Let \( g_1, \ldots, g_n \) be the generators of the free group \( F_n \). C. Akemann and P. Ostrand proved in [A-O] a formula for the norm of free operators, i.e. operators of the form \( \sum \alpha_i \lambda(g_i) \). This formula improved estimates of M. Leinert [L] and M. Bożejko [Bo]. It was previously known in the case of equal coefficients [K] and simpler proofs were found in [W] and [P-P]. In this note we show how a part of the proof of [P-P] can be generalized to obtain estimates for the operator-valued case, which improve the bounds in [H-P, Proposition 1.1] (see also [Bu] for related recent results).

**Theorem 1.** Let \( n \geq 2 \) and \( g_1, \ldots, g_n \) be the generators of the free group \( F_n \), and let further \( a_1, \ldots, a_n \) be some operators on a Hilbert space \( H \) which can be approximated by invertible operators. Then

\[
\left\| \sum_{i=1}^{n} \lambda(g_i) \otimes a_i \right\|_{\min} \leq \inf_{s > 0} \left( \left\| \sum_{i=1}^{n} (s^2 I + a_i a_i^*)^{1/2} - (n-2)sI \right\|^{1/2} \times \left\| \sum_{i=1}^{n} (s^2 I + a_i^* a_i)^{1/2} - (n-2)sI \right\|^{1/2} \right).
\]

**Remark.** If the Hilbert space \( H \) is finite-dimensional, any operator can be approximated by invertible ones. In the infinite-dimensional case this is not generally true; see [H, Problem 140]. However, we have

**Corollary 2.** For an arbitrary family of operators \( a_1, \ldots, a_n \) on a Hilbert space \( H \) we have

\[
\left\| \sum_{i=1}^{n} \lambda(g_i) \otimes a_i \right\|_{\min} \leq 2 \sqrt{1 - \frac{1}{n} \left( \frac{\| \sum_{i=1}^{n} a_i a_i^* \| + \| \sum_{i=1}^{n} a_i^* a_i \|}{2} \right)^{1/2}} \leq 2 \sqrt{1 - \frac{1}{n} \max \left\{ \| \sum_{i=1}^{n} a_i a_i^* \|^{1/2}, \| \sum_{i=1}^{n} a_i^* a_i \|^{1/2} \right\}}.
\]
The proof of (1) is an adaptation to the non-commutative situation of the first part of [P-P]. Denote by $\text{HS}(H)$ the space of Hilbert–Schmidt operators on the Hilbert space $H$ and by $\text{tr}$ the usual (unbounded) trace. We need the following version of the Cauchy–Schwarz inequality.

**Lemma 3.** For any $x_1, \ldots, x_n \in B(H)$ and $y_1, \ldots, y_n \in \text{HS}(H)$ we have the inequality

$$\left\| \sum_{i=1}^{n} x_i y_i \right\|_{\text{HS}} \leq \left( \sum_{i=1}^{n} x_i^* x_i \right)^{1/2} \text{tr} \left( \sum_{i=1}^{n} y_i^* y_i \right)^{1/2}.$$  

**Proof.** After writing the sum as

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_n \end{bmatrix}$$

the claim follows from the operator ideal property of $\text{HS}(H^n)$.

**Proof of Theorem 1.** We first consider the case where all the operators $a_i$ are invertible. The general case then follows by a topological argument.

Let $T = \sum_{i=1}^{n} \lambda(g_i) \otimes a_i$ act on the Hilbert space $\ell_2(F_n; \text{HS}(H))$. For every word $y \in F_n$, $i \in \{1, \ldots, n\}$ and any positive real number $s$ we define the following operator:

$$p_i(y, s) = a_i^{-1}((s^2 + a_i a_i^*)^{1/2} \mp s) = ((s^2 + a_i^* a_i)^{1/2} \mp s) a_i^{-1}$$

with

“−” if there is no cancellation in the word $g_i^{-1} y$, i.e. the first letter of $y$ is different from $g_i$,

“+” if the first letter of $y$ is $g_i$ and there is cancellation.

Here and in the following, scalars appearing in operator expressions mean the corresponding multiple of the identity operator, and the square root of a positive operator is always the unique positive square root. Then one can easily check that $p_i(y, s)$ is invertible and that its inverse is

$$p_i(y, s)^{-1} = ((s^2 + a_i a_i^*)^{1/2} \pm s) a_i^{-1} = a_i^{-1}((s^2 + a_i^* a_i)^{1/2} \pm s)$$

(note the change of sign).

Now pick $h \in \ell_2(F_n; \text{HS}(H))$ with finite support. In order to get the upper bound for $\|Th\|_2^2 = \sum_y \|Th(y)\|_{\text{HS}}^2$ we first give an estimate of

$$\|Th(y)\|_{\text{HS}}^2 = \left\| \sum_{i=1}^{n} a_i p_i(y, s)p_i(y, s)^{-1} h(g_i^{-1} y) \right\|_{\text{HS}}^2$$

for fixed $y$. Now the operator $a_i p_i(y, s)$ is positive and we can apply the above lemma by letting $x_i = (a_i p_i(y, s))^{1/2}$ be its (positive) square root.
and \( y_i = (a_i p_i(y, s))^{1/2} p_i(y, s)^{-1} h(g_i^{-1} y) \):

\[
\|T h(y)\|_{\text{HS}}^2 \leq \left\| \sum_{i=1}^{n} a_i p_i(y, s) \right\| \text{tr} \left( \sum_{i=1}^{n} h(g_i^{-1} y)^* p_i(y, s)^{*^{-1}} a_i h(g_i^{-1} y) \right). 
\]

Note that among the words \( \{g_1^{-1} y, \ldots, g_n^{-1} y\} \) there is at most one cancellation, so that there is always the “-” sign in \( p_i(y, s) \) except maybe for one case and since all the summands \( a_i p_i(y, s) \) are positive operators, we have the operator inequality

\[
\sum_{i=1}^{n} a_i p_i(y, s) \leq 2s + \sum_{i=1}^{n} ((s^2 + a_i a_i^*)^{1/2} - s) =: c_0(s).
\]

This upper bound does not depend on the word \( y \) and we can estimate the norm of \( T h \) as follows:

\[
\|T h\|_2^2 \leq \|c_0(s)\| \sum_{y} \text{tr} \left( \sum_{i=1}^{n} p_i(g_i y, s)^{*^{-1}} a_i h(y) h(y)^* \right)
\]

\[
\leq \|c_0(s)\| \sum_{y} \left\| \sum_{i=1}^{n} p_i(g_i y, s)^{*^{-1}} a_i \right\| \|h(y)\|_{\text{HS}}^2.
\]

Now \( p_i(g_i y, s) \) has always the “+” sign with at most one exception possible in the case when there is already cancellation in \( g_i y \). Thus \( p_i(g_i y, s)^{-1} \) has the “-” signs and with the formula for the inverse we get

\[
\sum_{i=1}^{n} p_i(g_i y, s)^{*^{-1}} a_i \leq 2s + \sum_{i=1}^{n} ((s^2 + a_i a_i^*)^{1/2} - s) =: c_1(s).
\]

The bound \( c_1(s) \) is again independent of \( y \) and we finally get the inequality we wanted: For all positive real numbers \( s \),

\[
\|T h\|_2^2 \leq \|c_0(s)\| \cdot \|c_1(s)\|.
\]

Let us now consider the case where the \( a_i \)'s are not invertible but approximable by invertible operators. Consider the topological space \( B(H)^n \) equipped with the product topology and define the functions

\[
B(H)^n \to \mathbb{R},
\]

\[
g : x = (a_1, \ldots, a_n) \mapsto \inf_{s \geq 0} \left( \|c_0(s)\| \cdot \|c_1(s)\| \right)^{1/2},
\]

\[
f : x = (a_1, \ldots, a_n) \mapsto \left\| \sum_{i=1}^{n} \lambda(g_i) \otimes a_i \right\|.
\]

Observe that \( g \) is upper semicontinuous, i.e. the set

\[
\{ x \in B(H)^n \mid g(x) < t \}
\]
is open for any $t \in \mathbb{R}$. Since $f$ is continuous, the set
\[
\{ x \in B(H)^n \mid g(x) - f(x) \geq 0 \}
\]
is closed and hence contains the closure of all the $n$-tuples of invertible operators.

Note that the infimum over all positive scalars $s$ could be replaced by an infimum over all positive operators $S$ which commute with the $a_i$’s. However, in view of the examples below this does not seem to improve the inequality very much.

**Proof of Corollary 2.** We will first prove the case where the $a_i$’s are approximable by invertibles. In fact we will show that the bound (1) is sharper than (2) just as in the commutative case (cf. [A-O]). We recall the following facts from non-commutative analysis. A function $f$ is called **operator-monotone** if for any positive selfadjoint operators $a, b,$
\[
 a \geq b \Rightarrow f(a) \geq f(b).
\]
It is **operator-concave** if the operator inequality
\[
f(\lambda a + (1 - \lambda)b) \geq \lambda f(a) + (1 - \lambda)f(b)
\]
holds for all positive operators $a, b$ and any $0 < \lambda < 1$. By Löwner’s theorem [M-O, p. 464], the function $t \mapsto t^\alpha$ is operator-monotone for $0 \leq \alpha \leq 1$ (see also [Ped]). Ando showed in [A] (see also [M]) that any operator-monotone function is necessarily operator-concave. We can now apply this to the function $t \mapsto \sqrt{t}$ and any sequence of positive operators $x_i = a_i^*a_i$:
\[
\frac{1}{n} \sum_{i=1}^{n} x_i^{1/2} \leq \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^{1/2},
\]
and our bound becomes
\[
\left( \|c_0(s)\| \cdot \|c_1(s)\| \right)^{1/2} \leq \left( 2 - n \right) s + \frac{n}{2} \left( \sqrt{\frac{s^2}{n} + \frac{1}{n} \left\| \sum_{i=1}^{n} a_i a_i^* \right\|} \right)^{1/2}
\times \left( 2 - n \right) s + n \left( \sqrt{\frac{s^2}{n} + \frac{1}{n} \left\| \sum_{i=1}^{n} a_i a_i^* \right\|} \right)^{1/2}
\leq (2 - n) s + \frac{n}{2} \left( \sqrt{\frac{s^2}{n} + \frac{1}{n} \left\| \sum_{i=1}^{n} a_i a_i^* \right\|} \right)
+ \left( \sqrt{\frac{s^2}{n} + \frac{1}{n} \left\| \sum_{i=1}^{n} a_i a_i^* \right\|} \right)
\]
\[ \leq (2 - n)s + n\sqrt{s^2 + \frac{1}{2n}\left(\sum_{i=1}^{n} a_i a_i^* + \sum_{i=1}^{n} a_i^* a_i\right)}. \]

The infimum of the last expression over all \( s \geq 0 \) is the expression in the claim.

Let us now consider general operators \( a_1, \ldots, a_n \) which are not necessarily approximable by invertible ones. Denote by \( P_l(H) \) the set of all finite-dimensional projections on \( H \). It is easy to see that the embedding

\[ \Phi : B(H) \to \bigoplus_{p \in P_l(H)} pB(H)p, \quad a \mapsto (pap)_{p \in P_l(H)}, \]

is a complete isometry, i.e. for any operators \( b_1, \ldots, b_n \) acting on some Hilbert space \( K \) we have

\[ \left\| \sum_{i=1}^{n} a_i \otimes b_i \right\|_{\min} = \left\| \sum_{i=1}^{n} \Phi(a_i) \otimes b_i \right\|_{\min}. \]

Now \( \Phi(a_i) \) lying in a direct sum of matrix algebras can be approximated by invertibles, so that inequality (2) holds when we replace \( a_i \) by \( \Phi(a_i) \). Next we use the fact that the right hand side of (2) comes from an operator space structure. Indeed, denoting by \( e_{ij} \) the canonical basis of \( M_n \), it is easy to see that

\[ \left\| \sum_{i=1}^{n} a_i \otimes e_{1i} \right\| = \left\| \sum_{i=1}^{n} a_i a_i^* \right\|^{1/2} \quad \text{and} \quad \left\| \sum_{i=1}^{n} a_i \otimes e_{i1} \right\| = \left\| \sum_{i=1}^{n} a_i^* a_i \right\|^{1/2}; \]

hence we have

\[ \left\| \sum_{i=1}^{n} \lambda(g_i) \otimes a_i \right\|_{\min} = \left\| \sum_{i=1}^{n} \lambda(g_i) \otimes \Phi(a_i) \right\|_{\min} \]

\[ \leq 2\sqrt{1 - \frac{1}{n}\left(\frac{\left\| \sum \Phi(a_i) \otimes e_{1i} \right\| + \left\| \sum \Phi(a_i) \otimes e_{i1} \right\|}{2}\right)^{1/2}} \]

\[ = 2\sqrt{1 - \frac{1}{n}\left(\frac{\left\| \sum a_i \otimes e_{1i} \right\| + \left\| \sum a_i \otimes e_{i1} \right\|}{2}\right)^{1/2}} \]

and this proves the claim. We do not see how to generalize (1) to non-approximable operators with a similar trick, since the assignment \( (a_i) \mapsto \left\| \sum |a_i| \right\| \) fails to be a norm and hence is not a complete invariant.

**Examples.** There is actually equality not only for scalar coefficients and it would be interesting to characterize such families of operators.

**Example 1 (Commuting normal operators).** As a simple consequence of Gel’fand’s theorem equality holds if the operators \( a_1, \ldots, a_n \) generate a commutative \( C^* \)-algebra.
Example 2. For unitaries $u_1, \ldots, u_n$ there is equality:

$$\left\| \sum_{i=1}^{n} \lambda(g_i) \otimes \alpha_i u_i \right\| = \left\| \sum_{i=1}^{n} \alpha_i \lambda(g_i) \right\|$$

$$= \min_{s \geq 0} 2s + \sum_{i=1}^{n} (\sqrt{s^2 + |\alpha_i|^2} - s)$$

$$= \min_{s \geq 0} (\|c_0(s)\| \cdot \|c_1(s)\|)^{1/2}.$$

The first identity is Fell’s lemma [F] applied to the left regular representation and the unitary representation which is uniquely determined by $\pi(g_i) = u_i$.

The next example uses the following simple identity. For any projection $p$ and positive real numbers $\sigma, \alpha$,

$$(\sigma I + (\sqrt{\sigma^2 + \alpha^2} - \sigma)p)^2 = \sigma^2 I + \alpha^2 p.$$  

Example 3. For the basis of the row-space $R_n$ and equal coefficients there is equality:

$$\left\| \sum_{i=1}^{n} \lambda(g_i) \otimes e_{1i} \right\| = \sqrt{n} = \min_{s \geq 0} (\|c_0(s)\| \cdot \|c_1(s)\|)^{1/2}.$$

However, if the coefficients are different there may be strict inequality, e.g.

$$\|\lambda(g_1) \otimes e_{11} + \lambda(g_2) \otimes e_{12} + 2\lambda(g_3) \otimes e_{13}\| = \sqrt{6} < \sqrt{8} = \min_{s \geq 0} (\|c_0(s)\| \cdot \|c_1(s)\|)^{1/2}.$$

The bound for the general operator $\sum_{i=1}^{n} \lambda(g_i) \otimes \alpha_i e_{1i}$ is determined by the norms

$$\|c_0(s)\| = \left\| 2s + \sum_{i=1}^{n} ((s^2 + |\alpha_i|^2 e_{11})^{1/2} - s) \right\|$$

$$= 2s + \sum_{i=1}^{n} (\sqrt{s^2 + |\alpha_i|^2} - s),$$

$$\|c_1(s)\| = \left\| 2s + \sum_{i=1}^{n} ((s^2 + |\alpha_i|^2 e_{i1})^{1/2} - s) \right\|$$

$$= \left\| 2s + \sum_{i=1}^{n} ((s^2 + |\alpha_i|^2)^{1/2} - s) e_{i1} \right\|$$

$$= s + \sqrt{s^2 + \max |\alpha_i|^2}.$$

We must find the minimum of the function

$$g : s \mapsto \|c_0(s)\| \cdot \|c_1(s)\|.$$
If $\alpha_1 = \ldots = \alpha_n = 1$ this is
\[ g(s) = (s + \sqrt{s^2 + 1})(2s + n(\sqrt{s^2 + 1} - s)), \]
with strictly positive derivative
\[ g'(s) = \frac{2(s + \sqrt{s^2 + 1})^2}{\sqrt{s^2 + 1}}, \]
and thus the minimum is attained at $s = 0$, which yields (5). For (6) where
$\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 2$ we consider
\[ g(s) = (2\sqrt{s^2 + 1} + \sqrt{s^2 + 4} - s)(s + \sqrt{s^2 + 4}), \]
which has the derivative
\[ g'(s) = \frac{(2s + 2\sqrt{s^2 + 4})(s\sqrt{s^2 + 4} + s^2 + 1)}{\sqrt{s^2 + 1}\sqrt{s^2 + 4}}. \]
This is again strictly positive and the infimum of $g$ is $g(0) = 8$.

**Example 4 (The Cuntz algebra).** In [H, Problem 140] it is shown that
the unilateral shift $S$ on $\ell_2$ cannot be approximated by invertible operators.
Consider the Cuntz algebra, which is generated by $n$ “free” copies of the
shift. A priori we cannot apply Theorem 1 to the sum
\[ \sum_{i=1}^{n} \lambda(g_i) \otimes \alpha_i S_i. \]
However, since this norm is trivially equal to $\left(\sum |\alpha_i|^2\right)^{1/2}$, the inequality
holds even in this case.

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**Added in proof.** We are working on an exact formula for norms of free operators
with matrix coefficients, which will be the subject of a forthcoming paper.

**REFERENCES**


58#2451.

[Bo] M. Bożejko, *On $\Lambda(p)$ sets with minimal constant in discrete noncommutative

[Bu] A. Buchholz, *Norm of convolution by operator-valued functions on free groups*,
preprint.

F. LEHNER


Fachbereich Mathematik
Universität Linz
4040 Linz, Austria
E-mail: lehner@caddo.bayou.uni-linz.ac.at

Current address:
IMADA
Odense Universitet
Campusvej 55
5230 Odense M, Denmark
E-mail: lehner@imada.ou.dk

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