big(\int\limits_{G^n} d\mu_n(g)(|g|^{-\gamma}\|(I-L(g_1))\dots(I-L(g_n))\varphi\|_p)^q \Big)^{1/q}$$

is finite where $g = (g_1,\dots,g_n)$, $|g| = |g_1|' + \dots + |g_n|'$ and $d\mu_n(g) = dg_1 \dots dg_n V(|g|)^{-n}$. Moreover, the norms

(12) $$\|\cdot\|_{p;\gamma,q}^{(n)}, \quad \|\cdot\|_{p;\gamma/m,q}^{(n,S)}, \quad \|\varphi\|_{p;\gamma/m,q}^{(n,H)} \quad \text{and} \quad \|\cdot\|_{p;\gamma,q}^{(n,L)}$$

are equivalent. It then follows by standard reasoning that all these spaces are independent of the choice of n, within the range $n > \gamma$.

All foregoing Lipschitz spaces, $L_{p;\gamma,q}^{(n)}$, $L_{p;\gamma/m,q}^{(n,S)}$, $L_{p;\gamma/m,q}^{(n,H)}$ and $L_{p;\gamma,q}^{(n,L)}$, can also be defined if $p = 1$ or $p = \infty$, as well as, with obvious modifications, if $q = \infty$. In those cases one can use the arguments used in all four steps of the proof of Theorem 3.2 in [EIR1] to deduce that the four Lipschitz spaces are equal and the four norms in (12) are equivalent.

5.2. *Bounded holomorphic functional calculus.* A second related topic is the existence of a bounded holomorphic functional calculus, in the sense of McIntosh [McI] (see also [CDMY] and [ADM]), for the homogeneous subcoercive operators H. Since these operators are automatically maximal

accretive on L_2 they have a bounded holomorphic calculus, with the sector of holomorphy dictated by the spectral properties of the individual H, on L_2 (see [ADM], Theorem G). But then the kernel bounds of Theorem 3.5 more than suffice to deduce that the H have a bounded holomorphic functional calculus on each of the L_p-spaces with $p \in (1, \infty)$ (see [DuR], Theorem 3.1 or Theorem 3.4). The transference arguments of Section 2 can then be used to derive alternative bounds on the norms $\|f(H)\|_{p \to p}$ of the holomorphic function f of H. The standard estimate is $\|f(H)\|_{p \to p} \leq c\|f\|_\infty$ with $\|f\|_\infty$ the supremum norm within the sector of holomorphy. But $f(H)$ can be expressed in the form $f(H) = L(K_f)$ where the kernel K_f is not necessarily integrable. This can, however, be arranged by regularization. Then arguing as in Section 4 one obtains bounds $\|f(H)\|_{p \to p} \leq c'\|f(\widetilde{H})\|_{\widetilde{p} \to \widetilde{p}}$ in terms of the homogeneous group.

On L_2 there is a connection between the holomorphic functional calculus and the Lipschitz spaces. The Hilbert space theory surrounding the functional calculus (see [ADM], Sections 3 and 4) ensures that the operators H satisfy quadratic estimates. In particular, the norm $\|\cdot\|_2$ is equivalent to the norms

$$\varphi \mapsto \left(\int_0^\infty dt\, t^{-1} (t^{n-\gamma} \|H^{n-\gamma} S_t \varphi\|_2)^2 \right)^{1/2}$$

for all $n > \gamma > 0$. Thus replacing φ by $H^\gamma \varphi$ one deduces that the norms $\varphi \mapsto \|H^\gamma \varphi\|_2$ and $\varphi \mapsto \|\varphi\|_{2;\gamma}^{(n,H)}$ are equivalent on $D(H^\gamma)$. Then of course the other Lipschitz norms are also equivalent to the norm on the fractional powers. This type of identification is not generally true on the other L_p-spaces [CDMY].

5.3. *Real second-order operators.* The kernel K of an mth order operator with complex coefficients is positive if, and only if, $m = 2$ and all coefficients are real (see [Rob], Section III.5). In that case one has Gaussian lower bounds

$$K_t(g) \geq at^{-D'/2} e^{-\omega t} e^{-b(|g|')^2 t^{-1}}$$

for some $a, b > 0$ and $\omega \geq 0$. These bounds have the correct small time behaviour. On nilpotent Lie groups we next prove Gaussian lower bounds with the correct large time behaviour, i.e., $\omega = 0$, which extends Lemma 3.4 to non-symmetric operators.

THEOREM 5.1. *If $m = 2$ and $c_\alpha \in \mathbb{R}$ for all α then there exist $a, b > 0$ such that*

$$K_t(g) \geq aV(t^{1/2})^{-1} e^{-b(|g|')^2 t^{-1}}$$

for all $t > 0$ and $g \in G$.

P r o o f. It follows from Lemma 3.4 that there exist $a, \omega > 0$ such that

$$\widetilde{K}_1(\widetilde{g}) \geq a \widetilde{K}_\omega^{\widetilde{\Delta}}(\widetilde{g})$$

for all $\widetilde{g} \in \widetilde{G}$, where $\widetilde{K}^{\widetilde{\Delta}}$ is again the kernel with respect to the sublaplacian. Then by scaling,

$$\widetilde{K}_t(\widetilde{g}) \geq a \widetilde{K}_{\omega t}^{\widetilde{\Delta}}(\widetilde{g})$$

for all $t > 0$ and $\widetilde{g} \in \widetilde{G}$. Next, if $\widetilde{g} \in \widetilde{G}$ then

$$\begin{aligned}
K_t(\pi(\widetilde{g})) &= \lim_{s \downarrow 0} K_{t+s}(\pi(\widetilde{g})) = \lim_{s \downarrow 0}(\widetilde{S}_t(\pi^* K_s))(\widetilde{g}) \\
&= \lim_{s \downarrow 0} \int_{\widetilde{G}} d\widetilde{h}\, \widetilde{K}_t(\widetilde{h}) K_s(\pi(\widetilde{h}^{-1}\widetilde{g})) \\
&\geq \lim_{s \downarrow 0} a \int_{\widetilde{G}} d\widetilde{h}\, \widetilde{K}_{\omega t}^{\widetilde{\Delta}}(\widetilde{h}) K_s(\pi(\widetilde{h}^{-1}\widetilde{g})) \\
&= \lim_{s \downarrow 0} a(K_{\omega t}^\Delta * K_s)(\pi(\widetilde{g})) = K_{\omega t}^\Delta(\pi(\widetilde{g}))
\end{aligned}$$

for all $t > 0$. Since π is surjective one establishes that $K_t \geq a K_{\omega t}^\Delta$ for all $t > 0$. Then the Gaussian lower bounds follow from Lemma 3.4 applied to K^Δ. ∎

As an easy consequence one has the following Harnack inequalities.

COROLLARY 5.2. *If $m = 2$ and the coefficients of the operator H are real then for all multi-indices α, β and all $v, \omega > 0$ there exists an $a_{\alpha,\beta;v,\omega} > 0$ such that*

$$\sup_{g \in B'(e;vt^{1/2})} |(A^\alpha B^\beta K_t)(g)| \leq a_{\alpha,\beta;v,\omega} t^{-(|\alpha|+|\beta|)/2} \inf_{g \in B'(e;vt^{1/2})} K_{\omega t}(g)$$

uniformly for all $t > 0$.

P r o o f. It follows from Lemma 3.3 and Theorem 3.5 that one has bounds

$$\sup_{g \in B'(e;vt^{1/2})} |(A^\alpha B^\beta K_t)(g)| \leq a_{\alpha,\beta} t^{-(|\alpha|+|\beta|)/2} V(t^{1/2})^{-1}$$

for all $v, t > 0$. But Theorem 5.1 gives complementary bounds

$$\inf_{g \in B'(e;vt^{1/2})} K_{\omega t}(g) \geq a V((\omega t)^{1/2})^{-1} e^{-bv^2/\omega}$$

for all $v, t > 0$. The Harnack inequalities follow by combination of these bounds. ∎

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