1. Introduction. Let $\mathcal{M}$ be a Kaehlerian manifold with almost complex structure $J$ and $\mathcal{M}$ be a Riemannian manifold isometrically immersed in $\mathcal{M}$. We denote by $T_x(\mathcal{M})$ and $T_x(\mathcal{M})^\perp$ the tangent space and the normal space of $\mathcal{M}$ respectively at a point $x$ of $\mathcal{M}$, then we call $\mathcal{M}$ a generic submanifold of $\mathcal{M}$. If $JT_x(\mathcal{M})^\perp \subset T_x(\mathcal{M})$ for any point $x$ of $\mathcal{M}$, then $\mathcal{M}$ is an anti-invariant (or totally real) submanifold of $\mathcal{M}$. If a generic submanifold is not anti-invariant, then we call it a proper generic submanifold. In [1] the second author proved that if the Ricci tensor $S$ of a compact $n$-dimensional generic minimal submanifold $\mathcal{M}$ of a complex projective space $\mathbb{P}^m$ satisfies $S(X,X) \geq (n-1)g(X,X) + 2g(PX,PX)$, then $\mathcal{M}$ is a real projective space $\mathbb{R}^n$, or $\mathcal{M}$ is the pseudo-Einstein real hypersurface $\pi(S((n+1)/2) \times S((n+1)/2)$, where $PX$ is the tangential part of $JX$ and $\pi$ denotes the projection with respect to the fibration $S^1 \to S^{2m+1} \to \mathbb{P}^m$, $S^k(r)$ being the $k$-dimensional Euclidean sphere with radius $r$. On the other hand, Maeda [2] studied an $n$-dimensional complete minimal real hypersurface $\mathcal{M}$ with $(n-1)g(X,X) \leq S(X,X) \leq (n+1)g(X,X)$, and proved that $\mathcal{M}$ is congruent to $\pi(S((n+1)/2) \times S((n+1)/2)$, (1/2)). The purpose of the present paper is to prove the following

**Theorem 1.** Let $\mathcal{M}$ be a compact $n$-dimensional proper generic minimal submanifold of a complex $m$-dimensional projective space $\mathbb{P}^m$. If the Ricci tensor $S$ of $\mathcal{M}$ satisfies $S(X,X) \geq (n-1)g(X,X)$ for any vector field $X$ tangent to $\mathcal{M}$, then $\mathcal{M}$ is a real hypersurface of $\mathbb{P}^m$, that is, $2m-n=1$.

2. Preliminaries. Let $\mathbb{P}^m$ denote the complex projective space of complex dimension $m$ (real dimension $2m$) equipped with the standard symmetric space metric $g$ normalized so that the maximum sectional curvature

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is four. We denote by $J$ the almost complex structure of $\mathbb{C}P^m$. Let $M$ be a real $n$-dimensional Riemannian manifold isometrically immersed in $\mathbb{C}P^m$. We denote by the same $g$ the Riemannian metric tensor field induced on $M$ from that of $\mathbb{C}P^m$. Covariant differentiation with respect to the Levi-Civita connection in $\mathbb{C}P^m$ (resp. $M$) will be denoted by $\nabla$ (resp. $\nabla$). Then the Gauss and Weingarten formulas are respectively given by

$$\nabla_X Y = \nabla_X Y + B(X,Y) \quad \text{and} \quad \nabla_X V = -A_V X + D_X V$$

for all vector fields $X,Y$ tangent to $M$ and every vector field $V$ normal to $M$, where $D$ denotes covariant differentiation with respect to the linear connection induced in the normal bundle $T(M) \perp$. $A$ and $B$ are both called the second fundamental forms of $M$, and are related by $g(B(X,Y),V) = g(A_V X,Y)$. For the second fundamental form $A$ we define its covariant derivative $\nabla_X A$ by

$$(\nabla_X A)V = \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

If $\text{Tr}A_V = 0$ for any vector field $V$ normal to $M$, then $M$ is said to be minimal, where $\text{Tr}$ denotes the trace of an operator. In the following, we assume that $M$ is a generic submanifold of $\mathbb{C}P^m$. Then the tangent space $T_x(M)$ is decomposed as follows:

$$T_x(M) = H_x(M) \oplus JT_x(M) \perp$$

at each point $x$ of $M$, where $H_x(M)$ denotes the orthogonal complement of $JT_x(M) \perp$ in $T_x(M)$. Then we see that $H_x(M)$ is a holomorphic subspace of $T_x(M)$. If $M$ is a real hypersurface of $\mathbb{C}P^m$, then $M$ is obviously a generic submanifold of $\mathbb{C}P^m$. In the following, we put $2m - n = p$, which is the codimension of $M$. For a vector field $X$ tangent to $M$, we put

$$JX = PX + FX,$$

where $PX$ is the tangential part of $JX$ and $FX$ the normal part of $JX$. Then $P$ is an endomorphism on the tangent bundle $T(M)$, and $F$ is a normal bundle valued 1-form on the tangent bundle $T(M)$. Then we see that $FPX = 0$ and $P^2 X = -X - JFX$. Moreover, we have

$$(\nabla_X P)Y = JB(X,Y) + A_F Y,$$

where we have put $(\nabla_X P)Y = \nabla_X (P Y) - P \nabla_X Y$ and $(\nabla_X F)Y = D_X (FY) - F \nabla_X Y$. For any vector field $U$ normal to $M$, we also have

$$\nabla_X U = -PA_U X + JD_X U,$$

$$B(X,U) = -FA_U X.$$

For all vector fields $U$ and $V$ normal to $M$, we obtain

$$A_U JV = A_V JU.$$
Let $R$ denote the Riemannian curvature tensor of $M$. Then we have the Gauss equation

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(PY, Z)PX - g(PX, Z)PY + 2g(X, PY)PZ + A_B(Y, Z)X - A_B(X, Z)Y.$$ 

The Codazzi equation of $M$ is given by

$$(\nabla_X A)V Y - (\nabla_Y A)V X = g(FX, V)PY - g(FY, V)PX - 2g(X, PY)JV.$$ 

We now define the curvature tensor $R^\perp$ of the normal bundle of $M$ by $R^\perp(X, Y) = [D_X, D_Y] - D_{[X, Y]}$. Then we have the Ricci equation

$$g(R^\perp(X, Y)U, V) = g([A_U, A_V]X, Y) + g(FY, U)g(FX, V) - g(FX, U)g(FY, V).$$ 

If $R^\perp$ vanishes identically, the normal connection of $M$ is said to be flat.

3. **Proof of the theorem.** From the Gauss equation the Ricci tensor $S$ of $M$ is given by

$$S(X, Y) = (n - 1)g(X, Y) + 3g(PX, PY) - \sum a g(A_a X, A_a Y)$$ 

for all vector fields $X$ and $Y$ tangent to $M$, where we have put $A_a = A_{v_a}$, $\{v_a\}$ being an orthonormal basis of the normal space of $M$. By assumption we have

$$S(X, X) = (n - 1)g(X, X) = 3g(PX, PX) - \sum a g(A_a X, A_a X) \geq 0.$$ 

Hence we obtain, for any vector field $V$ normal to $M$,

$$A_a JV = 0$$ 

for all $a$. This means that $A_U JV = 0$ for all vector fields $U$ and $V$ normal to $M$. Using the equation above, we find

$$(\nabla_X A)U JV + A_U \nabla_X JV = (\nabla_X A)U JV - A_U PA_V X = 0,$$ 

from which

$$g((\nabla_X A)U JV, Y) = g((\nabla_X A)U Y, JV) = g(A_U PA_V X, Y).$$ 

Thus we have, by the Codazzi equation,

$$2g(PX, Y)g(V, U) = g(A_U PA_V X, Y) + g(A_V PA_U X, Y).$$ 

In particular, we obtain

$$A_V PA_U X = PX.$$
for any vector field $X$ tangent to $M$ and any vector field $V$ normal to $M$. On the other hand, we have

$$S(PX, PX) = (n + 2)g(PX, PX) - \sum_a g(A_a PX, A_a PX),$$

from which

$$\sum_a g(A_a PX, A_a PX) = (n + 2)g(PX, PX) - S(PX, PX)$$

$$= \sum_a g(A_a PA_a X, PX) + (n + 2 - p)g(PX, PX) - S(PX, PX),$$

where we have put $p = 2m - n$, which is the codimension of $M$. Therefore we obtain

$$\frac{1}{2} \sum_a [P, A_a]^2 = (n + 2 - p)(n - p) - \sum_i S(\epsilon_i, \epsilon_i)$$

$$= (n + 2 - p)(n - p) - (n + 2)(n - p) + \sum_a \text{Tr} A_a^2$$

$$= -(n - p)p + \sum_a \text{Tr} A_a^2,$$

where $\{\epsilon_i\}$ denotes an orthonormal basis of the tangent space of $M$. By assumption we see

$$0 \leq \sum_i S(\epsilon_i, \epsilon_i) - (n - 1)(n - p) = 3(n - p) - \sum_a \text{Tr} A_a^2.$$ 

Hence we have

$$\sum_a \text{Tr} A_a^2 \leq 3(n - p).$$

Consequently, we conclude that

$$\frac{1}{2} \sum_a [P, A_a]^2 \leq -(n - p)p + 3(n - p) = (n - p)(3 - p).$$

Since $M$ is proper, we must have $n > p$. Hence we have $p \leq 3$. Suppose $p = 3$. Then $PA_a = A_a P$ for all $a$, and hence

$$A_a PA_a X = A_a^2 PX = PX$$

for all $a$. This implies that $\sum_a \text{Tr} A_a^2 = n - p$. Moreover, we have

$$S(PX, PX) = (n + 2)g(PX, PX) - \sum_a g(A_a PX, A_a PX)$$

$$= (n + 2 - p)g(PX, PX) = (n - 1)g(PX, PX),$$

$$S(JV, JV) = (n - 1)g(V, V).$$
Therefore, $M$ is Einstein. Since we have

$$g(A_aPA_bX, Y) + g(A_bPA_aX, Y) = 2g(PX, Y)g(v_a, v_b),$$

it follows that

$$g(A_aPX, A_bPX) = 0$$

for $a \neq b$. Suppose $A_aX = kX$ for $X \in PT_x(M)$. Then $A^2_aX = k^2X = X$. Hence we have $k = \pm 1 \neq 0$. Moreover, we obtain

$$0 = g(A_aX, A_bX) = kg(X, A_bX),$$

from which $g(A_aX, X) = 0$ for $b \neq a$. This is a contradiction to the fact $A^2_aX = X$ for all $a$ and for $X \in PT_x(M)$. Thus we must have $p = 3$. We next suppose that $p = 2$. Then

$$\sum_{a,i,j} g(\nabla_j Jv_a, e_i)g(e_j, \nabla_i Jv_a)$$

$$= \sum_{a,i,j} [g(PA_ae_j, e_i)g(e_j, PA_ae_i) - g(PA_ae_j, e_i)g(e_j, JD_i v_a)$$

$$- g(JD_j v_a, e_i)g(e_j, PA_a e_i) + g(JD_j v_a, e_i)g(e_j, JD_i v_a)]$$

$$= \sum_{a,i,j} g(PA_a e_j, A_a Pe_i) + \sum_{a,i,j} g(D_j v_a, Je_i)g(Je_j, D_i v_a)$$

$$= \sum_{a} \text{Tr}(PA_a)^2 + \sum_{a,b,c} g(D_{jb} v_a, v_c)g(v_b, D_{jc} v_a)$$

$$= \sum_{a} \text{Tr}(PA_a)^2 + \sum_{a,b} g(D_{jb} v_a, v_b)^2,$$

where we have put $\nabla_j, D_j, D_{jb}$ as $\nabla_{e_j}, D_{e_j}, D_{je_b}$ to simplify notation, and $a, b, c = 1, 2$. We also have

$$\sum_{a} (\text{div} Jv_a)^2 = \sum_{a,i,j} g(\nabla_j Jv_a, e_i)g(\nabla_j Jv_a, e_j)$$

$$= \sum_{a,i,j} g(JD_j v_a, e_i)g(JD_j v_a, e_j)$$

$$= \sum_{a,i,j} g(D_j v_a, Je_i)g(D_j v_a, Je_j) = \sum_{a,b} g(D_{jb} v_a, v_b)^2.$$

Generally, we have (cf. Yano [3])

$$\text{div}(\nabla X) - \text{div}((\text{div} X)X)$$

$$= S(X, X) + \sum_{i,j} g(\nabla_j X, e_i)g(e_j, \nabla_i X) - (\text{div}X)^2.$$
Using the equations above, we obtain

\[\sum_{a} \text{div}(\nabla_{Ja} Jv_{a}) - \sum_{a} \text{div}((\text{div} Jv_{a}) Jv_{a})\]

\[= \sum_{a} S(Jv_{a}, Jv_{a}) + \sum_{a} \text{Tr}(PA_{a})^{2}\]

\[= \sum_{a} (n-1) + \sum_{a} \text{Tr}(PA_{a})^{2}\]

\[= 2(n-1) + \frac{1}{2} \sum_{a} ||[P, A_{a}]||^{2} + \sum_{a} \text{Tr}(P^{2}A_{a}^{2})\]

\[= 2(n-1) - 2(n-2) + \sum_{a} \text{Tr}A_{a}^{2} + \sum_{a} \text{Tr}(P^{2}A_{a}^{2}) \geq 2.\]

If \(M\) is compact, the equation above gives a contradiction. Thus we have \(p \neq 2\). Therefore, we must have \(p = 1\), and hence \(M\) is a real hypersurface of \(CP^{m}\). This proves Theorem 1.

From Theorem 1 and a theorem of Maeda [2] we have

**Theorem 2.** Let \(M\) be a compact \(n\)-dimensional proper generic minimal submanifold of a complex \(m\)-dimensional projective space \(CP^{m}\). If the Ricci tensor \(S\) of \(M\) satisfies \((n-1)g(X, X) \leq S(X, X) \leq (n+1)g(X, X)\) for any vector field \(X\) tangent to \(M\), then \(M\) is a pseudo-Einstein real hypersurface \(\pi(S^{(n+1)/2}(\sqrt{1/2}) \times S^{(n+1)/2}(\sqrt{1/2})).\)

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