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REDUCTION OF THE CODIMENSION OF A GENERIC MINIMAL SUBMANIFOLD IMMERSED IN A COMPLEX PROJECTIVE SPACE

BY

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1. Introduction. Let $\overline{M}$ be a Kaehlerian manifold with almost complex structure J and M be a Riemannian manifold isometrically immersed in $\overline{M}$. We denote by $T_x(M)$ and $T_x(M)^\perp$ the tangent space and the normal space of M respectively at a point x of M. If $JT_x(M)^\perp \subset T_x(M)$ for any point x of M, then we call M a *generic submanifold* of $\overline{M}$. If $JT_x(M)^\perp = T_x(M)$, then M is an *anti-invariant* (or *totally real*) submanifold of $\overline{M}$. If a generic submanifold is not anti-invariant, then we call it a *proper generic submanifold*. In [1] the second author proved that if the Ricci tensor S of a compact n-dimensional generic minimal submanifold M of a complex projective space CP^m satisfies $S(X,X) \geq (n-1)g(X,X) + 2g(PX,PX)$, then M is a real projective space RP^n, or M is the pseudo-Einstein real hypersurface $\pi(S^{(n+1)/2}(\sqrt{1/2}) \times S^{(n+1)/2}(\sqrt{1/2}))$, where PX is the tangential part of JX and π denotes the projection with respect to the fibration $S^1 \to S^{2m+1} \to CP^m$, $S^k(r)$ being the k-dimensional Euclidean sphere with radius r. On the other hand, Maeda [2] studied an n-dimensional complete minimal real hypersurface M with $(n-1)g(X,X) \leq S(X,X) \leq (n+1)g(X,X)$, and proved that M is congruent to $\pi(S^{(n+1)/2}(\sqrt{1/2}) \times S^{(n+1)/2}(\sqrt{1/2}))$. The purpose of the present paper is to prove the following

THEOREM 1. *Let M be a compact n-dimensional proper generic minimal submanifold of a complex m-dimensional projective space CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$ for any vector field X tangent to M, then M is a real hypersurface of CP^m, that is*, $2m-n=1$.

2. Preliminaries. Let CP^m denote the complex projective space of complex dimension m (real dimension $2m$) equipped with the standard symmetric space metric g normalized so that the maximum sectional curvature

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is four. We denote by J the almost complex structure of CP^m. Let M be a real n-dimensional Riemannian manifold isometrically immersed in CP^m. We denote by the same g the Riemannian metric tensor field induced on M from that of CP^m. Covariant differentiation with respect to the Levi-Civita connection in CP^m (resp. M) will be denoted by $\overline{\nabla}$ (resp. ∇). Then the Gauss and Weingarten formulas are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y) \quad \text{and} \quad \overline{\nabla}_X V = -A_V X + D_X V$$

for all vector fields X, Y tangent to M and every vector field V normal to M, where D denotes covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^\perp$. A and B are both called the second fundamental forms of M, and are related by $g(B(X,Y),V) = g(A_V X, Y)$. For the second fundamental form A we define its covariant derivative $\nabla_X A$ by

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

If $\mathrm{Tr} A_V = 0$ for any vector field V normal to M, then M is said to be *minimal*, where Tr denotes the trace of an operator. In the following, we assume that M is a generic submanifold of CP^m. Then the tangent space $T_x(M)$ is decomposed as follows:

$$T_x(M) = H_x(M) \oplus JT_x(M)^\perp$$

at each point x of M, where $H_x(M)$ denotes the orthogonal complement of $JT_x(M)^\perp$ in $T_x(M)$. Then we see that $H_x(M)$ is a holomorphic subspace of $T_x(M)$. If M is a real hypersurface of CP^m, then M is obviously a generic submanifold of CP^m. In the following, we put $2m - n = p$, which is the codimension of M. For a vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part of JX and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$, and F is a normal bundle valued 1-form on the tangent bundle $T(M)$. Then we see that $FPX = 0$ and $P^2 X = -X - JFX$. Moreover, we have

$$(\nabla_X P)Y = JB(X,Y) + A_{FY} X, \qquad (\nabla_X F)Y = -B(X, PY),$$

where we have put $(\nabla_X P)Y = \nabla_X (PY) - P\nabla_X Y$ and $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$. For any vector field U normal to M, we also have

$$\nabla_X JU = -PA_U X + JD_X U, \qquad B(X, JU) = -FA_U X.$$

For all vector fields U and V normal to M, we obtain

$$A_U JV = A_V JU.$$

Let R denote the Riemannian curvature tensor of M. Then we have the Gauss equation

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY \\ &\quad + 2g(X,PY)PZ + A_{B(Y,Z)}X - A_{B(X,Z)}Y.\end{aligned}$$

The Codazzi equation of M is given by

$$(\nabla_X A)_V Y - (\nabla_Y A)_V X = g(FX,V)PY - g(FY,V)PX - 2g(X,PY)JV.$$

We now define the curvature tensor $R^{\perp}$ of the normal bundle of M by $R^{\perp}(X,Y) = [D_X, D_Y] - D_{[X,Y]}$. Then we have the Ricci equation

$$\begin{aligned}&g(R^{\perp}(X,Y)U,V) \\ &\quad = g([A_U, A_V]X,Y) + g(FY,U)g(FX,V) - g(FX,U)g(FY,V).\end{aligned}$$

If $R^{\perp}$ vanishes identically, the normal connection of M is said to be *flat*.

3. Proof of the theorem. From the Gauss equation the Ricci tensor S of M is given by

$$S(X,Y) = (n-1)g(X,Y) + 3g(PX,PY) - \sum_a g(A_a X, A_a Y)$$

for all vector fields X and Y tangent to M, where we have put $A_a = A_{v_a}$, $\{v_a\}$ being an orthonormal basis of the normal space of M. By assumption we have

$$S(X,X) - (n-1)g(X,X) = 3g(PX,PX) - \sum_a g(A_a X, A_a X) \geq 0.$$

Hence we obtain, for any vector field V normal to M,

$$A_a JV = 0$$

for all a. This means that $A_U JV = 0$ for all vector fields U and V normal to M. Using the equation above, we find

$$(\nabla_X A)_U JV + A_U \nabla_X JV = (\nabla_X A)_U JV - A_U P A_V X = 0,$$

from which

$$g((\nabla_X A)_U JV, Y) = g((\nabla_X A)_U Y, JV) = g(A_U P A_V X, Y).$$

Thus we have, by the Codazzi equation,

$$2g(PX,Y)g(V,U) = g(A_U P A_V X, Y) + g(A_V P A_U X, Y).$$

In particular, we obtain

$$A_V P A_V X = PX$$

for any vector field X tangent to M and any vector field V normal to M. On the other hand, we have

$$S(PX, PX) = (n+2)g(PX, PX) - \sum_a g(A_a PX, A_a PX),$$

from which

$$\begin{aligned}\sum_a g(A_a PX, A_a PX) &= (n+2)g(PX, PX) - S(PX, PX)\\ &= \sum_a g(A_a P A_a X, PX) + (n+2-p)g(PX, PX) - S(PX, PX),\end{aligned}$$

where we have put $p = 2m - n$, which is the codimension of M. Therefore we obtain

$$\begin{aligned}\frac{1}{2}\sum_a |[P, A_a]|^2 &= (n+2-p)(n-p) - \sum_i S(Pe_i, Pe_i)\\ &= (n+2-p)(n-p) - (n+2)(n-p) + \sum_a \operatorname{Tr} A_a^2\\ &= -(n-p)p + \sum_a \operatorname{Tr} A_a^2,\end{aligned}$$

where $\{e_i\}$ denotes an orthonormal basis of the tangent space of M. By assumption we see

$$0 \leq \sum_i S(Pe_i, Pe_i) - (n-1)(n-p) = 3(n-p) - \sum_a \operatorname{Tr} A_a^2.$$

Hence we have

$$\sum_a \operatorname{Tr} A_a^2 \leq 3(n-p).$$

Consequently, we conclude that

$$\frac{1}{2}\sum_a |[P, A_a]|^2 \leq -(n-p)p + 3(n-p) = (n-p)(3-p).$$

Since M is proper, we must have $n > p$. Hence we have $p \leq 3$. Suppose $p = 3$. Then $PA_a = A_a P$ for all a, and hence

$$A_a P A_a X = A_a^2 PX = PX$$

for all a. This implies that $\sum_a \mathrm{Tr} A_a^2 = n - p$. Moreover, we have

$$\begin{aligned}S(PX, PX) &= (n+2)g(PX, PX) - \sum_a g(A_a PX, A_a PX)\\ &= (n+2-p)g(PX, PX) = (n-1)g(PX, PX),\\ S(JV, JV) &= (n-1)g(V, V).\end{aligned}$$

Therefore, M is Einstein. Since we have

$$g(A_aPA_bX,Y)+g(A_bPA_aX,Y)=2g(PX,Y)g(v_a,v_b),$$

it follows that

$$g(A_aPX,A_bPX)=0$$

for $a\neq b$. Suppose $A_aX=kX$ for $X\in PT_x(M)$. Then $A_a^2X=k^2X=X$. Hence we have $k=\pm 1\neq 0$. Moreover, we obtain

$$0=g(A_aX,A_bX)=kg(X,A_bX),$$

from which $g(A_bX,X)=0$ for $b\neq a$. This is a contradiction to the fact $A_a^2X=X$ for all a and for $X\in PT_x(M)$. Thus we must have $p\neq 3$. We next suppose that $p=2$. Then

$$\begin{aligned}
&\sum_{a,i,j} g(\nabla_jJv_a,e_i)g(e_j,\nabla_iJv_a)\\
&\quad=\sum_{a,i,j}[g(PA_ae_j,e_i)g(e_j,PA_ae_i)-g(PA_ae_j,e_i)g(e_j,JD_iv_a)\\
&\qquad-g(JD_jv_a,e_i)g(e_j,PA_ae_i)+g(JD_jv_a,e_i)g(e_j,JD_iv_a)]\\
&\quad=-\sum_{a,j}g(PA_ae_j,A_aPe_j)+\sum_{a,i,j}g(D_jv_a,Je_i)g(Je_j,D_iv_a)\\
&\quad=\sum_a \mathrm{Tr}(PA_a)^2+\sum_{a,b,c}g(D_{Jb}v_a,v_c)g(v_b,D_{Jc}v_a)\\
&\quad=\sum_a \mathrm{Tr}(PA_a)^2+\sum_{a,b}g(D_{Jb}v_a,v_b)^2,
\end{aligned}$$

where we have put ∇_j, D_j, D_{Jb} as $\nabla_{e_j}, D_{e_j}, D_{Jv_b}$ to simplify notation, and $a,b,c=1,2$. We also have

$$\begin{aligned}
\sum_a(\mathrm{div}\,Jv_a)^2&=\sum_{a,i,j}g(\nabla_iJv_a,e_i)g(\nabla_jJv_a,e_j)\\
&=\sum_{a,i,j}g(JD_iv_a,e_i)g(JD_jv_a,e_j)\\
&=\sum_{a,i,j}g(D_iv_a,Je_i)g(D_jv_a,Je_j)=\sum_{a,b}g(D_{Jb}v_a,v_b)^2.
\end{aligned}$$

Generally, we have (cf. Yano [3])

$$\begin{aligned}
&\mathrm{div}(\nabla_XX)-\mathrm{div}((\mathrm{div}\,X)X)\\
&\qquad=S(X,X)+\sum_{i,j}g(\nabla_jX,e_i)g(e_j,\nabla_iX)-(\mathrm{div}\,X)^2.
\end{aligned}$$

Using the equations above, we obtain

$$\begin{aligned}\sum_a \operatorname{div}(\nabla_{Ja} Jv_a) - \sum_a \operatorname{div}((\operatorname{div} Jv_a)Jv_a) &= \sum_a S(Jv_a, Jv_a) + \sum_a \operatorname{Tr}(PA_a)^2 \\ &= \sum_a (n-1) + \sum_a \operatorname{Tr}(PA_a)^2 \\ &= 2(n-1) + \frac{1}{2}\sum_a |[P, A_a]|^2 + \sum_a \operatorname{Tr}(P^2 A_a^2) \\ &= 2(n-1) - 2(n-2) + \sum_a \operatorname{Tr} A_a^2 + \sum_a \operatorname{Tr}(P^2 A_a^2) \geq 2.\end{aligned}$$

If M is compact, the equation above gives a contradiction. Thus we have $p \neq 2$. Therefore, we must have $p = 1$, and hence M is a real hypersurface of CP^m. This proves Theorem 1.

From Theorem 1 and a theorem of Maeda [2] we have

THEOREM 2. *Let M be a compact n-dimensional proper generic minimal submanifold of a complex m-dimensional projective space CP^m. If the Ricci tensor S of M satisfies $(n-1)g(X,X) \leq S(X,X) \leq (n+1)g(X,X)$ for any vector field X tangent to M, then M is a pseudo-Einstein real hypersurface $\pi(S^{(n+1)/2}(\sqrt{1/2}) \times S^{(n+1)/2}(\sqrt{1/2}))$.*

REFERENCES

[1] M. Kon, *Generic minimal submanifolds of a complex projective space*, Bull. London Math. Soc. 12 (1980), 355–360.

[2] S. Maeda, *Real hypersurfaces of a complex projective space II*, Bull. Austral. Math. Soc. 29 (1984), 123–127.

[3] K. Yano, *On harmonic and Killing vector fields*, Ann. of Math. 55 (1952), 38–45.

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