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LOWER SEMICONTINUOUS DIFFERENTIAL INCLUSIONS. ONE-SIDED LIPSCHITZ APPROACH

BY

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Some properties of differential inclusions with lower semicontinuous right-hand side are considered. Our essential assumption is the one-sided Lipschitz condition introduced in [4]. Using the main idea of [10], we extend the well known relaxation theorem, stating that the solution set of the original problem is dense in the solution set of the relaxed one, under assumptions essentially weaker than those in the literature. Applications in optimal control are given.

1. Preliminaries. Refined lemma of Pliś. We investigate differential inclusions having the form

$$\dot{x}(t) \in F(t,x), \quad x(0) = x_0, \quad t \in [0,T] \quad (\text{usually } T = 1); \tag{1}$$

here $x \in E$ (a Banach space). It is well known that under some compactness type assumptions (1) admits a solution when $F(\cdot,\cdot)$ is almost lower semicontinuous (ALSC). However, the solution set may depend neither lower nor upper semicontinuously on parameters. Moreover, the solution set of the *relaxed system*

$$\dot{x}(t) \in \overline{\mathrm{co}}\, F(t,x), \quad x(0) = x_0, \tag{2}$$

is not closed in general. When $F(t,\cdot)$ is continuous the last system has compact solution set. When $\overline{\mathrm{co}}\, F(t,\cdot)$ is Lipschitz the solution set of (1) is dense in the solution set of (2). We extend this and related results to the case of lower semicontinuous and one-sided Lipschitz right-hand sides. Our results are based on a refined version of a lemma of Pliś [8]. The quasitrajectories introduced by Ważewski are essentially used in the paper. From the fact that the quasitrajectory set of (1) contains the relaxed solution set we obtain a short proof of the relaxation theorem.

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The paper consists of three sections. In this section we give the main definitions and notations and prove a refined version of the lemma of Pliś. The main results for (1) are presented in the second section. In the last section applications in optimal control are given.

Note that all the concepts not discussed in detail in the sequel can be found in [3]. Let E be a Banach space. Denote by $P_f(E)$ the set of all compact nonempty subsets of E. If $A \in P_f(E)$, then $\operatorname{clco} A$ (or $\overline{\operatorname{co}}\, A$) is the closed convex hull of A. A set valued map $F : E \to P_f(E)$ is said to be *Lower SemiContinuous* (LSC) if for every $x \in E$, $y \in F(x)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x') \cap U(y, \varepsilon) \neq \emptyset$ for every $x' \in U(x, \delta)$, where $U(x, \varepsilon)$ is the open ball with radius ε, centered at x. If for every $\varepsilon > 0$ there exists $I_\varepsilon \subset I$ with $\operatorname{meas}(I \setminus I_\varepsilon) \leq \varepsilon$ such that $F : I_\varepsilon \times E$ is LSC then F is called ALSC. For $x, y \in E$ we define $[x, y]_+ := \lim_{h \to 0^+} h^{-1}\{|x + hy| - |x|\}$. From [6], p. 7, we know that $[\cdot, \cdot]_+$ is upper semicontinuous as a real valued function and nonexpansive in the second variable. We set $d(b, A) = \inf_{a \in A} |b - a|$.

Definition 1.1. The multimap $F : I \times E \to P_f(E)$ is said to be *Almost Lower SemiContinuous* (ALSC) if for every $\varepsilon > 0$ there exists a compact set $I_\varepsilon \subset I$ with Lebesgue measure $\mu(I \setminus I_\varepsilon) \leq \varepsilon$ such that $F(\cdot, \cdot)$ is lower semicontinuous (LSC) on $I_\varepsilon \times E$.

We will use the following assumptions.

A1. $F : I \times E \to P_f(E)$ is ALSC and $|F(t, x)| \leq \lambda(t)\{1 + |x|\}$ for every $(t, x) \in I \times E$, where $\lambda(\cdot)$ is integrable and positive.

A2. There exists a Kamke function $u : I \times \mathbb{R}^+ \to \mathbb{R}$ such that for every $x, y \in E$ and every $f_x \in F(t, x)$ there exists $f_y \in F(t, y)$ with $[x - y, f_x - f_y]_+ \leq u(t, |x - y|)$.

We recall that a Carathéodory function $u(\cdot, \cdot)$ is called a *Kamke function* iff it is integrably bounded on bounded sets, $u(t, 0) \equiv 0$ and the unique solution of $\dot{s}(t) = u(t, s(t))$, $s(0) = 0$, is $s(t) \equiv 0$.

A3. There exists a Kamke function $w : I \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\lim_{h \to 0} \chi(F([t, t + h] \times A)) \leq w(t, \chi(A)) \quad \text{for every bounded } A \subset E.$$

Here $\chi(\cdot)$ denotes the Hausdorff measure of noncompactness.

Remark 1.1. A2 is the one-sided Lipschitz condition. For E^* strongly convex it becomes

$$\sigma(j(x - y), F(t, x)) - \sigma(j(x - y), F(t, y)) \leq u(t, |x - y|)|x - y|$$

where $\sigma(x, A) = \sup_{a \in A} \langle x, a \rangle$ denotes the support function and $j(x) = \{l \in E^* : \langle l, x \rangle = |x|^2 = |l|^2\}$ is the normalized duality map. More general dissipative conditions for ordinary differential equations are used in [6]. In the literature the following definition for one-sided Lipschitz multimaps is used:

For every $x, y \in E$ and every $f_x \in F(t,x), f_y \in F(t,y)$,

$$[x-y, f_x - f_y]_+ \le w(t, |x-y|).$$

However, if $F(t,\cdot)$ is LSC and one-sided Lipschitz then $F(\cdot,\cdot)$ is single valued.

THEOREM 1.1. *Suppose that F satisfies* A1–A3. *Let $\varepsilon > 0$ and let $f(\cdot)$ be a positive L_1-function. If $x(\cdot)$ is an AC function with $d(\dot{x}(t), F(t,x(t))) \le f(t)$ for a.e. $t \in I$, then there exists a solution $y(\cdot)$ of* (1) *such that $|x(t) - y(t)| \le r(t)$, where $r(0) = |x_0 - y_0|$ and $\dot{r}(t) = u(t,r) + f(t) + \varepsilon$.*

P r o o f. We claim that the following map is ALSC and nonempty compact valued:

$$\Gamma(t,w) := \mathrm{cl}\{v \in F(t,w) : [x(t)-w, \dot{x}(t)-v]_+ < u(t,|x(t)-w|) + f(t) + \varepsilon\}.$$

If $z \in F(t,x(t))$ and $|z - \dot{x}(t)| \le f(t)$, then there exists $v \in F(t,w)$ such that $[x(t)-w, z-v]_+ \le u(t,|x(t)-w|)$. Therefore $[x(t)-w, \dot{x}(t)-v]_+ \le f(t) + u(t,|x(t)-w|)$, i.e. $\Gamma(t,x)$ is nonempty compact valued.

From the Scorza Dragoni and Lusin's properties we see that for every $\varepsilon > 0$ there exists a compact set $I_\varepsilon \subset I$ with $\mu(I \setminus I_\varepsilon) \le \varepsilon$, such that $F(\cdot,\cdot)$ is LSC on $I_\varepsilon \times E$, $u(\cdot,\cdot)$ is continuous on $I_\varepsilon \times \mathbb{R}^+$ and $\dot{x}(\cdot), f(\cdot)$ are continuous on I_ε. Therefore it remains to show that $\Gamma(\cdot,\cdot)$ is LSC on $I_\varepsilon \times E$.

Let $l \in \Gamma(t,w)$ $(t \in I_\varepsilon)$ and let $[x(t)-w, \dot{x}(t)-l]_+ \le u(t,|x(t)-w|) + f(t) + \varepsilon - \gamma$, where $\gamma > 0$. Since $F(\cdot,\cdot)$ is LSC and $[\cdot,\cdot]_+$ is USC, there exist $\tau > t$ and $\mu > 0$ such that

$$[x(t') - w', \dot{x}(t') - l']_+ - [x(t)-w, \dot{x}(t) - l]_+ < \gamma$$

whenever $|l'-l| < \mu, |w'-w| < \mu$ and $t' \in I_\varepsilon \cap [t,\tau)$. Thus there exists $l' \in F(t',u')$ such that

$$[x(t') - u', \dot{x}(t') - l']_+ < u(t', |x(t') - u'|) + f(t') + \varepsilon.$$

Therefore $l' \in \Gamma(t', x(t'))$ and hence $\Gamma(\cdot,\cdot)$ is ALSC. ∎

2. Main results. In this section we prove the relaxation theorem and the continuous dependence for (1).

DEFINITION 2.1. The absolutely continuous (AC) function $x(\cdot)$ is said to be a *quasitrajectory* if there exists a sequence $\{x_i(\cdot)\}_{i=1}^{\infty}$ such that $d(\dot{x}_i(t), F(t,x_i(t))) \to 0$ for a.e. $t \in I$ and $x_i(\cdot) \to x(\cdot)$ uniformly on I.

The next theorem generalizes Theorem 3.3.1 of [9], where $\overline{\mathrm{co}}\, F(t,\cdot)$ is assumed to be Lipschitz, and Theorem 1 of [7], where $F(t,\cdot)$ is assumed to be Lipschitz.

THEOREM 2.1. *Under assumptions* A1–A3 *the solution set of* (1) *is dense in the quasitrajectory set of* (2).

P r o o f. The fact that the quasitrajectory sets of (1) and (2) coincide has a standard proof. In view of Theorem 1.1 we have to show that under assumptions A1 and A3 every relaxed solution is a quasitrajectory.

Let $x(\cdot)$ be a relaxed solution. The set $\bigcup_{t\in I} F(t,x(t))$ is precompact by A3. Therefore $F(\cdot,x(\cdot))$ is strongly measurable. Thus there exists a sequence $G_i : I \to P_f(E)$ of piecewise constant maps such that $d(G_i(t), F(t,x(t))) \to 0$ for a.e. $t \in I$. From the Lyapunov and Vitali theorems there exists a sequence $g_k(t) \in G_k(t)$ of piecewise constant functions with $y_k(t) = x_0 + \int_0^t g_k(t)\,d\tau$ which converges uniformly to $x(\cdot)$. By A1 and the dominated convergence theorem, $D^*_{\mathrm{H}}(F(t,y_k(t)),F(t,x(t))) \to 0$ a.e. on I. Here $D^*_{\mathrm{H}}(A,B) := \max_{a\in A}\min_{b\in B}|b-a|$. Thus $D^*_{\mathrm{H}}(F(t,y_k(t)),G_k(t)) \to 0$ a.e. on I and hence $d(\dot y_k(t),F(t,y_k(t))) \to 0$ for a.e. $t \in I$. ∎

Let E be reflexive. With the help of the recent result of Bressan and Staicu [1] one can prove Theorem 2.1 for

$$\dot x(t) \in Ax + F(t,x(t)), \qquad x(0) = x_0, \tag{3}$$

where A is m-accretive and generates a compact semigroup, while F satisfies A1–A2.

THEOREM 2.1′. *The set of integral solutions of* (3) (see [1]) *is dense in the quasitrajectory set.*

However, in general Banach spaces the quasitrajectory set of (3) does not coincide with the quasitrajectory set of the convexified problem. Let M be a metric space with a distance function $\varrho(\cdot,\cdot)$. For $\alpha \in M$ consider the differential inclusion

$$\dot x(t) \in F(t,x(t),\alpha), \qquad x(0) = x^\alpha. \tag{4}$$

PROPOSITION 2.1. *Suppose that* $F(\cdot,\cdot,\cdot)$ *is ALSC and satisfies* A2, A3 *uniformly in* α. *If* $\alpha \to \beta$ *and if* $x^\alpha \to x^\beta$ *then for every solution* x_β *of* (4β) *there exists a net* $x_\alpha(\cdot)$ *of solutions of* (4α) *converging uniformly to* $x_\beta(\cdot)$.

Proposition 2.1 follows easily from Theorem 1.1 and the proof is omitted.

R e m a r k 2.1. Obviously the quasitrajectory set of (4α) depends continuously on α whenever $F(t,x,\cdot)$ is continuous (in α).

When A2 does not hold Theorem 2.1 and Proposition 2.1 may not be true. Furthermore, there are multimaps F which are not Lipschitz but satisfy A2.

EXAMPLE 2.1. Let $\alpha \in \mathbb{R}$ and let

$$\dot x \in \{-1,1\}, \qquad \dot y \in |\alpha|x^2 - \sqrt[3]{y}.$$

Obviously $F(\cdot,\cdot,\cdot)$ satisfies all the conditions of Theorem 2.1 and Proposition 2.1 but F is not Lipschitz. When

$$\dot{x} \in \{-1, 1\}, \quad \dot{y} \in |\alpha| x^2 + \sqrt[3]{y},$$

Theorem 2.1 and Proposition 2.1 do not hold. Here A2 is not satisfied. ∎

Define $H(t,x) := \overline{\mathrm{co}} \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, F(t, x + \overline{U}(0,\varepsilon))$. If $\overline{\mathrm{co}}\, F(t,\cdot)$ is continuous then the quasitrajectory set coincides with the solution set of

$$\dot{x}(t) \in H(t, x(t)), \quad x(0) = x_0. \tag{5}$$

In the other case that need not be true.

EXAMPLE 2.2. Let $A \subset \mathbb{R}$ be open dense with $\mathrm{meas}(A) \le \varepsilon$. Define $F(x) := [0,1]$ for $x \in A$, and $F(x) = 0$ elsewhere. Consider the system

$$\dot{x}(t) \in F(x), \quad x(0) = 0.$$

Obviously $H(x) = [0,1]$, $x \in I$. But if $y(\cdot)$ is a quasitrajectory, then $y(t) \le \varepsilon$. On the other hand, $y(t) = t,\ t \in [0,1]$, is a solution of (5). ∎

For E^* uniformly convex, however, one can show that the two sets coincide. We will prove it for simplicity in the case of E a Hilbert space and $w(t,s) \equiv Ls$, since in the general case the method is the same.

THEOREM 2.2. *Let E be a Hilbert space and let $w(t,s) \equiv Ls$. Under assumptions* A1–A3 *the quasitrajectory set of* (1) *coincides with the solution set of* (5).

Proof. Note first that $H(\cdot,x)$ has a strongly measurable selector and $H(t,\cdot)$ is USC. Thus (5) has nonempty compact R_δ solution set. Let $x(\cdot)$ be AC with $\dot{x}(t) \in F(t, B(x,\varepsilon)) + B(0,\varepsilon)$, $x(0) = x_0$. Define the multimap

$$\Gamma(t,y) = \mathrm{cl}\{u \in F(t,y) : \langle x(t) - y, \dot{x}(t) - u\rangle < (L|x-y| + \delta)^2 + 2\delta\}.$$

Obviously $\Gamma(\cdot,\cdot)$ is nonempty compact valued and almost LSC (see the proof of Theorem 1.1). Let $\dot{y} \in \Gamma(t,y)$, $y(0) = x_0$. Therefore

$$\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t)\rangle < L|x(t) - y(t)|^2 + 2LM\delta + 2NL\delta + 2\delta.$$

Thus there exists a constant C such that $|x(t) - y(t)| \le C\delta^{1/2}\exp(L)$. It is straightforward to show (see [2]) that for every solution $x(\cdot)$ of (5) there exist sequences $\{\varepsilon_i\}_{i=1}^\infty$ and $\{x_i(\cdot)\}_{i=1}^\infty$ such that $\varepsilon_i \to 0$ and

$$\dot{x}_i(t) \in F(t, \overline{U}(x, \varepsilon_i)) + \overline{U}(0, \varepsilon_i). \ ∎$$

Remark 2.2. If $H(\cdot,\cdot)$ is compact valued then using the main idea of [5] one can prove Theorem 2.2 without A3. Indeed, let $P = \{p_i\}_{i=1}^n$ and $Q = \{q_j\}_{j=1}^m$ be two subdivisions of I with $P \subset Q$. Consider the following differential inclusions:

$$\dot{x}(t) \in H(t, x(p_i)), \quad x(0) = x_0, \tag{6}$$

$$\dot{y}(t) \in H(t, y(q_j)), \qquad y(0) = x_0. \tag{7}$$

It is easy to show that there exist constants M and N such that $|x(t)| \le M$ and $|H(t,x)| \le N$ for all solutions of (6) and (7) and all subdivisions P and Q. Let $x(\cdot)$ be a solution of (6). We set $\dot{y}(t) \equiv \dot{x}(t)$ on $[0, q_1]$. By virtue of A2 there exists a strongly measurable selection $f(t) \in H(t, y(q_1))$ such that

$$\langle x_0 - y(q_1), \dot{x}(t) - f(t)\rangle \le L|x_0 - y(q_1)|^2. \tag{8}$$

We let $y(t) = y(q_1) + \int_{q_1}^t f(s)\,ds$. Then

$$\begin{aligned}
&\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t)\rangle \\
&\qquad \le L|x_0 - y(q_1)|^2 + \{|x_0 - x(t)| + |y(q_1) - y(t)|\}|\dot{x}(t) - \dot{y}(t)| \\
&\qquad \le L|x(t) - y(t)|^2 + L\{|x_0 + x(t)| + |y(q_1) + y(t)|\} \\
&\qquad\qquad \times \{|x_0 - x(t)| + |y(q_1) - y(t)|\} \\
&\qquad\qquad + \{|x_0 - x(t)| + |y(q_1) - y(t)|\}|\dot{x}(t) - \dot{y}(t)|.
\end{aligned}$$

Thus

$$\begin{aligned}
\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t)\rangle &\le L|x(t) - y(t)|^2 + N(4LM + 2N)(t + t - q_1) \\
&\le L|x(t) - y(t)|^2 + \widetilde{C}t
\end{aligned}$$

where $\widetilde{C}$ is a constant (depending on N, L, M but not on t). One can continue in the same fashion. Thus for $q_j \in [p_i, p_{i+1})$ we get $\dot{y}(t)$ such that (8) holds with x_0 replaced by $x(p_i)$ and q_1 by q_j. Define $\delta = \max_i (p_i - p_{i-1})$, where $p_0 = 0$. Then

$$\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t)\rangle \le L|x(t) - y(t)|^2 + \widetilde{C}\delta$$

on I. If $\widehat{L} = \max(L, 0)$, then $|x(t) - y(t)|^2 \le 2\widetilde{C}\exp(\widehat{L}t)$. Thus there exists a constant C (not depending on δ) such that $|x(t) - y(t)| \le C\delta^{1/2}$. The latter implies that if R_{1P} is a compact subset of the solution set of (6) then there exists a compact subset R_{1Q} of the solution set of (7) with $D_{\mathrm{H}}(R_{1P}, R_{1Q}) \le C\delta^{1/2}$. Here the Hausdorff distance is in $P_f(C(I,E))$.

Now consider the sequence of subdivisions $P_i \subset P_{i+1}$ with $P_n = \{i/2^n\}_{i=1}^{2^n}$. Then $D_{\mathrm{H}}(R_{1P_k}, R_{1P_{k+1}}) \le C/2^k$. Thus $\{R_{1P_n}\}_{n=1}^{\infty}$ is a Cauchy sequence in $P_f(C(I,E))$. Let P be its limit. Then $P \not\equiv \emptyset$ is a compact subset of $C(I,E)$. It is standard to prove that every $x(\cdot) \in P$ is a solution of (5). Therefore the following proposition holds:

PROPOSITION 2.2. *If $\emptyset \not\equiv G(t,x) \subset H(t,x)$ is closed convex valued such that $G(\cdot,x)$ is strongly measurable and the support function $\sigma(l, G(t,\cdot))$ is USC for every $l \in E^*$ and a.e. $t \in I$, then the differential inclusion*

$$\dot{x}(t) \in G(t,x), \qquad x(0) = x_0,$$

has a solution.

Now it is easy to prove

THEOREM 2.2′. *Suppose all the conditions except* A3 *of Theorem* 2.2 *hold. Then the solution set of* (1) *is dense in the solution set of* (5).

P r o o f. Let $x(\cdot)$ be a solution of (5). As in the proof of Theorem 2.2 fix $\delta > 0$ and consider the multimap

$$\Gamma(t,x) = \mathrm{cl}\{u \in F(t,y) : \langle x(t) - y, \dot{x}(t) - u\rangle < (L|x-y| + \delta)^2 + 2\delta\}.$$

$\Gamma(\cdot,\cdot)$ is nonempty compact valued and almost LSC. Consequently, there exists a solution $y(\cdot)$ of

$$\dot{x}(t) \in \Gamma(t, x(t)), \quad x(0) = x_0.$$

It is standard to show that $|x(t) - y(t)| \le C\delta^{1/2}$, where C does not depend on δ. ∎

3. Applications in optimal control. In this section we consider optimal control systems having the form

$$\dot{x}(t) = f(t, x(t), u), \quad x(0) = x_0,\ u \in V. \tag{9}$$

Here $f : I \times E \times V \to E$ is a Carathéodory function, and E is a Banach space with uniformly convex dual E^*. When $f(t,\cdot,u)$ is continuous (not uniformly in u) the multimap $F(t,x) = \overline{\mathrm{co}}\, f(t,x,V)$ is LSC (not necessarily continuous) in x. Therefore it is not appropriate to associate the relaxed system with

$$x \in F(t,x), \quad x(0) = x_0. \tag{10}$$

The reason is that the solution set of the last system may be noncompact and nonclosed (see Example 3.1 below). So, we suppose:

B1. $f(\cdot,\cdot,u)$ is almost continuous, $f(t,x,\cdot)$ is continuous. V is a closed subset of a metric space, and $|f(t,x,V)| \le \lambda(t)\{1 + |x|\}$ for an integrable $\lambda(\cdot)$.

B2. $\langle j(x-y), f(t,x,u) - f(t,y,u)\rangle \le v(t,|x-y|)|x-y|$ for every $x, y \in E$. Here j is the duality map.

B3. $\lim_{h\to 0} \chi(f([t,t+h],A,V)) \le w(t,\chi(A))$.

Here v, w, χ are as in A1–A3. Define $G(t,x) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\{f(t, x + \overline{U}(0,\varepsilon), V)\}$. The *relaxed solutions* of (9) are the solutions of the system

$$x \in G(t,x), \quad x(0) = x_0. \tag{11}$$

From the results of the second section we conclude that the following theorem holds.

THEOREM 3.1. *Under assumptions* B1, B2 *the solution set of* (9) *is dense in the solution set of* (11), *i.e. the set of solutions is dense in the set of relaxed solutions.*

R e m a r k 3.1. When E^* is not uniformly convex one can consider (9) under B1–B3, defining the relaxed solutions as the quasitrajectories of (10).

EXAMPLE 3.1. Consider the system

$$\dot{x}(t) = u, \qquad x(0) = 0, \; u \in \{-1, 1\},$$

$$\dot{y}(t) = x^2 + f(y, v), \qquad y(0) = 0, \; v \in [1, \infty),$$

$$f(y, v) = \begin{cases} 0, & y \in (-\infty, -1/v], \\ -\sqrt[3]{vy} - 1, & y \in [-1/v, 1/v], \\ -2, & y \in [1/v, \infty). \end{cases}$$

This system satisfies B1, B2 and therefore Theorem 3.1 holds. Changing the signs on the right-hand side one obtains a system in $\mathbb{R}^2$ having the form $\dot{z} \in F(z)$, $z(0) = 0$. The solution set is dense in the solution set of $\dot{z} \in \overline{\mathrm{co}}\, F(z)$, $z(0) = 0$, but not in that of $\dot{z} \in G(z)$, $z(0) = 0$.

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