

*REPRESENTATION TYPE OF POSETS AND  
FINITE RANK BUTLER GROUPS*

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Some special classes of finite rank indecomposable Butler groups have recently been classified, up to quasi-isomorphism and near isomorphism, by complete sets of invariants as surveyed in [Arnold, Vinsonhaler, 93]. This leads to the question of determination of representation type, i.e. when is a classification even feasible? The approach in this paper is to use interpretations of classes of Butler groups as representations of finite partially ordered sets (posets) over various rings. In case the ring is a field, there are well known, suitable definitions of representation type. In particular, each poset has exactly one of finite, tame, or wild representation type, the representation type can be determined from the poset, and classification of indecomposable representations of a poset with finite or tame representation type is feasible.

For a ring  $R$  and finite poset  $S$ ,  $\text{rep}(S, R)$  denotes the category of finitely generated filtered  $R$ -representations of  $S$  [Simson, 92, 96]. The objects are  $U = (U_0, U_i : i \in S)$  such that  $U_0$  is a finitely generated submodule of a finitely generated free  $R$ -module, each  $U_i$  is a finitely generated submodule of  $U_0$ , and if  $i \leq j$  in  $S$ , then  $U_i$  is contained in  $U_j$ . A morphism  $f : U \rightarrow U'$  is an  $R$ -module homomorphism  $f : U_0 \rightarrow U'_0$  such that  $f(U_i)$  is contained in  $U'_i$  for each  $i$ . Direct sums are given by  $U \oplus V = (U_0 \oplus V_0, U_i \oplus V_i)$ . If  $k$  is a field, then  $\text{rep}(S, k)$  is also known as the category of  $S$ -spaces (or filtered  $k$ -linear representations of  $S$ ) [Simson, 92, Chapter 3].

As an illustration of an interpretation of Butler groups as representations, let  $T$  be a finite sublattice of the lattice of types (isomorphism classes of subgroups of  $\mathbb{Q}$ ) and  $B(T)$  the category of finite rank Butler groups with typeset contained in  $T$  [Butler, 65]. Each finite rank Butler group  $G$  is

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in  $B(T)$  for  $T$  the lattice generated by typeset  $G$ , known to be a finite set. There is a category equivalence from the quasi-homomorphism category  $B(T)_{\mathbb{Q}}$  of  $B(T)$  to the category  $\text{rep}(\text{JI}(T)^{\text{op}}, \mathbb{Q})$  of  $\mathbb{Q}$ -representations of  $\text{JI}(T)^{\text{op}}$ , the finite poset of join-irreducible elements of  $T$  with inverse ordering [Butler, 68, 87]. The quasi-isomorphism representation type of  $B(T)$  can then be determined from  $\text{JI}(T)$  as surveyed in [Arnold, 89]. Consequently, strongly indecomposables can be classified up to quasi-isomorphism in case  $B(T)_{\mathbb{Q}}$  has finite or tame representation type.

Representation type for the quasi-homomorphism category is not sufficient from the point of view of groups as an indecomposable group can decompose up to quasi-isomorphism. The focus of this paper is on the representation type of  $B(T)_p$ , the isomorphism at  $p$  category of  $B(T)$ . Isomorphism at  $p$  is the local version of near isomorphism as introduced in [Lady, 75]. This is relevant to representation type of indecomposable groups as a Butler group  $G$ , being a torsion-free abelian group of finite rank, is indecomposable if and only if  $G$  is indecomposable up to near isomorphism, i.e. isomorphism at  $p$  for each prime  $p$  [Arnold, 82].

Let  $\mathbb{Z}_{(p)}$  denote the localization of the integers at a prime  $p$  and  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)})$  the full subcategory of  $\text{rep}(S, R)$  consisting of representations  $(U_0, U_i : i \in S)$  with  $U_0$  a finitely generated free  $\mathbb{Z}_{(p)}$ -module and each  $U_i$  pure in, hence a summand of,  $U_0$ . Given a positive integer  $j$ , define  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  to be the full subcategory of  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)})$  with objects  $(U_0, U_i : i \in S)$  such that  $p^j U_0$  is contained in  $\sum_i U_i$ .

If  $T$  is a finite  $p$ -locally free lattice of types, then there is a category equivalence between  $B(T)_p$  and  $\text{rep}_{\text{fpure}}(\text{JI}(T)^{\text{op}}, \mathbb{Z}_{(p)})$ , the category of free, pure  $\mathbb{Z}_{(p)}$ -representations of  $\text{JI}(T)$  [Richman, 94]. Not much is known about the representation type of the category  $\text{rep}(S, R)$  of  $R$ -representations of a finite poset  $S$  in case  $R$  is not a field, but see [Plahotnik, 76], [Simson, 96], and [Files, Goebel, 96] and references therein.

Section 1 is devoted to the representation type of  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  for a finite poset  $S$ . The choice of appropriate definitions of tame and wild representation type for  $\mathbb{Z}_{(p)}$ -representations remains unclear. In the meantime, we use finite representation type to mean only finitely many isomorphism classes of indecomposables and introduce the notion of wild modulo  $p$  representation type in Section 1. One of the main results of Section 1 (Corollary 1.10) asserts that the category  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  has:

- (a) wild modulo  $p$  representation type if  $\text{width } S \geq 3$ ,  $j \geq 0$ ,
- (b) infinite representation type if  $\text{width } S = 2$  and  $S$  contains the poset  $(1, 2)$ ,  $j \geq 0$ ,
- (c) finite representation type if  $\text{width } S = 1$ ,  $j \geq 0$  or  $\text{width } S = 2$  and  $S$  does not contain the poset  $(1, 2)$ ,  $j \geq 0$ .

We conjecture that if  $\text{width } S = 2$  and  $S$  contains the poset  $(1, 2)$ , then  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  does not have wild modulo  $p$  representation type. Even if this conjecture is resolved in the affirmative, it remains unclear whether or not indecomposables can be classified.

Let  $S_n$  denote an antichain with  $n$  elements. As a special case of Corollary 1.10, together with known results for comparison, we have:

(a) For a field  $k$ ,  $\text{rep}(S_n, k)$  has finite representation type if  $n \leq 3$ , tame representation type if  $n = 4$ , and wild representation type if  $n \geq 5$ .

(b)  $\text{rep}_{\text{fpure}}(S_n, \mathbb{Z}_{(p)}, j)$  has finite representation type if  $n \leq 2$ ,  $j \geq 0$  and wild modulo  $p$  representation type if  $n \geq 3$ ,  $j \geq 0$ .

(c)  $\text{rep}(S_n, \mathbb{Z}_{(p)}, j)$  has finite representation type if  $n = 1$ ,  $j \geq 0$ , infinite representation type if  $n = 2$ ,  $j \geq 0$ , and wild modulo  $p$  representation type if  $n \geq 3$ ,  $j \geq 0$ .

(d) If  $S = (1, 2)$ , then  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, 0)$  has infinite representation type and each indecomposable has rank  $\leq 2$  (Theorem 1.8(a)).

(e) If  $S = (1, 2)$ , then  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)})$  has indecomposables of arbitrarily large rank (Theorem 1.8(b)).

Computations of indecomposables in  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)})$  cited in (d) and (e) for the “critical” poset  $S = (1, 2)$  demonstrate some anomalies that do not occur for fields. In particular, (d) illustrates a distinction between *rank finite representation type* (there is a bound on the ranks of indecomposables) and finite representation type as defined above. In fact,  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)})$  has rank finite representation type if and only if  $S$  does not contain the poset  $(1, 2)$  (Corollary 1.11).

In Section 2, computations of representation type in Section 1 are applied to isomorphism at  $p$  categories  $B(T, j)_p$  of Butler groups (Corollary 2.3). Specifically, the representation type of  $B(T, j)_p$  is the same as the representation type of  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  for  $S$  the finite poset of join-irreducible elements of  $T$  in case  $T$  is  $p$ -locally free. Thus, Corollary 1.10 applies directly. For example, let  $T_n$  be a finite  $p$ -locally free Boolean algebra of types with  $n$  atoms.

(a')  $B(T_n)_{\mathbb{Q}}$  has finite representation type if  $n \leq 3$ , tame representation type if  $n = 4$ , and wild representation type if  $n \geq 5$ .

(b')  $B(T_n, j)_p$  has finite representation type if  $n \leq 2$ ,  $j \geq 0$ , and wild modulo  $p$  representation type if  $n \geq 3$ ,  $j \geq 0$ .

While determining the representation type of  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$ , various indecomposable representations are constructed. Included are several examples illustrating how such representations can be translated into indecomposable Butler groups of finite rank (Examples 2.4 and 2.5).

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**1. Representation type of  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)})$  and  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$ .**

A category  $C$  has *finite representation type* if  $\text{Ind } C$  is finite, where  $\text{Ind } C$  is the set of isomorphism classes of indecomposables in  $C$ , and *rank finite representation type* if there is a finite bound on ranks of indecomposables in  $C$ .

Given  $U = (U_0, U_i) \in \text{rep}(S, k)$ ,  $k$  a field, define the *coordinate vector*  $\text{cdn } U = (u_0, u_i : i \in S)$ , where  $u_0 = \dim_k U_0$  and  $u_i = \dim_k U_i / \sum \{U_j : j < i \text{ in } S\}$  are non-negative integers [Simson, 92]. The category  $\text{rep}(S, k)$  has *tame representation type* provided that it has infinite representation type and for each coordinate vector  $w = (w_0, w_i)$  there are finitely many  $N_1, \dots, N_m \in \text{rep}(S, k[x])$  such that if  $U \in \text{rep}(S, k)$  is indecomposable with  $\text{cdn } U = w$ , then  $U$  is isomorphic to  $N_i \otimes_{k[x]} B$  for some finite-dimensional indecomposable  $k[x]$ -module  $B$  [Simson, 92]. In other words, indecomposables  $U$  in  $\text{rep}(S, k)$  are characterized up to isomorphism by  $\text{cdn } U$ , finitely many indecomposable  $k[x]$ -representations of  $S$ , and  $k[x]$ -modules  $k[x]/\langle g(x)^e \rangle$ ,  $g(x)$  an irreducible polynomial.

The category  $\text{rep}(S, k)$  has *wild representation type* if there is an exact  $k$ -linear functor  $\text{mod } k\langle x, y \rangle \rightarrow \text{rep}(k, S)$  that sends indecomposables to indecomposables and reflects isomorphism classes, where  $k\langle x, y \rangle$  is the polynomial ring with non-commuting indeterminates  $x$  and  $y$  and  $\text{mod } k\langle x, y \rangle$  is the category of  $k\langle x, y \rangle$ -modules having finite  $k$ -dimension (see [Simson, 92, 93]).

The following theorem, due to Drozd, Kleiner, and Nazarova, is well known (see [Simson, 92, Theorems 10.1 and 15.1]). The poset

$$\{a < b > c < d\}$$

is denoted by  $N$ . Let  $j \geq 0$ ,  $S$  a finite poset, and define a poset  $(S, j)$  to be the disjoint union of  $S$  and a chain  $C$  of length  $j$ . The ordering on  $(S, j)$  is exactly that on  $S$  and  $C$ , i.e. there are no relations between elements of  $C$  and  $S$ . Thus, for example,  $(i, j, k)$  denotes the disjoint union of 3 chains of length  $i$ ,  $j$ , and  $k$  respectively.

**THEOREM 1.1.** *Let  $S$  be a finite poset and  $k$  a field.*

- (a)  $\text{rep}(S, k)$  has exactly one of finite, tame, or wild representation type.
- (b)  $\text{rep}(S, k)$  has finite representation type if and only if  $S$  does not contain  $(1, 1, 1, 1)$ ,  $(2, 2, 2)$ ,  $(1, 3, 3)$ ,  $(1, 2, 5)$ , or  $(N, 4)$  as a subposet.
- (c)  $\text{rep}(S, k)$  has tame representation type if and only if  $S$  does not contain  $(1, 1, 1, 1, 1)$ ,  $(1, 1, 1, 2)$ ,  $(2, 2, 3)$ ,  $(1, 3, 4)$ ,  $(1, 2, 6)$ , or  $(N, 5)$  as a subposet.

Given a finite poset  $S$ , let  $S^* = (S, 1)$  be the disjoint union of  $S$  with a point  $*$ . An additive functor  $F$  from a category  $C$  to a category  $D$  is a *fully faithful embedding* if  $F : \text{Hom}_C(U, V) \rightarrow \text{Hom}_D(F(U), F(V))$  is an isomorphism for each  $U, V$  in  $C$ . If  $F$  is a fully faithful embedding, then  $F$  induces an injection from  $\text{Ind } C$  into  $\text{Ind } D$ .

LEMMA 1.2. *Suppose that  $S$  is a finite poset.*

(a) *If  $S'$  is a subposet of  $S$  and  $j' \leq j$ , then there is a fully faithful embedding*

$$\text{rep}_{\text{fpure}}(S', \mathbb{Z}_{(p)}, j') \rightarrow \text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j).$$

(b) *There is a fully faithful embedding*

$$\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j) \rightarrow \text{rep}(S^*, \mathbb{Z}_{(p)}, 0).$$

The image consists of all  $V = (V_0, V_i, V_* : i \in S)$  with each  $V_0 = \sum_i V_i$  a free  $\mathbb{Z}_{(p)}$ -module, each  $V_i$  pure in  $V_0$ , and  $V_* \cap V_i = p^j V_i$  for each  $i$  in  $S$ .

PROOF. (a) Given  $U = (U_0, U_i) \in \text{rep}_{\text{fpure}}(S', \mathbb{Z}_{(p)}, j')$ , define  $F(U) = (V_0, V_s : s \in S) \in \text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  by  $V_0 = U_0$ ,  $V_s = U_s$  if  $s \in S'$ ,  $V_s = \bigcap \{U_t : s < t, t \in S'\}$  if  $s \notin S'$  and there is some  $t \in S'$  with  $s < t$ , and  $V_s = U_0$  otherwise. It is routine to see that  $F$  is a fully faithful embedding, where  $F(f) = f$  (see [Simson, 92, Section 5.3]).

(b) The correspondence  $U = (U_0, U_i : i \in S) \rightarrow V = (\sum_i U_i, U_i, p^j U_0)$  is a functor, where  $f \rightarrow g = f|_{\sum U_i}$ . Note that  $V$  satisfies the given conditions. This functor is a fully faithful embedding, since if  $g : V \rightarrow V'$  is a representation morphism, then  $f = g/p^j : U \rightarrow U'$  is a unique extension of  $g$ .

Now let  $V = (V_0, V_i, V_* : i \in S) \in \text{rep}(S^*, \mathbb{Z}_{(p)}, 0)$  satisfy the given conditions and define  $U = (V_0 + (1/p^j)V_*, V_i : i \in S)$ . Then  $U_0 = V_0 + (1/p^j)V_*$  is finitely generated torsion-free, hence free, as a  $\mathbb{Z}_{(p)}$ -module, and  $p^j U_0$  is contained in  $\sum_i V_i = V_0$ . The fact that each  $V_i$  is pure in  $U_0$  follows from the assumption that  $V_* \cap V_i = p^j V_i$  for each  $i$ .

We next consider some cases for which  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  has finite representation type. The first theorem, and proof, is exactly as for fields. Given a finite poset  $S$ , the *width* of  $S$ ,  $w(S)$ , is the largest number of pairwise incomparable elements of  $S$ .

THEOREM 1.3. *Assume that  $w(S) = 1$ . Then the poset  $S$  is a chain and  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  has finite representation type for each  $j \geq 0$ . A complete list of  $\text{Ind } \text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  is  $(\mathbb{Z}_{(p)}, 0, \dots, 0, \mathbb{Z}_{(p)})$ ,  $(\mathbb{Z}_{(p)}, 0, \dots, 0, \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ ,  $\dots$ ,  $(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, \dots, \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ . Each indecomposable has endomorphism ring isomorphic to  $\mathbb{Z}_{(p)}$ .*

PROOF. Since  $w(S) = 1$ , any two elements of  $S$  are comparable, from which it follows that  $S$  is a chain, say  $S = \{1 < 2 < \dots < m\}$ . Let  $0 \neq U =$

$(U_0, U_1, \dots, U_m) \in \text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  with  $U_i$  contained in  $U_{i+1}$  for  $i \geq 1$  and write  $U_0 = U_m \oplus V_0$ . This is possible as pure submodules of finitely generated free  $\mathbb{Z}_{(p)}$ -modules are summands. Then

$$U = (U_m, U_1, \dots, U_{m-1}, U_m) \oplus (V_0, 0, \dots, 0).$$

But  $U$  is indecomposable with  $0 \neq p^j U_0$  contained in  $\sum_i U_i = U_m$ . Thus,  $V_0 = 0$  and  $U = (U_m, U_1, \dots, U_{m-1}, U_m)$ . Next write  $U_m = U_{m-1} \oplus V_m$ . Then

$$U = (U_{m-1}, U_1, \dots, U_{m-1}, U_{m-1}) \oplus (V_m, 0, \dots, 0, V_m).$$

Since  $U$  is indecomposable, either

$$\begin{aligned} U_{m-1} = 0 \quad \text{and} \quad U \text{ is isomorphic to } (\mathbb{Z}_{(p)}, 0, \dots, 0, \mathbb{Z}_{(p)}) \quad \text{or else} \\ V_m = 0 \quad \text{and} \quad U = (U_{m-1}, U_1, \dots, U_{m-1}, U_{m-1}). \end{aligned}$$

Continuing in this fashion completes the list of indecomposables. The endomorphism ring of each indecomposable is  $\mathbb{Z}_{(p)}$  since  $\mathbb{Z}_{(p)}$  is its own endomorphism ring.

Recall that  $S_n = (1, 1, \dots, 1)$  denotes the poset of  $n$  pairwise incomparable elements.

**THEOREM 1.4** ([Arnold, Dugas, 93B] or [Files, Goebel, 96]). *The category  $\text{rep}_{\text{fpure}}(S_2, \mathbb{Z}_{(p)}, j)$  has finite representation type for each  $j \geq 0$ . A complete list of representations in  $\text{Ind rep}_{\text{fpure}}(S_2, \mathbb{Z}_{(p)}, j)$  for  $j \geq 1$  is  $(\mathbb{Z}_{(p)}, 0, 0)$ ,  $(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ ,  $(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, 0)$ ,  $(\mathbb{Z}_{(p)}, 0, \mathbb{Z}_{(p)})$ , and  $U_b = (\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)} + \mathbb{Z}_{(p)}(1, 1)/p^j, \mathbb{Z}_{(p)} \oplus 0, 0 \oplus \mathbb{Z}_{(p)})$ .*

The case  $j = 0$  has an interpretation as a “matrix problem” over  $\mathbb{Z}_{(p)}$  (see [Simson, 92, Chapter 1] for a discussion of matrix problems over fields). In subsequent corollaries, some matrix problems are “solved” over  $\mathbb{Z}_{(p)}$  in the sense that representation type is determined.

**THEOREM 1.5.** *There is a fully faithful embedding*

$$H : \text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, 0) \rightarrow \text{rep}_{\text{fpure}}(S^*, \mathbb{Z}_{(p)}, 0)$$

with image  $H$  those  $V = (V_0 = \bigoplus_i V_i, V_i, V_* : i \in S)$  such that  $V_i \cap V_* = 0$  and  $V_i \oplus V_*$  is pure in  $V_0$  for each  $i \in S$ .

**PROOF.** Let  $U = (U_0, U_i : i \in S) \in \text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, 0)$  with  $U_0 = \sum_i U_i$  and define

$$H(U) = \left( \bigoplus_i U_i, U_i, U_* \right),$$

where  $U_*$  is the kernel of the epimorphism  $\pi : \bigoplus_i U_i \rightarrow U_0$  induced by inclusion of the  $U_i$ 's in  $U_0$ . Then  $H(U) \in \text{rep}_{\text{fpure}}(S^*, \mathbb{Z}_{(p)}, 0)$  with  $U_i \cap U_* = 0$  and  $U_i \oplus U_*$  pure in  $U_0$  for each  $i \in S$  as  $(\bigoplus_i U_i)/(U_i \oplus U_*)$ , being isomorphic to  $U_0/U_i$ , is torsion-free.

To see that  $H$  is a functor, let  $f : U \rightarrow U'$  be a representation morphism. Then  $g = \bigoplus f|_{U(i)} : \bigoplus_i U_i \rightarrow \bigoplus_i U'_i$  with  $f\pi = \pi'g$ . Thus,  $g : U_* = \ker \pi \rightarrow U'_* = \ker \pi'$  and  $H(f) = g : H(U) \rightarrow H(U')$  is a representation morphism. The reverse of this argument shows that  $H$  is a fully faithful embedding.

Finally, to see that image  $H$  is as described, let  $V = (V_0 = \bigoplus_i V_i, V_i, V_* : i \in S) \in \text{rep}_{\text{fpure}}(S^*, \mathbb{Z}_{(p)}, 0)$  with  $V_i \cap V_* = 0$  and  $V_i \oplus V_*$  pure in  $V_0$  for each  $i \in S$ . Define  $U = (U_0, U_i : i \in S)$  by  $U_0 = V_0/V_*$  and  $U_i = (V_i \oplus V_*)/V_*$  for each  $i \in S$ . Then  $U_0$  is a free  $\mathbb{Z}_{(p)}$ -module since  $V_*$  is pure in  $V_0$ ,  $U_0 = \sum_i U_i$ ,  $U_i$  is isomorphic to  $V_i$ , and  $U_i$  is pure in  $U_0$  since  $U_0/U_i$ , being isomorphic to  $V_0/(V_i \oplus V_*)$ , is torsion-free. It now follows that  $U \in \text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, 0)$  with  $H(U)$  isomorphic to  $V$ , as desired.

We say that  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  is *wild modulo  $p$*  if given two  $n \times n$   $\mathbb{Z}_{(p)}$ -matrices  $A$  and  $B$  with  $A \pmod{p}$  and  $B \pmod{p}$  non-zero  $\mathbb{Z}/p\mathbb{Z}$ -matrices, there is  $U \in \text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  and a  $\mathbb{Z}_{(p)}$ -algebra epimorphism

$$\text{End } U \rightarrow C(A \pmod{p}, B \pmod{p}),$$

where  $A \pmod{p}$  denotes the  $\mathbb{Z}/p\mathbb{Z}$ -matrix obtained by reducing the entries of  $A$  modulo  $p$  and  $C(A \pmod{p}, B \pmod{p})$  is the ring of all  $n \times n$   $\mathbb{Z}/p\mathbb{Z}$  matrices that commute with both  $A \pmod{p}$  and  $B \pmod{p}$ . This definition is motivated by the fact that if  $k$  is a field and  $\text{rep}(S, k)$  has wild representation type, then for each finite-dimensional  $k$ -algebra  $\Lambda$  there is some  $U$  in  $\text{rep}(S, k)$  with  $\text{End } U$  isomorphic to  $\Lambda$  [Brenner, 74]. Moreover, the latter condition is equivalent to the condition that for each pair of non-zero  $k$ -matrices  $A'$  and  $B'$  there is some  $U$  with  $\text{End } U$  isomorphic to  $C(A', B')$ .

LEMMA 1.6. *The category  $\text{rep}_{\text{fpure}}(S_3, \mathbb{Z}_{(p)}, 0)$  has wild modulo  $p$  representation type.*

PROOF. Note that  $S_3^* = S_4$ . Let  $A$  and  $B$  be an  $n \times n$   $\mathbb{Z}_{(p)}$ -matrices with  $A \pmod{p}$  and  $B \pmod{p}$  non-zero  $\mathbb{Z}/p\mathbb{Z}$ -matrices. Define  $U \in \text{rep}_{\text{fpure}}(S_3^*, \mathbb{Z}_{(p)}, 0)$  by  $U_0 = \mathbb{Z}_{(p)}^{7n} \oplus \mathbb{Z}_{(p)}^{6n} \oplus \mathbb{Z}_{(p)}^{5n}$ ,  $U_1 = \mathbb{Z}_{(p)}^{7n} \oplus 0 \oplus 0$ ,  $U_2 = 0 \oplus \mathbb{Z}_{(p)}^{6n} \oplus 0$ , and  $U_3 = 0 \oplus 0 \oplus \mathbb{Z}_{(p)}^{5n}$ , and  $U_* = (M_1, M_2, M_3)$  the free  $\mathbb{Z}_{(p)}$ -module with the rows of the following matrix as a basis:

$$\left[ \begin{array}{cccccc|cccc|cccc|cccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & -pA & p^2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & p^2I \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & -p^4I & 0 & p^6I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & p^5B \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & p^5I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & p^6I \\ 0 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p^6I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p^7I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{array} \right]$$

for  $I$  an  $n \times n$  identity matrix and  $0$  an  $n \times n$  matrix of 0's. Now  $U_i \cap U_* = 0$  and  $U_i \oplus U_*$  is pure in  $U_0$  for each  $1 \leq i \leq 3$ . Thus,  $U = H(V)$  for some  $V \in \text{rep}_{\text{fpure}}(S_3, \mathbb{Z}_{(p)}, 0)$  with  $\text{End } U$  isomorphic to  $\text{End } V$ , by Theorem 1.5.

A routine but quite lengthy computation shows that if  $A$  and  $B$  are invertible modulo  $p$ , then there is an epimorphism  $\text{End } U \rightarrow C(A \pmod{p}, B \pmod{p})$ . This computation is outlined in the Appendix. Finally, if  $A \pmod{p}$  and  $B \pmod{p}$  are non-zero, then there are elements  $a, b$  in  $\mathbb{Z}_{(p)}$  with  $A' = aI + A$  and  $B' = bI + B$  invertible modulo  $p$ . Since  $C(A' \pmod{p}, B' \pmod{p}) = C(A \pmod{p}, B \pmod{p})$ , the proof is now complete.

The rank of  $U = (U_0, U_i) \in \text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  is the rank of  $U_0$  as a (finitely generated) free  $\mathbb{Z}_p$ -module.

**THEOREM 1.7.** *Given  $n \geq 2$ , let  $C_n$  denote a chain with  $n$  elements so that  $C_n^* = (n, 1)$ .*

(a) *There is a fully faithful embedding  $F_j : \text{rep}_{\text{fpure}}(C_{n-1}^*, \mathbb{Z}_{(p)}, j) \rightarrow \text{rep}_{\text{fpure}}(C_n^*, \mathbb{Z}_{(p)}, 0)$  for each  $j$ .*

(b) *If  $V$  is an indecomposable in  $\text{rep}_{\text{fpure}}(C_n^*, \mathbb{Z}_{(p)}, 0)$  with rank  $> 1$ , then there is some  $j$  and indecomposable  $U$  in  $\text{rep}_{\text{fpure}}(C_{n-1}^*, \mathbb{Z}_{(p)}, j)$  with  $F_j(U) = V$ .*

**Proof.** (a) Let  $U = (U_0, U_1, \dots, U_{n-1}, U_*) \in \text{rep}_{\text{fpure}}(C_{n-1}^*, \mathbb{Z}_{(p)}, j)$  and define  $F_j(U) = V = (U_0, U_1, \dots, U_{n-1}, U_0, U_*) \in \text{rep}_{\text{fpure}}(C_n^*, \mathbb{Z}_{(p)}, 0)$ . It is routine to verify that, with  $F_j(f) = f$ ,  $F_j : \text{rep}_{\text{fpure}}(C_{n-1}^*, \mathbb{Z}_{(p)}, j) \rightarrow \text{rep}_{\text{fpure}}(C_n^*, \mathbb{Z}_{(p)}, 0)$  is a fully faithful embedding.

(b) Let  $V = (V_0, V_1, \dots, V_n, V_*)$  be an indecomposable representation in  $\text{rep}_{\text{fpure}}(C_n^*, \mathbb{Z}_{(p)}, 0)$  with rank  $> 1$  and  $V_0 = V_n + V_*$ . Then we have  $V_* = W_1 \oplus (V_* \cap V_n)$  for some  $W_1$ , since pure submodules of finitely generated  $\mathbb{Z}_{(p)}$ -modules are summands. Hence,  $V_0 = V_n + V_* = W_1 \oplus V_n$  and so

$$V = (W_1, 0, \dots, 0, 0, W_1) \oplus (V_n, V_1, \dots, V_{n-1}, V_n, V_n \cap V_*).$$

Since  $W_1$  is a direct sum of rank-1's and  $V$  is indecomposable with rank  $> 1$ , it follows that  $W_1 = 0$ ,  $V_0 = V_n$  contains  $V_*$ , and  $V = (V_n, V_1, \dots, V_{n-1}, V_n, V_*)$ .

In fact,  $p^j V_n$  is contained in  $V_{n-1} + V_*$  for some  $j$ . To see this, let  $W$  be the pure submodule of  $V_n$  generated by  $V_{n-1} + V_*$ . Then  $V_n = W' \oplus W$  for some  $W'$  and  $V = (W', 0, \dots, 0, W', 0) \oplus (W, V_1, \dots, V_{n-1}, W, V_*)$ . Again,  $W' = 0$  since  $V$  is indecomposable of rank  $> 1$  and  $V_n = W$ . Thus there is some  $j$  with  $p^j V_n$  contained in  $V_{n-1} + V_*$  as  $V_n$  is a finitely generated free  $\mathbb{Z}_p$ -module. It now follows that  $V = F_j(U)$  for  $U = (V_n, V_1, \dots, V_{n-1}, V_*)$  in  $\text{rep}_{\text{fpure}}(C_{n-1}^*, \mathbb{Z}_{(p)}, j)$ .

**THEOREM 1.8.** *Let  $S = (1, 2)$ .*

(a)  *$\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, 0)$  has infinite representation type and each indecomposable has rank  $\leq 2$ . A rank-2 indecomposable is of the form  $U = (U_0, \mathbb{Z}_{(p)} \oplus 0, U_0, 0 \oplus \mathbb{Z}_{(p)})$  and  $U_0 = \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)} + \mathbb{Z}_{(p)}(1, 1)/p^j$ .*

(b)  *$\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)})$  has indecomposables of arbitrarily large finite rank.*



Proof. (a) is a consequence of Theorems 1.4 and 1.7(b).

(b) We construct an indecomposable  $U = (U_0, U_1 = E \oplus X, U_2, U_3)$  in  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, 2n + 1)$  of rank  $2n$  as follows:

$$\begin{aligned} U_1 &= E \oplus X, & E &= \mathbb{Z}_{(p)}e_1 \oplus \dots \oplus \mathbb{Z}_{(p)}e_n, & X &= \mathbb{Z}_{(p)}x_1 \oplus \dots \oplus \mathbb{Z}_{(p)}x_n, \\ U_2 &= \mathbb{Z}_{(p)}f_1 \oplus \dots \oplus \mathbb{Z}_{(p)}f_n, & U_3 &= U_2 \oplus G_1, & G_1 &= \mathbb{Z}_{(p)}\varepsilon_1 \oplus \dots \oplus \mathbb{Z}_{(p)}\varepsilon_n, \\ & & \varepsilon_i &= (e_i + f_i)/p^{2i-1}, \\ U_0 &= E \oplus X \oplus G_2, & G_2 &= \mathbb{Z}_{(p)}\mu_1 \oplus \dots \oplus \mathbb{Z}_{(p)}\mu_n, \\ \mu_i &= (p(e_i + f_i) + e_{i+1} + f_{i+1} + p^{2i}x_i)/p^{2n+1} & \text{for } i \leq n-1, \\ \mu_n &= (p(e_n + f_n) + p^{2n}x_n)/p^{2n+1}. \end{aligned}$$

Notice that  $p^{2n+1}U_0$  is contained in  $E \oplus X \oplus U_2$ ,  $U_0 = U_1 \oplus G_2$ , and  $U_0 = U_2 \oplus X \oplus G_2$ . To see that  $U_3$  is pure in  $U_0$ , observe that

$$\begin{aligned} \mu_i &= ((e_i + f_i)/p^{2i-1} + (e_{i+1} + f_{i+1})/p^{2i} + x_i)/p^{2(n-i)+1} \\ &= (\varepsilon_i + p\varepsilon_{i+1} + x_i)/p^{2(n-i)+1} \end{aligned}$$

for  $1 \leq i \leq n-1$  and

$$\mu_n = (\varepsilon_n + x_n)/p.$$

It follows that  $U_3 = U_2 \oplus G_1$  is a pure submodule of  $U_0 = E \oplus X \oplus G_2$  since  $\varepsilon_1 + p\varepsilon_2, \dots, \varepsilon_{n-1} + p\varepsilon_n, \varepsilon_n$  is a basis of  $G_1$ .

Let  $f$  be a representation endomorphism of  $U$ , so that  $f(U_i)$  is contained in  $U_i$  for each  $i$ . Then  $f(E)$  is contained in  $E$  because  $U_3 \cap U_1 = E$ . Restriction induces an embedding of  $\text{End } U$  into  $\text{End}(U_1 \oplus U_2) = \text{End}(E \oplus U_2 \oplus X)$  with

$$f \rightarrow \varphi = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ \delta & 0 & \gamma \end{bmatrix}$$

for some  $n \times n$   $\mathbb{Z}_{(p)}$ -matrix where

- $\alpha = (\alpha_{ij})$  represents an endomorphism of  $E$  relative to the basis  $e_1, \dots, e_n$ ,
- $\beta = (\beta_{ij})$  represents an endomorphism of  $U_2$  relative to the basis  $f_1, \dots, f_n$ ,
- $\delta : X \rightarrow E$ , and
- $\gamma = (\gamma_{ij})$  represents an endomorphism of  $X$  relative to the basis  $x_1, \dots, x_n$ .

It is sufficient to prove that  $\varphi \pmod{p} = aI + N$  for some  $a \in \mathbb{Z}/p\mathbb{Z}$ , where  $I$  is the identity matrix and  $N$  is a nilpotent  $\mathbb{Z}/p\mathbb{Z}$ -matrix with zeros on the diagonal. To see this, first note that the embedding  $\text{End } U \rightarrow \text{End } V$ , for  $V = E \oplus U_2 \oplus X$ , is pure since  $E, U_2$ , and  $X$  are pure and fully invariant in  $U_0$ . Thus,  $\text{End } U/p \text{End } U \rightarrow \text{End } V/p \text{End } V$  is an embedding, whence

elements of  $\text{End } U/p\text{End } U$  are of the form  $aI + N$ ,  $N$  a nilpotent  $\mathbb{Z}/p\mathbb{Z}$ -matrix with zeros on the diagonal. If  $f$  is an idempotent endomorphism of  $U$ , then  $f \pmod p = aI + N$  is idempotent. A matrix multiplication shows that the diagonal elements of  $f \pmod p$  are  $a^2 = a$ , so that  $a$  is either 0 or 1. Since  $N$  is in the Jacobson radical of  $\text{End } U/p\text{End } U$  and  $p$  is in the Jacobson radical of  $\text{End } U$ ,  $f$  must be either 0 or 1. This shows that  $U$  is an indecomposable representation, as desired.

Now

$$\varepsilon_i \phi = (((e_i + f_i)/p^{2i-1})\alpha, ((e_i + f_i)/p^{2i-1})\beta, 0) = \sum_{j=1}^n r_{ij} \varepsilon_j + (e, u_2, 0),$$

for some  $(e, u_2) \in E \oplus U_2, r_{ij} \in \mathbb{Z}_p$ . This is because  $U_3 = G_1 \oplus E$  is the purification of  $E \oplus U_2$  in  $U_0$ . Evaluate and multiply both sides by  $p^{2i-1}$  to get

$$\sum_{j=1}^n (\alpha_{ij} e_j + \beta_{ij} f_j) \equiv \sum_{j=1}^n p^{2i-1-(2j-1)} (r_{ij} e_j + r_{ij} f_j) \pmod{p^{2i-1}(E \oplus U_2)}.$$

Equate coefficients of the  $e_i$ 's and  $f_i$ 's to see that

$$(*) \quad \alpha_{ij} \equiv p^{2(i-j)} r_{ij} \equiv \beta_{ij} \pmod{p^{2i-1}}.$$

In particular,  $\alpha_{ij} \equiv \beta_{ij} \equiv 0 \pmod p$  if  $i > j$ .

Similarly,

$$\mu_i \phi \equiv \sum_{j=1}^n s_{ij} \mu_j + (e, u_2, x) \quad \text{for some } (e, u_2, x) \in E \oplus U_2 \oplus X, \quad s_{ij} \in \mathbb{Z}_p,$$

since  $p^{2n+1}U_0 \leq E \oplus U_2 \oplus X$ . Evaluate and multiply by  $p^{2n+1}$  to get

$$\begin{aligned} & \sum_{j=1}^n (p\alpha_{ij} + \alpha_{i+1,j} + p^{2i}\delta_{ij})e_j + (p\beta_{ij} + \beta_{i+1,j})f_j + p^{2i}\gamma_{ij}x_j \\ & \equiv \sum_{j=1}^n ((ps_{ij} + s_{i,j-1})e_j + (ps_{ij} + s_{i,j-1})f_j + s_{ij}p^{2j}x_j) \pmod{p^{2n+1}(E \oplus U_2 \oplus X)}. \end{aligned}$$

Equating coefficients for each  $i \leq n - 1$  and  $j \leq n$  gives

- (i)  $p\alpha_{ij} + \alpha_{i+1,j} + p^{2i}\delta_{ij} \equiv ps_{ij} + s_{i,j-1} \pmod{p^{2n+1}}$ ,
- (ii)  $p\beta_{ij} + \beta_{i+1,j} \equiv ps_{ij} + s_{i,j-1} \pmod{p^{2n+1}}$ , and
- (iii)  $p^{2i}\gamma_{ij} \equiv p^{2j}s_{ij} \pmod{p^{2n+1}}$

and for  $i = n, j = n$ ,

- (iv)  $p\beta_{nn} + 0 \equiv ps_{nn} + s_{n,n-1} \pmod{p^{2n+1}}$  and
- (v)  $\gamma_{n,n-1} \equiv p^{-2}s_{n,n-1} \pmod{p^{2n+1}}$ .

From (ii) for  $j = i$ ,

$$p\beta_{ii} + \beta_{i+1,i} \equiv ps_{ii} + s_{i,i-1} \pmod{p^{2n+1}},$$

and from (ii) for  $j = i + 1$ ,

$$(vi) \quad p\beta_{i,i+1} + \beta_{i+1,i+1} \equiv ps_{i,i+1} + s_{ii} \pmod{p^{2n+1}}.$$

Combining these equations gives

$$p\beta_{ii} + \beta_{i+1,i} \equiv p(p\beta_{i,i+1} + \beta_{i+1,i+1} - ps_{i,i+1}) + s_{i,i-1} \pmod{p^{2n+1}}$$

and so

$$(vii) \quad p(\beta_{ii} - \beta_{i+1,i+1}) \equiv -\beta_{i+1,i} + p^2\beta_{i,i+1} - p^2s_{i,i+1} + s_{i,i-1} \pmod{p^{2n+1}}.$$

By (\*),

$$\beta_{i+1,i} \equiv p^2r_{i+1,i} \pmod{p^{2i-1}},$$

whence  $p^2$  divides  $\beta_{i+1,i}$ . Also, (iii) gives

$$p^{2i}\gamma_{ij} \equiv p^{2(i-1)}s_{i,i-1} \pmod{p^{2n+1}}$$

so  $s_{i,i-1} \equiv 0 \pmod{p^2}$ . Using the above two equations in (vii) yields

$$\beta_{ii} \equiv \beta_{i+1,i+1} \pmod{p} \quad \text{for each } i \leq n - 1.$$

By (iii),

$$\gamma_{ij} \equiv p^{2(j-i)}s_{ij} \pmod{p^{2n+1}},$$

so that

$$\gamma_{ij} \equiv 0 \pmod{p} \quad \text{for } j > i, \quad \gamma_{ii} \equiv s_{ii} \pmod{p^{2(n-i)+1}}.$$

Combining this with (vi) gives

$$\gamma_{ii} \equiv s_{ii} \equiv \beta_{i+1,i+1} \pmod{p} \quad \text{for all } i \leq n - 1.$$

Finally,  $p^2$  divides  $s_{n,n-1}$  by (v) and so  $\beta_{nn} \equiv s_{nn} \equiv \gamma_{nn} \pmod{p}$  by (iv).

It now follows that there is an  $a \in \mathbb{Z}_{(p)}$  with  $a \equiv \alpha_{ii} \equiv \beta_{ii} \equiv \gamma_{ii} \pmod{p}$  for all  $i$  and

$$a \pmod{p} I + N \equiv \varphi \pmod{p} \equiv \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ \delta & 0 & \gamma \end{bmatrix} \pmod{p}$$

for some  $\mathbb{Z}/p\mathbb{Z}$ -matrix  $N$  with zeros on the diagonal. Since  $\alpha \pmod{p}$  and  $\beta \pmod{p}$  are upper triangular matrices and  $\gamma \pmod{p}$  is a lower triangular matrix, we conclude that  $N$  is nilpotent as desired.

**THEOREM 1.9.** *Let  $S$  be a poset of width 2 such that  $S$  does not contain the poset  $(1, 2)$ . For each  $j \geq 0$ ,  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  has finite representation type and each indecomposable has rank  $\leq 2$ .*

**Proof.** Let  $a$  and  $b$  be pairwise incomparable elements of  $S$ . Then the poset  $S$  is the disjoint union of  $X, Y = \{a, b\}$ , and  $Z$ , where  $x \leq a$  and  $x \leq b$

for each  $x \in X$  and  $a \leq z$ ,  $b \leq z$  for each  $z \in Z$ . This is a consequence of the hypothesis that the poset  $(1, 2)$  is not contained in  $S$ .

Let  $V = (V_0, V_i)$  be an indecomposable in  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)})$  and  $W$  the purification of  $V_a + V_b$  in  $V_0$ . Then  $W$  is a summand of  $V_0$  contained in  $V_z$  for each  $z$  in  $Z$  and  $V_x$  contained in  $V_a \cap V_b$  for each  $x$  in  $X$ , say  $V_0 = W \oplus U$ . Also,  $V_z = W \oplus U \cap V_z$  for each  $z$  in  $Z$  and

$$V = (U, U_x = 0, U_a = 0, U_b = 0, V_z \cap U : x \in X, z \in Z) \\ \oplus (W, V_x, V_a, V_b, V_z \cap W = W : x \in X, z \in Z).$$

Since  $V$  is indecomposable, either  $W = 0$  or else  $U = 0$  and  $V_0 = W = V_z$  for each  $z$  in  $Z$ . If  $W = 0$ , the proof is concluded by induction on the cardinality of  $S$ .

Next, assume  $U = 0$  and write  $V_0 = V_z = W = U \oplus (V_a \cap V_b)$  for each  $z$  in  $Z$ . Then  $V_a = (U \cap V_a) \oplus (V_a \cap V_b)$ ,  $V_b = (U \cap V_b) \oplus (V_a \cap V_b)$ , and  $V_x$  is contained in  $V_a \cap V_b$  for  $x$  in  $X$ . So,

$$V = V' \oplus V'' = (U, U_x = 0, U \cap V_a, U \cap V_b, U_z = U) \\ \oplus (V_a \cap V_b, V_x, V_a \cap V_b, V_a \cap V_b, V_a \cap V_b \cap V_z = V_a \cap V_b).$$

Since  $V$  is indecomposable, either  $V = V'$  or  $V = V''$ . In the first case, if  $X$  is non-empty, then  $V$  is induced by a representation of the proper subposet  $Y \cup Z$  and so the proof is concluded by induction on the cardinality of  $S$ . In the second case,  $W = V_a \cap V_b$ . Thus,  $V_a = V_b$  and if  $Z$  is non-empty, then  $V = V''$  is induced by a representation of the proper subposet  $X \cup Y$ . Once again, the proof follows by induction. We are left with the case where  $X = Z = \emptyset$ , which is just the case  $S = S_2$ . Now apply Theorem 1.4.

**COROLLARY 1.10.** *The category  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, j)$  has:*

- (a) *wild modulo  $p$  representation type if  $w(S) \geq 3$ ,  $j \geq 0$ ,*
- (b) *infinite representation type if  $w(S) = 2$ ,  $S \geq (1, 2)$ ,  $j \geq 0$ ,*
- (c) *finite representation type if  $w(S) = 1$ ,  $j \geq 0$  or  $w(S) = 2$ ,  $S$  does not contain  $(1, 2)$ ,  $j \geq 0$ .*

**Proof.** The wild modulo  $p$  condition comes from Lemmas 1.6 and 1.2(a).

The finiteness conditions arise from Theorems 1.3 and 1.9 while the infinite condition for  $w(S) = 2$  is Theorem 1.8.

We next examine some conditions for rank finite representation type.

**COROLLARY 1.11.** *The category  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)})$  has rank finite representation type if and only if  $S$  does not contain  $(1, 2)$ .*

**Proof.** A consequence of Theorems 1.8, 1.9, and Lemma 1.2.

**THEOREM 1.12.** *If  $S = (1, 3)$  or  $(2, 2)$ , then  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, 0)$  has rank infinite representation type.*

PROOF. The case of  $S = (1, 3)$  is a consequence of Theorems 1.8(b) and 1.7.

Let  $S = (2, 2)$ . We construct an indecomposable  $U = (U_0, U_1, U_2, U_3, U_0)$  in  $\text{rep}_{\text{fpure}}(S, \mathbb{Z}_{(p)}, 2n - 1)$  with rank  $U_0 = 2n$  as follows:

$$\begin{aligned} U_1 &= \mathbb{Z}_{(p)}(p^{n-1}e_1 + \dots + pe_{n-1} + e_n) \leq U_2 = \mathbb{Z}_{(p)}e_1 \oplus \dots \oplus \mathbb{Z}_{(p)}e_n, \\ U_3 &= \mathbb{Z}_{(p)}f_1 \oplus \dots \oplus \mathbb{Z}_{(p)}f_n \\ &\leq U_0 = U_2 \oplus \mathbb{Z}_{(p)}(e_1 + f_1)/p \oplus \dots \oplus \mathbb{Z}_{(p)}(e_n + f_n)/p^{2n-1}. \end{aligned}$$

Note that each  $U_i$  is pure in  $U_0$ .

Let  $f$  be a representation endomorphism of  $U$ . Then  $U_2 \oplus U_3$  is a fully invariant submodule of  $U_0$  with  $p^{2n-1}U_0$  contained in  $U_2 \oplus U_3$ . The restriction of  $f$  to  $U_2 \oplus U_3$  is  $g = (\alpha, \beta)$  with  $\alpha = (a_{ij})$  an  $n \times n$   $\mathbb{Z}_{(p)}$ -matrix representing an endomorphism of  $U_2$  relative to the basis  $e_1, \dots, e_n$  and  $\beta = (b_{ij})$  an  $n \times n$   $\mathbb{Z}_{(p)}$ -matrix representing an endomorphism of  $U_3$  relative to the basis  $f_1, \dots, f_n$ .

Now

$$\begin{aligned} f((e_i + f_i)/p^{2i-1}) &= (((e_i + f_i)/p^{2i-1})\alpha, ((e_i + f_i)/p^{2i-1})\beta) \\ &= (a_{i1}e_1/p^{2i-1} + \dots + a_{in}e_n/p^{2i-1}, b_{i1}f_1/p^{2i-1} + \dots + b_{in}f_n/p^{2i-1}) \\ &= r_{i1}(e_1 + f_1)/p + \dots + r_{in}(e_n + f_n)/p^{2n-1} + (y_2, y_3) \end{aligned}$$

for some  $y_i$  in  $U_i$ ,  $r_{ij} \in \mathbb{Z}_{(p)}$ . Equate coefficients of the  $e_i$ 's and  $f_i$ 's to see that if  $i > j$ , then  $a_{ij} \equiv b_{ij} \equiv 0 \pmod p$ .

Next, let  $e = p^{n-1}e_1 + \dots + pe_{n-1} + e_n$ , whence  $U_1 = \mathbb{Z}_{(p)}e$  is preserved by  $f$ . Then  $e\alpha = re$  for some  $r$  in  $\mathbb{Z}_{(p)}$ . Equating coefficients yields  $rp^{n-j} = p^{n-1}a_{1j} + \dots + a_{nj}$  for each  $j$ . It follows that  $r \equiv a_{jj} \pmod p$  for each  $j$ . Thus,  $g \pmod p = (\alpha, \beta) \pmod p$  is of the form

$$r \pmod p I + M = \begin{bmatrix} \alpha \pmod p & 0 \\ 0 & \beta \pmod p \end{bmatrix}$$

for an upper triangular  $\mathbb{Z}/p\mathbb{Z}$ -matrix  $M$  with zeros on the diagonal. Since  $M$  is nilpotent, it follows that 0 and 1 are the idempotents of  $\text{End } U$ , just as in the proof of Theorem 1.8. This shows that  $U$  is an indecomposable representation of rank  $2n$ .

Following is a computation of representation types for  $\text{rep}(S, \mathbb{Z}_{(p)}, j)$ , although these categories of representations are not directly applicable to Butler groups.

COROLLARY 1.13. *The category  $\text{rep}(S, \mathbb{Z}_{(p)}, j)$  has:*

- (a) *finite representation type if  $S = S_1$ ,  $j \geq 0$ ,*

- (b) *infinite representation type* if  $w(S) = 1$ ,  $S \neq S_1$ ,  $j \geq 0$  or  $w(S) = 2$ ,  $j \geq 0$ ,  
 (c) *wild modulo  $p$  representation type* if  $w(S) \geq 3$ ,  $j \geq 0$ .

**PROOF.** Indecomposables in  $\text{rep}(S_1, \mathbb{Z}_{(p)}, j)$  are of the form  $(\mathbb{Z}_{(p)}, p^j \mathbb{Z}_{(p)})$  as a consequence of the stacked basis theorem. If  $S$  is a chain of length 2, then there are infinitely many non-isomorphic indecomposables  $(\mathbb{Z}_{(p)}, p^m \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$  in  $\text{rep}(S, \mathbb{Z}_{(p)}, 0)$ . Furthermore,  $\text{rep}(S_2, \mathbb{Z}_{(p)}, j)$  has infinite representation type for  $j \geq 0$  [Arnold, Dugas 93B] or [Files, Goebel 96]. Finally, apply Corollary 1.10 for the wild modulo  $p$  case.

**2. Representation type of Butler groups.** A *type* is an isomorphism class of a *rank-1 group*, a subgroup of the additive group of rationals. The set of all types form a lattice [Fuchs, 73]. A *Butler group*  $G$  is a pure subgroup of a finite direct sum  $C_1 \oplus \dots \oplus C_m$  of rank-1 groups  $C_i$  [Butler, 65]. If  $G$  is a Butler group, then *typeset*  $G$ , the set of types of pure rank-1 subgroups of  $G$ , is contained in the lattice of types generated by the types of the  $C_i$ 's. In particular, typeset  $G$  is finite if  $G$  is a Butler group.

Let  $T$  be a finite sublattice of the lattice of all types with least element  $\tau_0$  and  $B(T)$  the category of finite rank Butler groups with typeset contained in  $T$ . Fix a prime  $p$ . The lattice  $T$  is  *$p$ -locally free* if  $X_{(p)} \approx \mathbb{Z}_{(p)}$  for each rank-1 group  $X$  with type  $X \in T$ . For  $j \geq 0$ , let  $B(T, j)$  be the full subcategory of  $B(T)$  consisting of those  $G$  with  $p^j G$  contained in  $\sum \{G(\tau) : \tau \in \text{JI}(T)\}$ . The isomorphism at  $p$  category of  $B(T, j)$  is denoted by  $B(T, j)_p$ ; the objects are groups in  $B(T, j)$  and morphisms are  $\text{Hom}(G, H)_p = \text{Hom}(G, H) \otimes \mathbb{Z}_{(p)}$  (see [Arnold, 82]).

The first proposition shows that, up to isomorphism at  $p$ , there is essentially no loss of generality in considering  $B(T, j)$ . The proof is essentially as in [Arnold, Dugas, 95B].

**PROPOSITION 2.1.** *If  $G$  is in  $B(T)$  and has no rank-1 summand of type  $\tau_0$ , then  $G$  is isomorphic at  $p$  to a group  $H \in B(T, j)$  for some  $j \geq 0$ .*

**PROOF.** First observe that  $\sum \{G(\sigma) : \sigma \in \text{JI}(T)\} = G^*(\tau_0)$ . This is because  $\tau_0 \notin \text{JI}(T)$  and if  $\tau \in T \setminus \text{JI}(T)$ , then  $\tau \geq \sigma$  for some  $\sigma \in \text{JI}(T)$ , whence  $G(\tau)$  is contained in  $G(\sigma)$ . Since  $G$  is a Butler group,  $G = G(\tau_0) = G' \oplus G^\#(\tau_0)$  with  $G'$   $\tau_0$ -homogeneous completely decomposable and  $G^\#(\tau_0)/G^*(\tau_0)$  finite, where  $G^\#(\tau_0)$  is the purification of  $G^*(\tau_0)$  [Butler, 65]. By the assumption on  $G$ ,  $G' = 0$  and so  $G/G^*(\tau_0)$  is finite. Now let  $H$  be a subgroup of  $G$  with  $H/G^*(\tau_0)$  the  $p$ -component of  $G/G^*(\tau_0)$ . Since  $G/H$  is finite with order prime to  $p$ ,  $G$  is isomorphic to  $H$  at  $p$ . Moreover, for some  $i$ ,  $p^i(H/G^*(\tau_0)) = 0$  so that  $p^{2i}H$  is contained in  $p^i G^*(\tau_0)$  and  $H \in B(T, 2i)$ , as desired.

THEOREM 2.2 [Richman, 94]. *If  $T$  is  $p$ -locally free, then there is a category equivalence  $F$  from  $B(T, j)_p$  to  $\text{rep}_{\text{fpure}}(\text{JI}(T)^{\text{op}}, \mathbb{Z}_{(p)}, j)$  given by  $F(G) = (G_p, G(\tau)_p : \tau \in \text{JI}(T))$ .*

COROLLARY 2.3. *If  $T$  is  $p$ -locally free and  $S = \text{JI}(T)^{\text{op}}$ , then  $B(T, j)_p$  has:*

- (a) *wild modulo  $p$  representation type if  $w(S) \geq 3, j \geq 0$ ,*
- (b) *infinite representation type if  $w(S) = 2, S \geq (1, 2), j \geq 0$ ,*
- (c) *finite representation type if  $w(S) = 1, j \geq 0$  or  $w(S) = 2, S$  does not contain  $(1, 2), j \geq 0$ .*

PROOF. Apply Corollary 1.10 and Theorem 2.2.

The next example demonstrates how to transcribe representations in  $\text{rep}(\text{JI}(T)^{\text{op}}, \mathbb{Z}_{(p)}, j)$  into groups in  $B(T, j)_p$ .

EXAMPLE 2.4. If  $T_3$  is a  $p$ -locally free Boolean algebra of types with three atoms, then  $B(T_3, 2)_p$  is wild modulo  $p$ .

PROOF. Let  $\tau_1, \tau_2, \tau_3$  be the atoms of  $T_3$  and choose subgroups  $A_i$  of  $\mathbb{Q}$  with type  $A_i = \tau_i, 1 \in A_i$ , and  $1/p \notin A_i$  for each  $i$ . Given two  $n \times n$   $\mathbb{Z}$ -matrices  $A$  and  $B$  with  $A \pmod{p}$  and  $B \pmod{p}$  invertible  $k$ -matrices, define  $G \leq \mathbb{Q}^n \oplus \mathbb{Q}^n \oplus \mathbb{Q}^n \oplus \mathbb{Q}^n$  by

$$G = A_1^{2n} \oplus A_2^{2n} + (1 + 0 + 0 + 1)A_3^n + (0 + 1 + 0 + 1)A_3^n + (1/p^2)(1 + 0 + A + 1)\mathbb{Z}^n + (1/p)(0 + 1 + B + 0)\mathbb{Z}^n.$$

Then  $U = (G_p, G(\tau_1)_p, G(\tau_2)_p, G(\tau_3)_p) \in \text{rep}_{\text{fpure}}(\mathbb{Z}_{(p)}, S_3, 2)$  with

$$\begin{aligned} U_0 &= \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n \\ &\quad + (1/p^2)(1 + 0 + A + 1)\mathbb{Z}_{(p)}^n + (1/p)(0 + 1 + B + 0)\mathbb{Z}_{(p)}^n, \\ U_1 &= \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n \oplus 0 \oplus 0, \quad U_2 = 0 \oplus 0 \oplus \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n, \\ U_3 &= (1 + 0 + 1 + 0)\mathbb{Z}_{(p)}^n \oplus (0 + 1 + 0 + 1)\mathbb{Z}_{(p)}^n. \end{aligned}$$

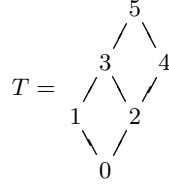
By Lemma 1.2(b),  $\text{End } U$  is isomorphic to  $\text{End } V$  for  $V = (V_0, V_1, V_2, V_3, V_*)$  with

$$\begin{aligned} V_0 &= \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n, \\ V_1 &= \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n \oplus 0 \oplus 0, \quad V_2 = 0 \oplus 0 \oplus \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n, \\ V_3 &= (1 + 0 + 1 + 0)\mathbb{Z}_{(p)}^n \oplus (0 + 1 + 0 + 1)\mathbb{Z}_{(p)}^n, \\ V_* &= p^2 U_0 = p^2(\mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n \oplus \mathbb{Z}_{(p)}^n) \\ &\quad + (1 + 0 + A + 1)\mathbb{Z}_{(p)}^n + p(0 + 1 + B + 0)\mathbb{Z}_{(p)}^n. \end{aligned}$$

A lengthy but straightforward computation, working modulo  $p$  and  $p^2$ , shows that there is a  $\mathbb{Z}_{(p)}$ -algebra epimorphism  $\text{End } V \rightarrow C(A \pmod{p}, B$

(mod  $p$ ). The idea is to write a representation endomorphism  $f : V_0 \rightarrow V_0$  and interpret the equations resulting from  $f(V_i) \leq V_i$  for each  $i$  and  $f(V_*) \leq V_*$ ; the details of this computation are not included. This completes the proof.

EXAMPLE 2.5. Let  $T$  be a  $p$ -locally finite lattice of types with



For example, designate  $0 = \text{type } \mathbb{Z}$ ,  $1 = \text{type } \mathbb{Z}[3]$ ,  $2 = \text{type } \mathbb{Z}[5]$ ,  $3 = \text{type } \mathbb{Z}[3] + \mathbb{Z}[5]$ ,  $4 = \text{type } \mathbb{Z}[5] + \mathbb{Z}[7]$ , and  $5 = \text{type } \mathbb{Z}[3] + \mathbb{Z}[5] + \mathbb{Z}[7]$ , where  $\mathbb{Z}[n]$  is the subgroup of  $\mathbb{Q}$  generated by  $\{1/n^i : 1 \leq i\}$ , and  $p = 11$ .

Let  $A_i$  be subgroups of  $\mathbb{Q}$  with  $1_{\mathbb{Q}} \in A_i$ ,  $\text{type } A_i = i$ , and  $1_{\mathbb{Q}}/p \notin A_i$ .

(a)  $B(T, 0)_p$  has infinite representation type but each indecomposable has rank  $\leq 2$ . The indecomposables in  $B(T, 0)_p$  of rank 2 are of the form  $G = A_4 \oplus A_3 + \mathbb{Z}(1, 1)/p^j$ .

(b)  $B(T)_p$  has indecomposables of arbitrarily large finite rank.

Proof. (a) Note that

$$\text{JI}(T)^{\text{op}} = \begin{array}{c} 1 \quad 2 \\ | \\ 4 \end{array}$$

Now apply Theorems 2.2 and 1.8 to see that indecomposables have rank  $\leq 2$ . In fact, if  $G$  is as defined above, then

$$\begin{aligned} (G_p, G(1)_p, G(4)_p, G(2)_p) &= (G_p, G(1)_p, G(4)_p, G_p) \\ &= (U_0, 0 \oplus \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)} \oplus 0, U_0) \end{aligned}$$

for  $U_0 = \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)} + \mathbb{Z}_{(p)}(1, 1)/p^j$ , as desired.

(b) is a consequence of Theorems 2.2 and 1.8(b).

**3. Appendix.** Let  $U = (U_0, U_1, U_2, U_3)$  be as defined in Lemma 1.6 and  $f : U_0 \rightarrow U_0$  a representation endomorphism of  $U$ . Since  $f : U_i \rightarrow U_i$  for  $1 \leq i \leq 3$ ,  $f$  can be represented as  $(\alpha, \beta, \gamma)$  for  $\alpha = (a_{ij})$  a  $7 \times 7$  matrix with entries  $a_{ij}$  an  $n \times n$   $\mathbb{Z}_{(p)}$ -matrix,  $\beta = (b_{ij})$  is a  $6 \times 6$  matrix with entries  $b_{ij}$  an  $n \times n$   $\mathbb{Z}_{(p)}$ -matrix, and  $\gamma = (g_{ij})$  is a  $5 \times 5$  matrix with entries  $g_{ij}$  an  $n \times n$   $\mathbb{Z}_{(p)}$ -matrix. Since  $f$  is a  $\mathbb{Z}_{(p)}$ -homomorphism and  $\alpha, \beta, \gamma$  are restrictions of  $f$ , it follows that  $M_2\beta = M_1\alpha M_1^{-1}M_2$  is a  $7 \times 6$  matrix with



entries  $n \times n$  matrices labelled by  $\text{II}(i, j)$  and  $M_3\gamma = M_1\alpha M_1^{-1}M_3$  is a  $7 \times 5$  matrix with entries  $n \times n$  matrices labelled by  $\text{III}(i, j)$ .

A direct computation of the entries  $\text{II}(i, j)$  and  $\text{III}(i, j)$  yields the following equations, where the equations with \* involve matrices on the main diagonal.

$$\begin{aligned}
& * \text{II}(1, 1) : b_{11} = -pa_{12}A + p^5a_{14} - p^4a_{13} + a_{11} \\
& \text{II}(1, 2) : b_{12} = p^2a_{12} \\
& \text{II}(1, 3) : b_{13} = p^6a_{13} \\
& \text{II}(1, 4) : b_{14} = a_{15}p^{-2} \\
& \text{II}(1, 5) : b_{15} = a_{16}p^{-6} \\
& \text{II}(1, 6) : b_{16} = a_{17}p^{-7} \\
& * \text{II}(2, 1) : -pAb_{11} + p^2b_{21} = -pa_{22}A + p^5a_{24} - p^4a_{23} + a_{21} \\
& * \text{II}(2, 2) : p^2b_{22} - pAb_{12} = p^2a_{22} \\
& \text{II}(2, 3) : -pAb_{13} + p^2b_{23} = p^6a_{23} \\
& \text{II}(2, 4) : -pAb_{14} + p^2b_{24} = a_{25}p^{-2} \\
& \text{II}(2, 5) : -pAb_{15} + p^2b_{25} = a_{26}p^{-6} \\
& \text{II}(2, 6) : -pAb_{16} + p^2b_{26} = a_{27}p^{-7} \\
& * \text{II}(3, 1) : -p^4b_{11} + p^6b_{31} = -pa_{32}A + p^5a_{34} - p^4a_{33} + a_{31} \\
& \text{II}(3, 2) : p^6b_{32} - p^4b_{12} = p^2a_{32} \\
& * \text{II}(3, 3) : p^6b_{33} - p^4b_{13} = p^6a_{33} \\
& \text{II}(3, 4) : p^6b_{34} - p^4b_{14} = a_{35}p^{-2} \\
& \text{II}(3, 5) : p^6b_{35} - p^4b_{15} = a_{36}p^{-6} \\
& \text{II}(3, 6) : p^6b_{36} - p^4b_{16} = a_{37}p^{-7} \\
& * \text{II}(4, 1) : p^5b_{11} = -pa_{42}A + p^5a_{44} - p^4a_{43} + a_{41} \\
& \text{II}(4, 2) : p^5b_{12} = p^2a_{42} \\
& \text{II}(4, 3) : p^5b_{13} = p^6a_{43} \\
& \text{II}(4, 4) : p^5b_{14} = a_{45}p^{-2} \\
& \text{II}(4, 5) : p^5b_{15} = a_{46}p^{-6} \\
& \text{II}(4, 6) : p^5b_{16} = a_{47}p^{-7} \\
& \text{II}(5, 1) : b_{41} = -p^3a_{52}A + p^7a_{54} - p^6a_{53} + p^2a_{51} \\
& \text{II}(5, 2) : b_{42} = p^4a_{52} \\
& \text{II}(5, 3) : b_{43} = p^8a_{53} \\
& * \text{II}(5, 4) : b_{44} = a_{55} \\
& \text{II}(5, 5) : b_{45} = a_{56}p^{-4}
\end{aligned}$$

$$\begin{aligned}
& \text{II}(5, 6) : b_{46} = a_{57}p^{-5} \\
& \text{II}(6, 1) : b_{51} = -p^7 a_{62}A + p^{11}a_{64} - p^{10}a_{63} + p^6 a_{61} \\
& \text{II}(6, 2) : b_{52} = p^8 a_{62} \\
& \text{II}(6, 3) : b_{53} = p^{12}a_{63} \\
& \text{II}(6, 4) : b_{54} = p^4 a_{65} \\
& * \text{II}(6, 5) : b_{55} = a_{66} \\
& \text{II}(6, 6) : b_{56} = a_{67}p^{-1} \\
& \text{II}(7, 1) : b_{61} = -p^8 a_{72}A + p^{12}a_{74} - p^{11}a_{73} + p^7 a_{71} \\
& \text{II}(7, 2) : b_{62} = p^9 a_{73} \\
& \text{II}(7, 3) : b_{63} = p^{13}a_{73} \\
& \text{II}(7, 4) : b_{64} = p^5 a_{75} \\
& \text{II}(7, 5) : b_{65} = pa_{76} \\
& * \text{II}(7, 6) : b_{66} = a_{77} \\
& \text{III}(1, 1) : g_{41} = a_{15}p^{-2} + a_{12} \\
& \text{III}(1, 2) : g_{42} = a_{16}p^{-6} + a_{13} \\
& \text{III}(1, 3) : g_{43} = a_{17}p^{-7} + a_{14} \\
& * \text{III}(1, 4) : g_{44} = a_{11} \\
& \text{III}(1, 5) : g_{45} = p^5 a_{13}B + p^6 a_{14} + p^2 a_{12} \\
& * \text{III}(2, 1) : g_{11} + p^2 g_{51} = a_{25}p^{-2} + a_{22} \\
& \text{III}(2, 2) : p^2 g_{52} + g_{12} = a_{26}p^{-6} + a_{23} \\
& \text{III}(2, 3) : p^2 g_{53} + g_{13} = a_{27}p^{-7} + a_{24} \\
& \text{III}(2, 4) : p^2 g_{54} + g_{14} = a_{21} \\
& * \text{III}(2, 5) : p^2 g_{55} + g_{15} = p^5 a_{23}B + p^6 a_{24} + p^2 a_{22} \\
& \text{III}(3, 1) : p^5 Bg_{51} + g_{21} = a_{35}p^{-2} + a_{32} \\
& * \text{III}(3, 2) : p^5 Bg_{52} + g_{22} = a_{36}p^{-6} + a_{33} \\
& \text{III}(3, 3) : p^5 Bg_{53} + g_{23} = a_{37}p^{-7} + a_{34} \\
& \text{III}(3, 4) : p^5 Bg_{54} + g_{24} = a_{31} \\
& * \text{III}(3, 5) : p^5 Bg_{55} + g_{25} = p^5 a_{33}B + p^6 a_{34} + p^2 a_{32} \\
& \text{III}(4, 1) : p^6 g_{51} + g_{31} = a_{45}p^{-2} + a_{42} \\
& \text{III}(4, 2) : p^6 g_{52} + g_{32} = a_{46}p^{-6} + a_{43} \\
& * \text{III}(4, 3) : p^6 g_{53} + g_{33} = a_{47}p^{-7} + a_{44} \\
& \text{III}(4, 4) : p^6 g_{54} + g_{34} = a_{41} \\
& * \text{III}(4, 5) : p^6 g_{55} + g_{35} = p^5 a_{43}B + p^6 a_{44} + p^2 a_{42}
\end{aligned}$$

$$\begin{aligned}
& *III(5, 1) : g_{11} = p^2 a_{52} + a_{55} \\
& III(5, 2) : g_{12} = p^2 a_{53} + a_{56} p^{-4} \\
& III(5, 3) : g_{13} = p^2 a_{54} + a_{57} p^{-5} \\
& III(5, 4) : g_{14} = p^2 a_{51} \\
& III(5, 5) : g_{15} = p^7 a_{53} B + p^8 a_{54} + p^4 a_{52} \\
& III(6, 1) : g_{21} = p^6 a_{62} + p^4 a_{65} \\
& *III(6, 2) : g_{22} = p^6 a_{63} + a_{66} \\
& III(6, 3) : g_{23} = p^6 a_{64} + a_{67} p^{-1} \\
& III(6, 4) : g_{24} = p^6 a_{61} \\
& III(6, 5) : g_{25} = p^{11} a_{63} B + p^{12} a_{64} + p^8 a_{62} \\
& III(7, 1) : g_{31} = p^7 a_{72} + p^5 a_{75} \\
& III(7, 2) : g_{32} = p^7 a_{73} + p a_{76} \\
& *III(7, 3) : g_{33} = p^7 a_{74} + a_{77} \\
& III(7, 4) : g_{34} = p^7 a_{71} \\
& III(7, 5) : g_{35} = p^{12} a_{73} B + p^{13} a_{74} + p^9 a_{72}
\end{aligned}$$

The next step involves a listing of consequences of the above equations, where  $\equiv$  without any modifier denotes congruence mod  $p$ . Relations between main diagonal elements are enclosed in a box for easy identification.

$$\begin{aligned}
II(1, 1) & \Rightarrow \boxed{b_{11} \equiv a_{11}} \\
II(2, 1), III(2, 4), \text{ and } III(5, 4) & \Rightarrow \boxed{Ab_{11} \equiv a_{22}A} \\
II(3, 1) \text{ and } III(3, 4) & \Rightarrow -p^4 b_{11} \equiv -pa_{32}A - p^4 a_{33} + a_{31} \pmod{p^5} \\
& III(3, 4) \Rightarrow a_{31} \equiv g_{24} \pmod{p^5} \\
& III(6, 4) \Rightarrow g_{24} \equiv 0 \pmod{p^6} \Rightarrow a_{31} \equiv 0 \pmod{p^5} \\
& \qquad \qquad \qquad \Rightarrow -p^4 b_{11} \equiv -p^4 a_{33} - pa_{32}A \pmod{p^5} \\
II(3, 2) & \Rightarrow pa_{32} \equiv -p^3 b_{12} \pmod{p^5} \\
II(1, 2) & \Rightarrow b_{12} \equiv 0 \pmod{p^2} \Rightarrow pa_{32} \equiv 0 \pmod{p^5} \Rightarrow \boxed{b_{11} \equiv a_{33}} \\
II(3, 3) \text{ and } II(1, 3) & \Rightarrow \boxed{b_{33} \equiv a_{33}} \\
II(4, 1) & \Rightarrow p^5 b_{11} = p^5 a_{44} - pa_{42}A - p^4 a_{43} + a_{41} \\
& III(4, 4) \Rightarrow a_{41} = p^6 g_{54} + g_{34} \\
& III(7, 4) \Rightarrow g_{34} \equiv 0 \pmod{p^7} \Rightarrow a_{41} \equiv 0 \pmod{p^6} \\
II(4, 3) & \Rightarrow p^6 a_{43} = p^5 b_{13} = p^{11} a_{13} \Rightarrow p^4 a_{43} = p^9 a_{13} \\
& III(4, 1) \Rightarrow a_{42} \equiv g_{31} - a_{45} p^{-2} \pmod{p^5} \\
& III(7, 1) \Rightarrow g_{31} \equiv 0 \pmod{p^5}, \quad II(4, 4) \Rightarrow a_{45} p^{-2} \equiv 0 \pmod{p^5}
\end{aligned}$$

So  $a_{42} \equiv 0 \pmod{p^5}$ , and  $p^5 b_{11} \equiv p^5 a_{44} - p^4 a_{43} \pmod{p^6}$ .

Finally,  $\text{II}(4, 5), \text{II}(4, 2) \Rightarrow a_{43} \equiv g_{32} \pmod{p^2} \Rightarrow \boxed{b_{11} \equiv a_{44}}$

$\text{II}(5, 4), \text{II}(6, 5), \text{II}(7, 6) \Rightarrow \boxed{b_{44} \equiv a_{55}}, \boxed{b_{55} \equiv a_{66}}, \boxed{b_{66} \equiv a_{77}}$

$\text{III}(1, 4) \Rightarrow \boxed{g_{44} \equiv a_{11}}$

$\text{III}(2, 1)$  and  $\text{II}(2, 4) \Rightarrow \boxed{g_{11} \equiv a_{22}}$

$\text{III}(2, 5)$  and  $\text{III}(5, 5) \Rightarrow \boxed{g_{55} \equiv a_{22}}$

$\text{III}(3, 2)$  and  $\text{II}(3, 5) \Rightarrow \boxed{g_{22} \equiv a_{33}}$

$\text{III}(3, 5)$  and  $\text{III}(6, 5) \Rightarrow p^5 B g_{55} \equiv p^5 a_{33} B + p^2 a_{32} \pmod{p^6}$

$\text{II}(1, 2)$  and  $\text{II}(3, 2) \Rightarrow p^2 a_{32} \equiv 0 \pmod{p^6} \Rightarrow \boxed{B g_{55} \equiv a_{33} B}$

$\text{III}(4, 3)$  and  $\text{II}(4, 6) \Rightarrow \boxed{g_{33} \equiv a_{44}}$

$\text{III}(4, 5) \Rightarrow p^6 g_{55} + g_{35} = p^5 a_{43} B + p^6 a_{44} + p^2 a_{42}$

$\text{III}(7, 5) \Rightarrow g_{35} \equiv 0 \pmod{p^9}$

$\text{II}(4, 3)$  and  $\text{II}(1, 3) \Rightarrow p^5 a_{43} = p^4 b_{13} = p^{10} a_{13} \equiv 0 \pmod{p^{10}}$

$\text{II}(4, 2)$  and  $\text{II}(1, 2) \Rightarrow p^2 a_{42} = p^5 b_{12} = p^7 a_{12} \Rightarrow \boxed{g_{55} \equiv a_{44}}$

$\text{III}(5, 1) \Rightarrow \boxed{g_{11} \equiv a_{55}}$

$\text{III}(6, 2) \Rightarrow \boxed{g_{22} \equiv a_{66}}$

$\text{III}(7, 3) \Rightarrow \boxed{g_{33} \equiv a_{77}}$

$\text{II}(2, 2) \Rightarrow p^2 b_{22} - p A b_{12} = p^2 a_{22}$

$\text{II}(1, 2) \Rightarrow b_{12} = p^2 a_{12} \Rightarrow \boxed{b_{22} \equiv a_{22}}$

We now see that all diagonal elements are congruent mod  $p$  to a fixed  $n \times n \mathbb{Z}/p\mathbb{Z}$  matrix  $a$  with  $aA \equiv Aa$  and  $aB \equiv Ba$ . Then

$$\begin{bmatrix} \alpha \pmod{p} & 0 & 0 \\ 0 & \beta \pmod{p} & 0 \\ 0 & 0 & \gamma \pmod{p} \end{bmatrix} = a \pmod{p} I + N$$

for some nilpotent  $\mathbb{Z}/p\mathbb{Z}$ -matrix  $N$  with zero diagonal elements. In particular, an inspection of the equations for  $\text{II}$  and  $\text{III}$  shows that  $\alpha \pmod{p}$  is of the form

$$\begin{bmatrix} a & * & * & * & 0 & 0 & 0 \\ 0 & a & * & * & 0 & 0 & 0 \\ 0 & 0 & a & * & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 \\ * & * & * & * & a & 0 & 0 \\ * & * & * & * & * & a & 0 \\ * & * & * & * & * & * & a \end{bmatrix},$$

$\beta \pmod{p}$  is of the form

$$\begin{bmatrix} a & 0 & 0 & * & * & * \\ * & a & 0 & * & * & * \\ * & * & a & * & * & * \\ 0 & 0 & 0 & a & * & * \\ 0 & 0 & 0 & 0 & a & * \\ 0 & 0 & 0 & 0 & 0 & a \end{bmatrix}$$

and  $\gamma \pmod{p}$  is of the form

$$\begin{bmatrix} a & * & * & 0 & 0 \\ 0 & a & * & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ * & * & * & a & 0 \\ * & * & * & * & a \end{bmatrix}.$$

A matrix computation shows that  $N$  is nilpotent. It now follows, as in the proof of Theorem 1.8, that there is a  $\mathbb{Z}_{(p)}$ -algebra epimorphism  $\text{End } U \rightarrow C(A \pmod{p}, B \pmod{p})$ .

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