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ON COMPACT ELEMENTS IN SOLVABLE LIE GROUPS

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1. Introduction. In considering solvmanifolds, L. Auslander investigated *rigid* Lie groups, i.e., simply connected solvable Lie groups where all elements x of their Lie algebra meet the following condition: $\operatorname{spec}\operatorname{ad}(x) \subseteq i\mathbb{R}$ (cf. [2]). In a sense, these groups are opposite to those solvable Lie groups whose exponential map is a homeomorphism. While in the latter case the structure of the group is uniquely determined by the structure of its Lie algebra, the infinitesimal properties of rigid Lie groups do not reflect all global properties. We will consider a class of Lie groups which are close to rigid Lie groups, namely Lie groups generated by their compact elements: If G is a real Lie group and $g \in G$, then g will be called *compact* if it is an element of a compact subgroup of G. The set of all compact elements of G is denoted by $\operatorname{comp} G$. It is exactly the union of all compact subgroups of G. In general, $\operatorname{comp} G$ is not a subgroup. The subgroup which is generated by the compact elements will be denoted by $\kappa(G) := \langle \operatorname{comp} G \rangle$. Since the group $\kappa(G)$ contains a maximal torus, it is closed. If H is a subgroup of G with $H \supseteq \kappa(G)$ we get $\kappa(H) \subseteq \kappa(G)$. On the other hand, all compact subgroups of G are in H and $\kappa(H) = \kappa(G)$. If we set $H = \kappa(G)$ we get $\kappa(\kappa(G)) = \kappa(G)$. Our aim is to characterize solvable Lie groups G such that $\kappa(G) = G$ and those satisfying $\overline{\operatorname{comp} G} = G$.

The Lie algebra of a Lie group G is denoted by $\mathbf{L}(G)$ or, simply, by $\mathfrak{g}$. We remark that in a solvable Lie algebra $\mathfrak{g}$ the commutator subalgebra $\mathfrak{g}'$ is a nilpotent ideal. Analogously, in a solvable connected Lie group G the commutator subgroup G' is a nilpotent normal subgroup. Furthermore, if G is connected then so is G'. We denote the intersection of the descending central series of $\mathfrak{g}$ by $\mathcal{C}^\infty$ and the center of $\mathfrak{g}$ by $\mathfrak{z}$. Since the compact connected subgroups of a solvable Lie group are tori and all maximal tori are conjugate under $\widetilde{\Gamma} = I_{\exp \mathcal{C}^\infty}$, where I_g denotes the inner conjugation with g, we get $\operatorname{comp} G = \widetilde{\Gamma}.T$ for a maximal torus T.

It is clear that we also have to look at Lie algebras in which we shall call an element compact if it lies in a *compactly embedded subalgebra* $\mathfrak{t}$, which

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means that $\overline{e^{\operatorname{ad}\mathfrak{t}}}$ is compact in $\operatorname{Aut}(\mathfrak{g})$. We note that $\langle e^{\operatorname{ad}\mathfrak{g}}\rangle$ is a solvable analytic subgroup of $\operatorname{Aut}(\mathfrak{g})$. Thus its closure is solvable as well. But the only compact subgroups of a solvable Lie group are tori, which are, in particular, abelian. So, each compactly embedded subalgebra of a solvable Lie algebra is abelian. Compact elements in Lie algebras are exactly those elements where $\operatorname{ad}(x)$ is semisimple and has only pure imaginary eigenvalues. The set of all compactly embedded subalgebras of $\mathfrak{g}$ is denoted by $\operatorname{comp}\mathfrak{g}$. In particular, $\operatorname{comp}\mathfrak{g}$ is the union of all compactly embedded subalgebras of $\mathfrak{g}$. Since $\operatorname{comp}\mathfrak{g}$, in general, is not a subalgebra, we set $\kappa(\mathfrak{g}) = \langle\operatorname{comp}\mathfrak{g}\rangle$. Let $\Gamma := e^{\operatorname{ad}\mathcal{C}^\infty} \subseteq \operatorname{Inn}(\mathfrak{g})$. In [6] it is proved that in the solvable case we have $\operatorname{comp}\mathfrak{g} = \Gamma.\mathfrak{k}$ for a maximal compactly embedded subalgebra $\mathfrak{k}$ and that $\kappa(\mathfrak{g}) = \operatorname{span}\operatorname{comp}\mathfrak{g}$. In Theorem 3.2, we shall show that $\kappa(\kappa(\mathfrak{g})) = \kappa(\mathfrak{g})$. In [6] and [9] one may find more information about compact elements in solvable Lie algebras.

In Section 4, we shall show that the solvable connected Lie groups G which are generated by their compact elements are characterized by $\mathfrak{g} = \kappa(\mathfrak{g}) = \langle\Gamma.\mathfrak{t}\rangle$ where $\mathfrak{t}$ is the Lie algebra of a maximal torus of G. This statement is equivalent to the conditions that $\mathfrak{g} = \mathfrak{t} + \mathfrak{g}'$ and $\mathfrak{z}\cap\mathfrak{k} \subseteq \langle\Gamma.\mathfrak{t}\rangle$ and $\mathfrak{k} \supseteq \mathfrak{t}$. Furthermore, in Section 5, we shall characterize Lie algebras where $\operatorname{comp}\mathfrak{g}$ is dense, and in Section 6 Lie groups where $\operatorname{comp} G$ is dense. For this, we need Cartan subgroups, which we now introduce.

An element x in a Lie algebra $\mathfrak{g}$ is called *regular* if the nilspace $\mathfrak{g}^0(\operatorname{ad} x)$ has minimal dimension. This dimension is called the *rank* of $\mathfrak{g}$. Remember that this nilspace is a Cartan algebra and each Cartan algebra can be written in this manner. The set of all regular elements of $\mathfrak{g}$ is denoted by $\operatorname{reg}\mathfrak{g}$. We remark that $\operatorname{reg}\mathfrak{g}$ is open and dense in $\mathfrak{g}$. An element $y \in \mathfrak{g}$ is called *exp-regular* if the exponential function is regular in y. The set of all exp-regular elements of $\mathfrak{g}$ is denoted by $\operatorname{reg}\exp$. The set $\operatorname{reg}\exp$ is open and dense in $\mathfrak{g}$ as well. An element $g \in G$ is called *regular* if the nilspace $N(\operatorname{Ad}(g) - \operatorname{id})$ of $\operatorname{Ad}(g) - \operatorname{id}$ has minimal dimension, i.e., if $\dim N(\operatorname{Ad}(g) - \operatorname{id})$ is equal to the rank of the Lie algebra $\mathbf{L}(G)$. The set of all regular elements of G is denoted by $\operatorname{Reg} G$. Analogously, $\operatorname{Reg} G$ is open and dense in G (cf. [8, Lemma 4]).

Now we define Cartan subgroups: Let $N(\mathfrak{h}) = \{g \in G : \operatorname{Ad}(g).\mathfrak{h} = \mathfrak{h}\}$ denote the normalizer of $\mathfrak{h}$ in G. Then $N(\mathfrak{h})$ acts on the root space Λ on the right via $(\lambda, g) \mapsto \lambda \circ \operatorname{Ad}(g)$. We set $C(\mathfrak{h}) = \{g \in N(\mathfrak{h}) : \lambda \circ \operatorname{Ad}(g) = \lambda$ for all $\lambda \in \Lambda\}$. We say that a subgroup H of a connected Lie group G is a *Cartan group* if $\mathbf{L}(H)$ is a Cartan algebra and $H = C(\mathbf{L}(H))$. In solvable Lie groups, Cartan subgroups H are connected (cf. [8, Proposition 19]), hence $H = \exp\mathbf{L}(H)$, and are conjugate under $\widetilde{\Gamma}$.

We shall prove that the connected solvable Lie groups with a centerfree Lie algebra whose set of compact elements is dense are characterized by the fact that their Cartan subgroups are exactly the maximal tori. If the

Lie algebra has a center the condition is that $\mathfrak{g} = \overline{\Gamma.\mathfrak{t}}$ where $\mathfrak{t}$ is the Lie algebra of a maximal torus of G. This is equivalent to the fact that Cartan subgroups of $G/\exp\mathfrak{z}$ are maximal tori and that $\mathfrak{z} \subseteq \overline{\Gamma.\mathfrak{t}}$.

2. The weight decomposition. As a useful tool, we need the *weight decomposition* of a Lie algebra with respect to some nilpotent subalgebra $\mathfrak{n}$. First, we assume that $\mathfrak{g}$ is a complex Lie algebra. Let $\mathfrak{n}^*$ denote the dual space of $\mathfrak{n}$, i.e., all linear maps from $\mathfrak{n}$ into $\mathbb{C}$. A $\lambda \in \mathfrak{n}^*$ is called a *weight* if there is an $0 \neq x \in \mathfrak{g}$ and an $n \in \mathbb{N}$ such that $(\mathrm{ad}(u) - \lambda(u)\,\mathrm{id})^n x = 0$ for all $u \in \mathfrak{n}$. If λ is not the trivial map it is called a *root*. The set of all roots is denoted by Λ. We define $\mathfrak{g}^\lambda = \{x \in \mathfrak{g} : (\forall u \in \mathfrak{n})(\exists n \in \mathbb{N})\ (\mathrm{ad}(u) - \lambda(u)\,\mathrm{id})^n x = 0\}$. By [3, VII.1.3, Proposition 9], we can decompose $\mathfrak{g}$ as follows: $\mathfrak{g} = \mathfrak{g}^0 + \sum_{\lambda\in\Lambda} \mathfrak{g}^\lambda$. This is the *weight decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{n}$. We note that each $\lambda \in \Lambda$ maps $u \in \mathfrak{n}$ onto an eigenvalue of $\mathrm{ad}\, u$.

In the real case, the situation is more complicated. But in view of our subject, we can confine ourselves to compactly embedded abelian subalgebras (cf. [4, III.6]). Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{k}$ be a compactly embedded abelian subalgebra. Now we turn to the complexification $\mathfrak{g}_{\mathbb{C}}$ and consider the weight decomposition with respect to $\mathfrak{k}_{\mathbb{C}}$. Since each $\mu \in \mathrm{spec}\,\mathrm{ad}\, k$ for all $k \in \mathfrak{k}_{\mathbb{C}}$ is purely imaginary, $-\mu$ is in $\mathrm{spec}\,\mathrm{ad}\, k$ as well. Thus, if λ is a weight then so is $-\lambda$. If we set $\omega = -i\lambda|_{\mathfrak{k}}$ then ω is a real linear form on $\mathfrak{k}$. We denote by Ω the set $\{\omega \in \mathfrak{k}^* : (\exists \lambda \in \Lambda)\ \omega = -i\lambda|_{\mathfrak{k}}\}$. Now we set $\mathfrak{g}^\omega = \mathfrak{g} \cap (\mathfrak{g}_{\mathbb{C}}^\lambda \oplus \mathfrak{g}_{\mathbb{C}}^{-\lambda})$. We observe that $\mathfrak{g}^\omega = \mathfrak{g}^{-\omega}$. So, we can decompose Ω into Ω^+ and Ω^- where $\Omega^- = -\Omega^+$. (Note that $\omega \neq 0$ for all $\omega \in \Omega$.) The decomposition $\mathfrak{g} = \mathfrak{g}^0 \oplus \sum_{\omega\in\Omega^+} \mathfrak{g}^\omega$ is called the *real weight decomposition* of $\mathfrak{g}$. We abbreviate $\sum_{\omega\in\Omega^+} \mathfrak{g}^\omega$ to $\mathfrak{g}^+$, which is also called the *Fitting one-component*. Note that the Fitting one-component depends on $\mathfrak{k}$. We observe that $\mathfrak{g}^0 = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$ where $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$ denotes the centralizer of $\mathfrak{k}$ in $\mathfrak{g}$.

Next we introduce a *complex structure* on $\mathfrak{g}^+$. Let V be a real vector space. We call a linear map I a *complex structure on* V if $I^2 = -\,\mathrm{id}$. On $\mathfrak{g}^+$ we can define a complex structure as follows: Let u be any element of $\mathfrak{k}$ and $x \in \mathfrak{g}^+$. Then we can write x as $\sum_{\omega\in\Lambda^+} x^\omega$ and define $I : \mathfrak{g}^+ \to \mathfrak{g}^+$, $x \mapsto \sum_{\omega\in\Lambda^+} [u, x^\omega]/\omega(u)$. Note that this definition does not depend on u.

3. $\kappa(\mathfrak{g})$ in a solvable Lie algebra. Let $\mathfrak{g}$ be a real solvable Lie algebra. First, we show that $\kappa(\kappa(\mathfrak{g})) = \kappa(\mathfrak{g})$.

3.1. LEMMA. *Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{u}$ a subalgebra with* $\mathrm{comp}\,\mathfrak{g} \subseteq \mathfrak{u}$. *Then* $\mathrm{comp}\,\mathfrak{g} \subseteq \mathrm{comp}\,\mathfrak{u}$.

Proof. If $x \in \mathrm{comp}\,\mathfrak{g}$ then x is compact in $\mathfrak{u}$ as well. Thus $\mathrm{comp}\,\mathfrak{g} \subseteq \mathrm{comp}\,\mathfrak{u}$. ■

3.2. THEOREM. *In a solvable Lie algebra $\mathfrak{g}$ we have $\kappa(\kappa(\mathfrak{g})) = \kappa(\mathfrak{g})$.*

P r o o f. By the preceding lemma, we get

$$\kappa(\mathfrak{g}) = \operatorname{span}\operatorname{comp}\mathfrak{g} \subseteq \operatorname{span}\operatorname{comp}\kappa(\mathfrak{g}) = \kappa(\kappa(\mathfrak{g})).$$

The inclusion $\kappa(\kappa(\mathfrak{g})) \subseteq \kappa(\mathfrak{g})$ is clear. ∎

Now we characterize solvable Lie algebras which are generated by their compact elements.

3.3. Lemma. *Let $\mathfrak{g}$ be a Lie algebra.*

(i) *If $\varphi : \mathfrak{g} \to \mathfrak{g}$ is an automorphism then $\varphi(\operatorname{comp}\mathfrak{g}) \subseteq \operatorname{comp}\mathfrak{g}$.*
(ii) *The subalgebra $\kappa(\mathfrak{g})$ is a characteristic ideal of $\mathfrak{g}$.*

P r o o f. (i) Let $\mathfrak{t}$ be a compactly embedded subalgebra of $\mathfrak{g}$. Then $\varphi(\mathfrak{t})$ is compactly embedded in $\varphi(\mathfrak{g}) = \mathfrak{g}$. Thus $\varphi(\operatorname{comp}\mathfrak{g}) \subseteq \operatorname{comp}\mathfrak{g}$.

(ii) Condition (i) implies

$$\varphi(\kappa(\mathfrak{g})) = \varphi(\operatorname{span}\operatorname{comp}\mathfrak{g}) = \operatorname{span}\varphi(\operatorname{comp}\mathfrak{g}) \subseteq \operatorname{span}\operatorname{comp}\mathfrak{g} = \kappa(\mathfrak{g}).$$

In particular, $\kappa(\mathfrak{g})$ is invariant under inner derivations. ∎

3.4. Lemma. *Let $\mathfrak{g}$ be a Lie algebra and $\operatorname{der}\mathfrak{g}$ the set of all derivations of $\mathfrak{g}$. If $x \in \operatorname{comp}\mathfrak{k}$ then $(\operatorname{der}\mathfrak{g}).x \subseteq \kappa(\mathfrak{g})$.*

P r o o f. Assume that $\delta \in \operatorname{der}\mathfrak{g}$. Then $e^{t\delta} \in \operatorname{Aut}(\mathfrak{g})$ for all $t \in \mathbb{R}$. The invariance of $\operatorname{comp}\mathfrak{g}$ implies $e^{t\delta}x \in \mathfrak{g}$ for all $t \in \mathbb{R}$. So $t^{-1}(e^{t\delta}x - x) \in \kappa(\mathfrak{g})$ for all $t \in \mathbb{R}$. By a limiting process we get

$$\delta(x) = \lim_{t\to 0,\, t\neq 0} \frac{1}{t}(e^{t\delta}x - x) \in \kappa(\mathfrak{g}).$$

Thus $\operatorname{der}(\mathfrak{g}).x \subseteq \kappa(\mathfrak{g})$. ∎

3.5. Lemma. *Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{k}$ a maximal compactly embedded abelian subalgebra. Then $\mathfrak{g}^+ \subseteq \operatorname{comp}\mathfrak{g}$.*

P r o o f. We know that $\mathfrak{g}^+ = \sum_{\omega\in\Omega^+} \mathfrak{g}^\omega$. Assume that $x \in \mathfrak{g}^\omega$, $r \in \mathbb{R}$, $\omega \neq 0$. We find $u \in \mathfrak{k}$ with $\omega(u) = r$. Now let I be a complex structure on $\mathfrak{g}^+$. Lemma 3.4 implies $rIx = \omega(u)Ix = [u,x] = -\operatorname{ad}(x)(u) \in \operatorname{der}(\mathfrak{g}).\mathfrak{k} \subseteq \kappa(\mathfrak{g})$. Thus we have $\mathbb{R}Ix \subseteq \kappa(\mathfrak{g})$, and because of $I\mathfrak{g}^\omega = \mathfrak{g}^\omega$ we get $\mathfrak{g}^\omega \subseteq \kappa(\mathfrak{g})$. ∎

Using these facts we prove the following theorem which gives information about the structure of $\kappa(\mathfrak{g})$.

3.6. Theorem. *Let $\mathfrak{g}$ be a solvable Lie algebra and suppose that $\mathfrak{k}$ is a compactly embedded abelian subalgebra. We have $\kappa(\mathfrak{g}) = \mathfrak{k} + \langle\mathfrak{g}^+\rangle$.*

P r o o f. By Lemma 3.5 we have $\mathfrak{g}^+ \subseteq \kappa(\mathfrak{g})$. Since $\kappa(\mathfrak{g})$ is a subalgebra we get $\langle\mathfrak{g}^+\rangle \subseteq \kappa(\mathfrak{g})$. Of course, $\mathfrak{k}$ lies in $\kappa(\mathfrak{g})$, hence $\mathfrak{k} + \langle\mathfrak{g}^+\rangle \subseteq \kappa(\mathfrak{g})$. On the other hand, we have $[\mathfrak{g}^0, \mathfrak{k} + \langle\mathfrak{g}^+\rangle] = [\mathfrak{g}^0, \langle\mathfrak{g}^+\rangle] \subseteq \langle\mathfrak{g}^+\rangle \subseteq \mathfrak{k} + \langle\mathfrak{g}^+\rangle$. Thus, $\mathfrak{k} + \langle\mathfrak{g}^+\rangle$ is an ideal in $\mathfrak{g}$. So, $\operatorname{comp}\mathfrak{g} = \Gamma.\mathfrak{k} \subseteq \mathfrak{k} + \langle\mathfrak{g}^+\rangle$ and, consequently, $\kappa(\mathfrak{g}) = \operatorname{span}\Gamma.\mathfrak{k} \subseteq \mathfrak{k} + \langle\mathfrak{g}^+\rangle$. ∎

3.7. Corollary. *In a solvable Lie algebra* $\kappa(\mathfrak{g}) = \mathfrak{g}$ *holds if and only if* $\mathfrak{g} = \mathfrak{k} + \langle \mathfrak{g}^+ \rangle$.

4. $\kappa(G)$ **in a solvable Lie group.** If we investigate Lie groups, we cannot simply transfer the results for the Lie algebras. The reason is that, in general, the exponential function need not map compact elements of the Lie algebra to compact elements of the Lie group.

4.1. Proposition. *In any Lie group* G, *we have* $\exp^{-1}(\operatorname{comp} G) \subseteq \operatorname{comp} \mathfrak{g}$.

Proof. Assume that $x \in \exp^{-1}(\operatorname{comp} G)$. Then $\exp x$ is contained in a torus T. Thus, there is a maximal compactly embedded subalgebra $\mathfrak{t}$ with $\exp x \in \exp \mathfrak{t}$. Hence, $\operatorname{ad}(x)$ is semisimple and has only pure imaginary eigenvalues. Thus $x \in \operatorname{comp} \mathfrak{g}$. ∎

It is possible to find conditions for $G = \kappa(G)$ and $G = \overline{\operatorname{comp} G}$ which can be tested by the Lie algebras. Before we turn to this problem we prove the analogous case of Theorem 3.2. The next proposition is very useful to reduce a problem given on the level of Lie groups to the level of Lie algebras.

4.2. Proposition. *Let* $(A_i)_{i \in I}$ *be a family of analytic subgroups of the Lie group* G. *Then* $\langle \bigcup_{i \in I} \mathbf{L}(A_i) \rangle = \mathbf{L}(\langle \bigcup_{i \in I} A_i \rangle)$.

Proof. (i) Assume that $x \in \bigcup_{i \in I} \mathbf{L}(A_i)$. Then there is an $i \in I$ with $x \in \mathbf{L}(A_i)$. This means that there is an $i \in I$ with $\exp \mathbb{R}x \subseteq A_i$. Therefore $\exp \mathbb{R}x \subseteq \langle \bigcup_{i \in I} A_i \rangle$ and this yields $x \in \mathbf{L}(\langle \bigcup_{i \in I} A_i \rangle)$. Thus $\bigcup_{i \in I} \mathbf{L}(A_i) \subseteq \mathbf{L}(\langle \bigcup_{i \in I} A_i \rangle)$ and hence $\langle \bigcup_{i \in I} \mathbf{L}(A_i) \rangle \subseteq \mathbf{L}(\langle \bigcup_{i \in I} A_i \rangle)$.

(ii) We prove that for all Lie algebras $\mathfrak{m}$ with $\mathfrak{m} \supseteq \bigcup_{i \in I} \mathbf{L}(A_i)$ we have $\mathfrak{m} \supseteq \mathbf{L}(\langle \bigcup_{i \in I} A_i \rangle)$. First, we note that $\mathfrak{m} \supseteq \mathbf{L}(A_i)$ for all $i \in I$. So $\langle \exp \mathfrak{m} \rangle \supseteq A_i$ holds because the A_i are analytic. Consequently, we get $\langle \exp \mathfrak{m} \rangle \supseteq \langle \bigcup_{i \in I} A_i \rangle$. Furthermore, $\langle \exp \mathfrak{m} \rangle$ is analytic. This yields $\mathfrak{m} = \mathbf{L}(\langle \exp \mathfrak{m} \rangle) \supseteq \mathbf{L}(\langle \bigcup_{i \in I} A_i \rangle)$. So, $\mathbf{L}(\langle \bigcup_{i \in I} A_i \rangle)$ is the smallest Lie algebra which contains $\bigcup_{i \in I} \mathbf{L}(A_i)$, and hence is equal to $\langle \bigcup_{i \in I} \mathbf{L}(A_i) \rangle$. ∎

4.3. Theorem. *Let* G *be a solvable connected Lie group and* T *a maximal torus of* G. *Then* $G = \kappa(G)$ *if and only if* $\mathfrak{g} = \kappa(\mathfrak{g}) = \langle \Gamma . \mathfrak{t} \rangle$.

Proof. We assume that $G = \kappa(G)$. Since T is a maximal torus in G, we have $G = \langle \widetilde{\Gamma}.T \rangle$. By Proposition 4.2 we get $\mathfrak{g} = \mathbf{L}(\langle \widetilde{\Gamma}.T \rangle) = \langle \Gamma.\mathfrak{t} \rangle$. On the other hand, we have $\kappa(\mathfrak{g}) = \langle \Gamma.\mathfrak{k} \rangle$, where $\mathfrak{k}$ is a maximal compactly embedded Lie algebra. Thus $\mathfrak{t} \subseteq \mathfrak{k}$ yields $\mathfrak{g} = \langle \Gamma.\mathfrak{t} \rangle \subseteq \kappa(\mathfrak{g})$. Now suppose that $\mathfrak{g} = \kappa(\mathfrak{g}) = \langle \Gamma.\mathfrak{t} \rangle$. Proposition 4.2 yields $\langle \Gamma.\mathfrak{t} \rangle = \mathbf{L}(\langle \widetilde{\Gamma}.T \rangle) = \mathbf{L}(\kappa(G))$. ∎

Now we look for an equivalent condition to $\mathfrak{g} = \langle \Gamma.\mathfrak{t} \rangle$.

4.4. PROPOSITION. *Let* $\mathfrak{g}$ *be a solvable Lie algebra with* $\mathfrak{g} = \kappa(\mathfrak{g})$, $\mathfrak{k}$ *a maximal compactly embedded subalgebra and* $\mathfrak{t}$ *a subalgebra of* $\mathfrak{k}$. *Then* $\langle \Gamma.\mathfrak{t}\rangle = \mathfrak{g}$ *if and only if* $\mathfrak{g} = \mathfrak{t} + \mathfrak{g}'$ *and* $\mathfrak{z} \cap \mathfrak{g}' \subseteq \langle \Gamma.\mathfrak{t}\rangle$.

Proof. We only have to prove the "only if" direction. Since $\mathfrak{k}$ is compactly embedded and $\mathfrak{g}'$ is nilpotent, we have $\mathfrak{k}\cap\mathfrak{g}' \subseteq \mathfrak{z}$, hence $\mathfrak{k}\cap\mathfrak{g}' = \mathfrak{z}\cap\mathfrak{g}'$. As $\mathfrak{k}+\mathfrak{g}' = \mathfrak{t}+\mathfrak{g}'$ we get $\mathfrak{k} = \mathfrak{t}+(\mathfrak{k}\cap\mathfrak{g}')$. It follows that $\Gamma.\mathfrak{k} \subseteq \Gamma.\mathfrak{t}+\Gamma.(\mathfrak{k}\cap\mathfrak{g}') = \Gamma.\mathfrak{t} + (\mathfrak{z}\cap\mathfrak{g}')$. Thus $\mathfrak{g} = \kappa(\mathfrak{g}) \subseteq \langle \Gamma.\mathfrak{t}\rangle$. ∎

5. The case $\mathfrak{g} = \overline{\operatorname{comp}\mathfrak{g}}$.

Our aim is to find conditions that are equivalent to $G = \overline{\operatorname{comp} G}$. To this end, we first consider the analogous problem for Lie algebras.

5.1. THEOREM. *Let* $\mathfrak{g}$ *be a solvable Lie algebra and* $\mathfrak{k}$ *a maximal compactly embedded subalgebra. Moreover*, *assume* $\overline{\operatorname{comp}\mathfrak{g}} = \mathfrak{g}$. *Then* $\mathfrak{k} = \mathfrak{g}^0 = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$. *Consequently*, $\mathfrak{k}$ *is a Cartan subalgebra.*

Proof. Since $\kappa(\mathfrak{g}) \supseteq \overline{\operatorname{comp}\mathfrak{g}} = \mathfrak{g}$, we have $\mathfrak{g} = \mathfrak{k}+\mathfrak{g}'$. Since $\mathfrak{g}'$ is nilpotent and $\operatorname{reg}\mathfrak{g}$ is dense in $\mathfrak{g}$, there is a regular $t \in \mathfrak{k}$. We define the analytic map $\varphi : \mathfrak{g}^0 \times \mathfrak{g}^+ \to \mathfrak{g}, (s,x) \mapsto e^{\operatorname{ad} x}(t+s)$. This map has the derivative

$$d\varphi(s_0,x_0) : \mathfrak{g}^0 \times \mathfrak{g}^+ \to \mathfrak{g}^0 \times \mathfrak{g}^+,\ (s,x) \mapsto (e^{\operatorname{ad} x_0}s, (d\exp(x_0)\circ(\operatorname{ad} -t))x).$$

It is invertible for $x_0 = 0$ because $(d\exp(x_0)\circ \operatorname{ad}(-t))$ is invertible. Thus, there are open zero-neighborhoods $U \subseteq \mathfrak{g}^0$, $V \subseteq \mathfrak{g}^+$ and an open neighborhood W of t in $\mathfrak{g}$ such that φ is a local diffeomorphism from $U\times V$ to W. Since $\operatorname{comp}\mathfrak{g}$ is dense in $\mathfrak{g}$ and W is an open neighborhood of t in $\mathfrak{g}$ we see that $W\cap\operatorname{comp}\mathfrak{g}$ is dense in W. Since φ is a diffeomorphism, $\varphi^{-1}(W)$ is dense in $U\times V$, and $\varphi(\varphi^{-1}(W)) \subseteq \operatorname{comp}\mathfrak{g}$. Now suppose that $(u,v)\in\varphi^{-1}(W)$. Then we have $e^{\operatorname{ad} v}(t+u) \in \operatorname{comp}\mathfrak{g}$ and hence $t+u \in \operatorname{comp}\mathfrak{g}\cap\mathfrak{g}^0$.

Now we show $\operatorname{comp}\mathfrak{g}\cap\mathfrak{g}^0 = \mathfrak{k}$. Assume that $x \in (\operatorname{comp}\mathfrak{g})\cap\mathfrak{g}^0$. Then $[x,\mathfrak{k}] = \{0\}$. Thus $\mathfrak{k}_1 := \mathbb{R}x + \mathfrak{k}$ is compactly embedded and $\mathfrak{k}_1 \supseteq \mathfrak{k}$. The maximality of $\mathfrak{k}$ implies $x \in \mathfrak{k}$. This in turn implies $t+u \in \mathfrak{k}$ and hence $u \in \mathfrak{k}$. If we consider the projection $\pi_U : U\times V \to U, (u,v)\mapsto u$, we see that $\pi_U(\varphi^{-1}(W))$ is dense in $U \subseteq \mathfrak{k}$. This implies $\mathfrak{g}^0 \subseteq \mathfrak{k}$. ∎

The next theorem is based upon [5, Prop. IV.4.4].

5.2. THEOREM. *Let* $\mathfrak{g}$ *be a solvable Lie algebra and* $\mathfrak{k}$ *a maximal compactly embedded subalgebra. Then the following claims are equivalent*:

(i) $\mathfrak{k}$ *is a Cartan subalgebra. In particular*, *all Cartan subalgebras of* $\mathfrak{g}$ *are compactly embedded.*

(ii) $\operatorname{comp}\mathfrak{g}$ *is dense.*

(iii) $\mathfrak{k} = \mathfrak{g}^0 = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$, *that is*, $\mathfrak{k}$ *is its own centralizer.*

(iv) *We have* $\mathfrak{g} = \kappa(\mathfrak{g})$ *and for one* (*and hence for every*) *Cartan subalgebra* $\mathfrak{h}$ *we have* $\mathfrak{h}\cap\mathfrak{g}' \subseteq \mathfrak{z}$.

If these conditions are satisfied then $\operatorname{comp}\mathfrak{g}$ *is a neighborhood of each of its regular points. Moreover*, $\mathfrak{g}' = (\mathfrak{z} \cap \mathfrak{g}') + \mathfrak{g}^+$.

Proof. (i)⇒(ii). Each regular point is contained in a Cartan subalgebra, hence in a subalgebra conjugate to $\mathfrak{k}$. But the regular points are dense. Thus $\operatorname{comp}\mathfrak{g}$ is dense.

(ii)⇒(iii). This follows from Theorem 5.1.

(iii)⇒(i). Since $\mathfrak{k}$ is an abelian subalgebra only containing semisimple elements, by [3, VII, §2, No. 3, Prop. 10] each Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$ is a Cartan subalgebra of $\mathfrak{g}$. Furthermore, we get $\mathfrak{k} \subseteq \mathfrak{h} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$. Thus $\mathfrak{k}$ is a Cartan subalgebra. Since in a solavble Lie algebra all Cartan subalgebras are conjugate each of them is compactly embedded.

(i)⇒(iv). If $\mathfrak{k}$ is a Cartan subalgebra we have $\mathfrak{g} = \mathfrak{k} + \mathfrak{g}'$, hence $\mathfrak{g} = \kappa(\mathfrak{g})$. Furthermore, we have $\mathfrak{k} \cap \mathfrak{g}' \subseteq \mathfrak{z}$.

(iv)⇒(i). We have $\mathfrak{g} = \mathfrak{k} + \mathfrak{g}'$. Moreover, there is a Cartan subalgebra $\mathfrak{h} \supseteq \mathfrak{k}$. Thus $\mathfrak{h} = \mathfrak{k} + (\mathfrak{h} \cap \mathfrak{g}') \subseteq \mathfrak{k} + \mathfrak{z} \subseteq \mathfrak{k}$. Thus $\mathfrak{k}$ is a Cartan subalgebra.

Now we prove the last claim. We observe that $\mathfrak{g} = \mathfrak{k} + \mathfrak{g}^+$ and $\mathfrak{g}^+ \subseteq \mathfrak{g}'$. Thus $\mathfrak{g}' = \mathfrak{g}^+ + (\mathfrak{g}' \cap \mathfrak{k})$. But since $\mathfrak{g}'$ consists of nilpotent and $\mathfrak{k}$ consists of semisimple elements we have $\mathfrak{g}' \cap \mathfrak{k} = \mathfrak{g}' \cap \mathfrak{z}$. ■

6. The case $\overline{\operatorname{comp} G} = G$. We look for necessary and sufficient conditions for $\overline{\operatorname{comp} G} = G$. Of course, a necessary condition is that $\overline{\operatorname{comp} G} = \kappa(G)$. Moreover, if G is a Lie group we have $\overline{\operatorname{comp} G} \subseteq \overline{\exp \operatorname{comp}\mathfrak{g}}$.

6.1. Lemma. *In every Lie group*,

$$\overline{\operatorname{comp} G} \cap \operatorname{Reg} G = \operatorname{comp} G \cap \operatorname{Reg} G.$$

Proof. Let $g = \exp x_0$ be regular in G. By [7, Lemma 1.4], x_0 is regular and exp-regular in $\mathfrak{g}$. Then there is a Cartan subalgebra $\mathfrak{h}$ which contains x_0. Now, let U be a zero-neighborhood in $\mathfrak{h}$ and V a zero-neighborhood in the Fitting one-component $\mathfrak{g}^+$ of $\mathfrak{g}$. We define the function $f : U \times V \to W$, $(x, y) \mapsto (\exp y) g \exp x (\exp(-y)) = \exp e^{\operatorname{ad} y}(x_0 * x)$, where W is an open neighborhood of g and f is a local diffeomorphism.

Moreover, let g be in $\overline{\operatorname{comp} G}$. Then $g = \lim_{n\to\infty} g_n$, where the g_n can be chosen in such a way that $g_n \in W \cap \operatorname{comp} G \cap \operatorname{Reg} G$. Since f is a diffeomorphism we get $g_n = (\exp y_n) g \exp x_n (\exp(-y_n)) \in \operatorname{comp} G$ for proper x_n and y_n. Since g_n converges to g, y_n and x_n converge to 0. Since $g \exp x_n$ is in H for a Cartan subgroup $H = \exp \mathfrak{h}$, we have $\exp(-y_n) g_n \exp y_n = g \exp x_n$. The left side is in $\operatorname{comp} G$. Thus $g \exp x_n \in (\operatorname{comp} G) \cap H = \operatorname{comp} H$. This group is closed and central in H, which yields

$$\lim_{n\to\infty} g \exp x_n = g \lim_{n\to\infty} \exp x_n = g \in \overline{\operatorname{comp} H} = \operatorname{comp} H \subseteq \operatorname{comp} G.$$

Thus we get $\overline{\operatorname{comp} G} \cap \operatorname{Reg} G = \operatorname{comp} G \cap \operatorname{Reg} G$. ■

Let H be a Cartan subgroup of a Lie group G. By [7, Theorem 3.4] we get $\exp^{-1}(H) \cap \operatorname{reg}\exp = \mathfrak{h} \cap \operatorname{reg}\exp$.

6.2. LEMMA. *We have* $\exp^{-1}(\overline{\operatorname{comp} G} \cap \operatorname{Reg} G) \subseteq \operatorname{comp}\mathfrak{g} \cap \operatorname{reg}\mathfrak{g}$.

Proof. First, we note that

$$\begin{aligned}\exp^{-1}(\overline{\operatorname{comp} G} \cap \operatorname{Reg} G) &= \exp^{-1}((\operatorname{comp} G) \cap \operatorname{Reg} G)\\ &= \exp^{-1}(\operatorname{comp} G) \cap \exp^{-1}(\operatorname{Reg} G).\end{aligned}$$

Proposition 4.1 and [7, Lemma 1.4] imply the claim. ∎

6.3. PROPOSITION. *Let G be a solvable connected Lie group and T a maximal torus. Then $G = \overline{\operatorname{comp} G}$ implies $\mathfrak{g} = \overline{\Gamma.\mathfrak{t}}$.*

Proof. By Lemmas 6.1 and 6.2, $G = \overline{\operatorname{comp} G}$ implies $\exp^{-1}(\operatorname{Reg} G) \subseteq \exp^{-1}(\operatorname{comp} G)$. On the other hand,

$$\exp^{-1}(\operatorname{comp} G) = \{x \in \mathfrak{g} : \exp x \text{ lies in a torus}\} = \Gamma.\mathfrak{t}.$$

Thus, we get $\Gamma.\mathfrak{t} \supseteq \exp^{-1}(\operatorname{Reg} G) = \operatorname{reg}\exp \cap \operatorname{reg}\mathfrak{g}$ by [7, Lemma 1.4]. By [5, Lemma IV.4.6] and the fact that $\operatorname{reg}\mathfrak{g}$ is open and dense in $\mathfrak{g}$ we deduce that $\operatorname{reg}\exp \cap \operatorname{reg}\mathfrak{g}$ is open and dense in $\mathfrak{g}$. So, $\overline{\Gamma.\mathfrak{t}} = \mathfrak{g}$. ∎

6.4. COROLLARY. *Assume that G is a Lie group with $\overline{\operatorname{comp} G} = G$. Then $\mathfrak{g} = \overline{\operatorname{comp}\mathfrak{g}}$.*

6.5. THEOREM. *Let G be a connected solvable Lie group and T a maximal torus. Assume that $\mathfrak{g}$ is centerfree. The following statements are equivalent*:

(i) $G = \overline{\operatorname{comp} G}$.
(ii) $\mathfrak{g} = \overline{\Gamma.\mathfrak{t}}$.
(iii) *T is a Cartan subgroup of G.*

Proof. (i)⇒(ii) is Proposition 6.3.

(ii)⇒(iii). Since $\mathfrak{g} = \overline{\Gamma.\mathfrak{t}}$, we have $\mathfrak{g} = \mathfrak{t} + \mathfrak{g}'$. Moreover, there is a maximal compactly embedded Lie algebra $\mathfrak{k} \supseteq \mathfrak{t}$, and $\mathfrak{t} + \mathfrak{g}' = \mathfrak{k} + \mathfrak{g}'$. Thus we get $\mathfrak{k} = \mathfrak{t} + (\mathfrak{k} \cap \mathfrak{g}') = \mathfrak{t} + (\mathfrak{z} \cap \mathfrak{g}') = \mathfrak{t}$. So, $\mathfrak{t}$ is maximal compactly embedded and hence $\mathfrak{g} = \overline{\operatorname{comp}\mathfrak{g}}$. By Theorem 5.2, $\mathfrak{t}$ is a Cartan subalgebra and hence T is a Cartan subgroup of G.

(iii)⇒(i). If T is a Cartan subgroup, we have $\overline{\operatorname{comp} G} = \overline{\widetilde{\Gamma}.T} = G$. In a solvable Lie group the Cartan subgroups are conjugate and connected. Hence $\widetilde{\Gamma}.T \supseteq \operatorname{Reg} G$. But $\overline{\operatorname{Reg} G} = G$ by [8, Lemma 4] and so $\overline{\widetilde{\Gamma}.T} = G$. ∎

If the Lie algebra is not centerfree the problem is more difficult.

6.6. LEMMA. *Let G be a solvable connected Lie group, $\mathfrak{z}$ the center of $\mathfrak{g}$ and $\pi : G \to G/\exp\mathfrak{z}$ the natural map. Then $\pi(\operatorname{comp} G) = \operatorname{comp}\pi(G)$.*

Proof. Since the image of a compact set under π is compact, we have $\pi(\operatorname{comp} G) \subseteq \operatorname{comp}\pi(G)$. On the other hand, let U be a maximal torus of

$\pi(G)$. That means that there is a closed subgroup T of G with $T \supseteq \exp\mathfrak{z}$ such that $\pi(T) = U$. In particular, T is abelian. By [1, Theorem 9.16] there is a closed connected T_1 with $T \cong T_1 \times \exp\mathfrak{z}$. It follows that $\pi(T) \cong T_1$, hence T_1 is compact. Consequently, $\operatorname{comp}\pi(G) \subseteq \pi(\operatorname{comp} G)$. It follows that $\operatorname{comp}\pi(G) = \pi(\operatorname{comp} G)$. ■

6.7. THEOREM. *Let G be a connected solvable Lie group and T a maximal torus. Then $\overline{\operatorname{comp} G} = G$ if and only if $\overline{\Gamma.\mathfrak{t}} = \mathfrak{g}$.*

Proof. By Proposition 6.3, $\overline{\operatorname{comp} G} = G$ implies $\overline{\Gamma.\mathfrak{t}} = \mathfrak{g}$. Now suppose that $\overline{\Gamma.\mathfrak{t}} = \mathfrak{g}$. We reduce the problem to the centerless case of Theorem 6.5. First, for each $g \in \overline{\operatorname{comp} G}$ there is a sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in \exp^{-1}(\operatorname{comp} G)$ such that $g = \lim_{n\to\infty} \exp y_n$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. For each $z \in \mathfrak{z}$ we have

$$\exp z \cdot g = \exp z \cdot \lim_{n\to\infty} \exp y_n = \lim_{n\to\infty} \exp(z + y_n)$$
$$\in \overline{\exp\mathfrak{g}} = \overline{\exp\overline{\Gamma.\mathfrak{t}}} = \overline{\exp\Gamma.\mathfrak{t}} = \overline{\widetilde{\Gamma}.T} = \overline{\operatorname{comp} G}.$$

Thus $\overline{\operatorname{comp} G} = \exp\mathfrak{z} \cdot \overline{\operatorname{comp} G}$.

Now we consider the natural map $\pi : G \to G/\exp\mathfrak{z}$. By Lemma 6.6 we have $\pi(\overline{\operatorname{comp} G}) \subseteq \overline{\operatorname{comp}\pi(G)}$. Furthermore,

$$\operatorname{comp}\pi(G) = \pi(\operatorname{comp} G) \subseteq \pi(\overline{\operatorname{comp} G}) \subseteq \overline{\pi(\operatorname{comp} G)} = \overline{\operatorname{comp}\pi(G)}.$$

Since $\overline{\operatorname{comp} G}$ is saturated we deduce that $\pi(\overline{\operatorname{comp} G})$ is closed in $\pi(G)$, hence $\overline{\operatorname{comp}\pi(G)} = \pi(\overline{\operatorname{comp} G})$. Now suppose that $\overline{\operatorname{comp}\pi(G)} = \pi(G)$. Then

$$G = \pi^{-1}(\overline{\operatorname{comp}\pi(G)}) = \pi^{-1}(\pi(\overline{\operatorname{comp} G})) = \overline{\operatorname{comp} G} \cdot \exp\mathfrak{z} = \overline{\operatorname{comp} G}.$$

Next, we prove that $d\pi(1)(\mathfrak{g})$ is centerfree. For this, assume that $z \in \pi^{-1}(\mathfrak{z}(d\pi(1)(\mathfrak{g})))$. Then $[z, x] \in \mathfrak{z}$ for all $x \in \mathfrak{g}$. Since $\operatorname{comp}\mathfrak{g} = \mathfrak{g}$, the Cartan subalgebras are compactly embedded, hence abelian. We consider the root decomposition belonging to a Cartan subalgebra $\mathfrak{k}$. Then $z = z_0 + \sum_{\omega\in\Omega^+} z_\omega$ with $z_0 \in \mathfrak{k}$ and $z_\omega \in \mathfrak{g}^\omega$. If $x \in \mathfrak{k}$ we have $[x, z_0] \in \mathfrak{k}$ and $[x, z_\omega] \in \mathfrak{g}^\omega$. Since $\mathfrak{z} \subseteq \mathfrak{k}$ we get $[x, z_\omega] = 0$. Since this is true for each $x \in \mathfrak{k}$ we get $z_\omega = 0$ for all $\omega \in \Omega$. This implies $z \in \mathfrak{k}$. Now let $x_\omega \in \mathfrak{g}^\omega$. Then $[z, x_\omega] \in \mathfrak{g}^\omega \cap \mathfrak{z} = \{0\}$. Since $\mathfrak{k}$ is abelian, we get $[z, \mathfrak{k}] = \{0\}$. It follows that $z \in \mathfrak{z}$. So $d\pi(1)(\mathfrak{g})$ is centerfree. Furthermore, $d\pi(1)(\overline{\Gamma.\mathfrak{t}}) = \overline{\Gamma.d\pi(1)(\mathfrak{t})} = d\pi(1)(\mathfrak{g})$. By Theorem 6.5 we have $\pi(G) = \overline{\operatorname{comp}\pi(G)}$. ■

6.8. COROLLARY. *If G is a solvable connected Lie group and $\mathfrak{z}$ the center of $\mathfrak{g}$ then $G = \overline{\operatorname{comp} G}$ if and only if the Cartan subgroups of $G/\exp\mathfrak{z}$ are tori and $\mathfrak{z} \subseteq \overline{\Gamma.\mathfrak{t}}$ with the Lie algebra $\mathfrak{t}$ of a torus of G.*

An example shows that there are Lie groups whose sets of compact elements are dense and whose Cartan subgroups are not tori:

6.9. EXAMPLE. We consider the oscillator group $G = T \ltimes H$ with Heisenberg group H and torus T. The Lie algebra is given by the Lie brackets $[t, x_1] = x_2$, $[t, x_2] = -x_1$ and $[x_1, x_2] = z$. Then $\mathfrak{t} = \mathbb{R}t$ and $\mathfrak{z} = \mathbb{R}z$. Moreover, $\mathcal{C}^\infty = \operatorname{span}\{x_1, x_2, z\}$. In particular, $G/\exp\mathfrak{z}$ is isomorphic to a covering of the motion group of the real plane and $T\exp\mathfrak{z}/\exp\mathfrak{z}$ is a torus and a Cartan subgroup of $G/\exp\mathfrak{z}$. Thus $\operatorname{comp}(G/\exp\mathfrak{z})$ is dense in $G/\exp\mathfrak{z}$. Moreover, let $z \in \mathfrak{z}$. We remark that

$$\frac{-t}{n^2} + \frac{x_2}{n} - \frac{x_1}{n} + z = \frac{e^{n^2 \operatorname{ad}(x_1+x_2)}(-t)}{n^2} \in \Gamma.\mathfrak{t}.$$

Thus

$$z = \lim_{n\to\infty} \frac{e^{n^2 \operatorname{ad}(x_1+x_2)}(-t)}{n^2} \in \overline{\Gamma.\mathfrak{t}}.$$

So, the set of the compact elements of G is dense although $Z(H) \cong \mathbb{R}$.

The next example shows that there are Lie groups whose Cartan subalgebras are of the form $T\exp\mathfrak{z}$ with a maximal torus T and where $G \neq \overline{\operatorname{comp} G}$.

6.10. EXAMPLE. We look at a Lie group $G = \mathbb{T} \ltimes \mathbb{R}^6$ with the following Lie brackets: $[t, x_1] = x_2$, $[t, x_2] = -x_1$, $[t, y_1] = y_2$, $[t, y_2] = -y_1$, $[x_1, y_1] = [x_2, y_2] = z$ and $[x_1, x_2] = u$. Then $\mathfrak{z} = \operatorname{span}\{z, u\}$ and a Cartan subgroup is equal to the product of a maximal torus and $\exp\mathfrak{z}$. But for $\alpha_i \in \mathbb{R}$ with $i = 1, \ldots, 4$ we get

$$\begin{aligned}&\frac{e^{n^2 \operatorname{ad}(\alpha_1 x_1+\alpha_2 x_2+\alpha_3 y_1+\alpha_4 y_2)}(-t)}{n^2}\\ &\quad= \frac{-t}{n^2} + \frac{\alpha_1 x_2 - \alpha_2 x_1 + \alpha_3 y_2 - \alpha_4 y_1}{n} + \alpha_1^2 u - 2\alpha_1\alpha_4 z + \alpha_2^2 u + 2\alpha_2\alpha_3 z.\end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{e^{n^2 \operatorname{ad}(\alpha_1 x_1+\alpha_2 x_2+\alpha_3 y_1+\alpha_4 y_2)}(-t)}{n^2} = \alpha_1^2 u - 2\alpha_1\alpha_4 z + \alpha_2^2 u + 2\alpha_2\alpha_3 z.$$

This shows that $z \notin \overline{\Gamma.\mathfrak{t}}$ and $G \neq \overline{\operatorname{comp} G}$.

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