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TUBULAR MUTATIONS

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1. Introduction

1.1. In the study of vector bundles over projective algebraic varieties, mutations have been used since the early beginnings until today; see for instance the papers of Atiyah [1] dealing with indecomposable bundles on an elliptic curve, or Drezet [4] treating exceptional bundles on the projective plane. The concept of mutations with respect to exceptional bundles was formalized by Gorodentsev and Rudakov [6] and successfully applied in the classification of all exceptional bundles on the projective plane [13]. Later Bondal generalized this concept investigating mutations with respect to exceptional objects in suitable triangulated categories [3].

As was shown in [11] mutations with respect to a finite Auslander–Reiten orbit, called *tubular mutations* further on, play an important role in the classification of indecomposable vector bundles on a weighted projective line of genus one as in the case of an elliptic curve (compare [1] and [9]), and can also be used to classify the indecomposable modules over a tubular canonical algebra (compare [12]). In the existing literature tubular mutations are only established as partially defined functors, though it was pointed out in [10] that their natural environment are the associated derived categories and though it was clear how mutations would act on objects. Previous methods did not allow the extension of the action of tubular mutations to all morphisms, a problem solved in this paper. Observe that we are dealing here with mutations not with respect to an exceptional object but with respect to the direct sum of all objects of a finite Auslander–Reiten orbit. The advantage of considering mutations as derived equivalences is that it allows the study of the whole automorphism group of the derived category of coherent sheaves on a weighted projective line of genus one, equivalently of the derived category of modules over a tubular algebra; in particular, this allows the description of the relations between the mutations and Ringel's shrinking functors [12]. We will investigate these problems in a forthcoming paper.

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2. Main result

2.1. Let k be an algebraically closed field. Further, let $\mathbf{p} = (p_1, \ldots, p_t)$ be a weight sequence of positive integers p_i and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t)$ a parameter sequence of pairwise distinct elements of $\mathbb{P}^1(k)$ such that $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$ and $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ the attached weighted projective line [5]. Alternatively such a weighted projective line $\mathbb{X}$ with data $\mathbf{p}, \boldsymbol{\lambda}$ can be viewed as the usual projective line together with a parabolic structure determined by these weights [14], as follows from work of Lenzing [10]. We put $p = \text{l.c.m.}(p_1, \ldots, p_t)$. Denote by $\mathbf{L}(\mathbf{p})$ the corresponding rank one abelian group on generators $\vec{x}_1, \ldots, \vec{x}_t$ with relations $p_1\vec{x}_1 = \ldots = p_t\vec{x}_t$ $(:= \vec{c})$. Geigle and Lenzing introduced in [5] the category $\operatorname{coh}(\mathbb{X})$ (resp. $\operatorname{Qcoh}(\mathbb{X})$) of $\mathbf{L}(\mathbf{p})$-graded coherent (resp. quasi-coherent) sheaves on $\mathbb{X}$ with structure sheaf $\mathcal{O}$ and dualizing sheaf $\omega = \mathcal{O}((t-2)\vec{c} - \sum_{i=1}^{t} \vec{x}_i)$.

2.2. For an abelian category $\mathcal{A}$ we denote by $C^{\mathrm{b}}(\mathcal{A})$ the category of bounded complexes over $\mathcal{A}$, and by $K^{\mathrm{b}}(\mathcal{A})$ (resp. $D^{\mathrm{b}}(\mathcal{A})$) the corresponding homotopy (resp. derived) category. Moreover, if $\mathcal{A}'$ is a full abelian subcategory of $\mathcal{A}$ then $C^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ (resp. $K^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$, $D^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$) denotes the full subcategory of $C^{\mathrm{b}}(\mathcal{A})$ (resp. $K^{\mathrm{b}}(\mathcal{A})$, $D^{\mathrm{b}}(\mathcal{A})$) formed by all complexes with cohomology in $\mathcal{A}'$. Then we have localization functors $\kappa : K^{\mathrm{b}}(\mathcal{A}) \to D^{\mathrm{b}}(\mathcal{A})$ and $\kappa : K^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A}) \to D^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$.

We denote the translation functor of a triangulated category by $X \mapsto X[1]$.

2.3. Let $\mathcal{C}$ be a triangulated k-category. Recall that an object $A \in \mathcal{C}$ is called *exceptional* if $\operatorname{Hom}_{\mathcal{C}}(A, A[i]) = 0$ for $i \neq 0$ and $\operatorname{Hom}_{\mathcal{C}}(A, A) = k$. In [3] Bondal defined left mutations with respect to an exceptional object $A \in \mathcal{C}$ by forming triangles

$$\operatorname{Hom}^\bullet(A, X) \otimes A \xrightarrow{\operatorname{can}_X} X \to L_A(X)$$

for all objects $X \in \mathcal{C}$, where $\operatorname{Hom}^\bullet(A, X) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(A, X[j])[-j]$ is considered as a complex with zero differential, and the maps can_X are the canonical maps corresponding to the identity endomorphisms. In this case it is easy to check that L_A is a functor.

2.4. Let $\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$ be the canonical algebra [12] attached to the data $\mathbf{p}, \boldsymbol{\lambda}$. It was proved in [5] that there is a triangle-equivalence $\mathcal{D}^{\mathrm{b}}(\operatorname{coh}(\mathbb{X})) \xrightarrow{\simeq} \mathcal{D}^{\mathrm{b}}(\operatorname{mod}(\Lambda))$ where $\operatorname{mod}(\Lambda)$ denotes the category of finite-dimensional right Λ-modules and $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$.

For a weighted projective line $\mathbb{X}$ the *virtual genus* $g_{\mathbb{X}}$ is defined by

$$g_{\mathbb{X}} = 1 + \frac{1}{2}\left((t-2)p - \sum_{i=1}^{t} \frac{p}{p_i}\right).$$

We are interested in weighted projective lines of genus one. It is easy to see that $g_{\mathbb{X}} = 1$ if and only if the weight sequence is up to permutation one of the following: $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$. Furthermore, this is equivalent to the fact that $\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$ is a tubular algebra in the sense of [12]. Finally, by [8] each tubular algebra is derived equivalent to a tubular canonical algebra.

If $\mathbb{T}$ is a nonsingular elliptic curve over k we denote by $\mathrm{coh}(\mathbb{T})$ (resp. $\mathrm{Qcoh}(\mathbb{T})$) the category of coherent (resp. quasi-coherent) sheaves and put $p = 1$. The categories $\mathrm{coh}(\mathbb{X})$ and $\mathrm{coh}(\mathbb{T})$ admit Auslander–Reiten sequences, the Auslander–Reiten translation is denoted by τ. It is given by tensoring with the dualizing sheaf, which equals $\omega = \mathcal{O}((t-2)\vec{c} - \sum_{i=1}^{t} \vec{x}_i)$ in the case $\mathrm{coh}(\mathbb{X})$ and $\mathcal{O}_{\mathbb{T}}$ in the case $\mathrm{coh}(\mathbb{T})$.

Recall that for a weighted projective line $\mathbb{X}$ of genus one the *Auslander–Reiten components* of $\mathrm{coh}(\mathbb{X})$ are tubes in the sense of [12]. Also when $\mathbb{T}$ is a nonsingular elliptic curve the Auslander–Reiten components of $\mathrm{coh}(\mathbb{T})$ are tubes, in this case all of them are homogeneous. An indecomposable sheaf lying at the mouth of a tube is called *quasi-simple*. Note that in our situation a sheaf is quasi-simple if and only if it is μ-stable.

THEOREM 2.5. *Let $\mathbb{Y}$ be a weighted projective line of genus one or a nonsingular elliptic curve, and $\mathcal{U}$ a τ-orbit of a sheaf in* $\mathrm{coh}(\mathbb{Y})$. *Then there exists an equivalence $L : \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y})) \to \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y}))$ and a natural transformation $\eta : \mathrm{id} \to L$ such that*

$$\bigoplus_{U \in \mathcal{U}} \mathrm{Hom}^{\bullet}(U, X) \otimes U \xrightarrow{\mathrm{can}_X} X \xrightarrow{\eta_X} L(X)$$

is a distinguished triangle for each object $X \in \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y}))$.

Proof. Step 1. Let $\mathcal{U}$ be an arbitrary τ-orbit of a quasi-simple sheaf in $\mathrm{coh}(\mathbb{Y})$. We first show the existence of a functor L such that $L(X)$ appears in a distinguished triangle as above.

We consider the functor $F : \mathrm{Qcoh}(\mathbb{Y}) \to \mathrm{Qcoh}(\mathbb{Y})$ defined on objects by

$$F(X) = \bigoplus_{U \in \mathcal{U}} \mathrm{Hom}(U, X) \otimes U$$

and the morphism of functors $\alpha : F \to \mathrm{id}_{\mathrm{Qcoh}(\mathbb{Y})}$ which is given by the canonical maps. Obviously, F extends to a functor $\overline{F} : \mathcal{K}^{\mathrm{b}}(\mathrm{Qcoh}(\mathbb{Y})) \to \mathcal{K}^{\mathrm{b}}(\mathrm{Qcoh}(\mathbb{Y}))$ and α extends to a morphism of functors $\overline{\alpha} : \overline{F} \to \mathrm{id}_{\mathcal{K}^{\mathrm{b}}(\mathrm{Qcoh}(\mathbb{Y}))}$. Define $\bar{L} : \mathcal{K}^{\mathrm{b}}(\mathrm{Qcoh}(\mathbb{Y})) \to \mathcal{K}^{\mathrm{b}}(\mathrm{Qcoh}(\mathbb{Y}))$ as the mapping cone of $\overline{\alpha}$, thus for $X^{\bullet} = (X^n, d^n) \in \mathcal{K}^{\mathrm{b}}(\mathrm{Qcoh}(\mathbb{Y}))$ we get $\bar{L}(X^{\bullet})^n = F(X^{n+1}) \oplus X^n$ and the differential in $\bar{L}(X^{\bullet})$ is given by

$$\begin{pmatrix} -F(d^{n+1}) & 0 \\ \alpha(X^{n+1}) & d^n \end{pmatrix} : F(X^{n+1}) \oplus X^n \to F(X^{n+2}) \oplus X^{n+1}.$$

Clearly, $\bar{L}$ is a functor.

Let $\mathcal{I}$ be the full subcategory of $\mathrm{Qcoh}(\mathbb{Y})$ consisting of all injective quasicoherent sheaves, and let $\mathcal{K}^{\mathrm{b}}(\mathcal{I})$ be the full subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{Qcoh}(\mathbb{Y}))$ formed by all complexes of objects from $\mathcal{I}$, and let $\mathcal{K}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathcal{I})$, be the full subcategory of $\mathcal{K}^{\mathrm{b}}(\mathcal{I})$ having all cohomology sheaves in $\mathrm{coh}(\mathbb{Y})$. We show that if $I^\bullet \in \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathcal{I})$, then $\bar{L}(I^\bullet) \in \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathrm{Qcoh}(\mathbb{Y}))$. Let $I^\bullet = (I^n, d^n) \in \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathcal{I})$. It is sufficient to show that $\overline{F}(I^\bullet)$ has coherent cohomology. We define $B^n = \mathrm{im}(d^{n-1})$, $Z^n = \ker(d^n)$, $H^n = Z^n/B^n$ and $F'(X) = \bigoplus_{j=1}^{p} \mathrm{Ext}^1(\tau^j\mathcal{O}, X) \otimes \tau^j\mathcal{O}$ for $X \in \mathrm{coh}(\mathbb{Y})$. Observe that $F(Z^n) \simeq \ker(F(d^n))$, $F'(B^n) = 0$ and $F(H^n) \simeq F(Z^n)/F(B^n)$ for all $n \in \mathbb{Z}$. Now, for the exact sequence

$$0 \to F(B^n)/\mathrm{im}(F(d^{n-1})) \to F(Z^n)/\mathrm{im}(F(d^{n-1})) \to F(Z^n)/F(B^n) \to 0$$

we see that $F(B^n)/\mathrm{im}(F(d^{n-1})) \simeq F'(Z^{n-1}) \simeq F'(H^{n-1})$ and $F(Z^n)/F(B^n)$ are coherent, therefore the middle term is coherent, too. Hence the complex $\overline{F}(I^\bullet)$ has coherent cohomology.

Thus by restriction we obtain a functor

$$L' : \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathcal{I}) \to \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathrm{Qcoh}(\mathbb{Y})).$$

Now the composition

$$\mathcal{K}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathcal{I}) \overset{i}{\hookrightarrow} \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathrm{Qcoh}(\mathbb{Y})) \overset{\kappa}{\to} \mathcal{D}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathrm{Qcoh}(\mathbb{Y}))$$

is an equivalence; let ϕ be a quasi-inverse of $\kappa \circ i$. Then we consider the composition

$$\mathcal{D}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathrm{Qcoh}(\mathbb{Y})) \overset{\phi}{\to} \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathcal{I}) \overset{L'}{\to} \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathrm{Qcoh}(\mathbb{Y})) \overset{\kappa}{\to} \mathcal{D}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathrm{Qcoh}(\mathbb{Y})),$$

and identifying $\mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y}))$ with $\mathcal{D}^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathrm{Qcoh}(\mathbb{Y}))$ we get a functor $L : \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y})) \to \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y}))$. Moreover, the obvious natural transformation $\mathrm{id}_{\mathcal{K}^{\mathrm{b}}(\mathrm{Qcoh}(\mathbb{Y}))} \to \overline{F}$ induces a natural transformation $\eta : \mathrm{id}_{\mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y}))} \to L$. The existence of triangles as stated in the theorem is a consequence of the definition of L.

Step 2. Next we show that $L = L_{\mathcal{U}}$ is an equivalence in the particular case that $\mathcal{U}$ is the τ-orbit of the structure sheaf. For this we apply Beilinson's Lemma [2] saying that if $G : \mathcal{C} \to \mathcal{D}$ is an exact functor between triangulated categories and if $\mathcal{X} = \{X_i\}_{i\in I}$ is a generating system (in the sense of triangulated categories) of $\mathcal{C}$ such that $\{G(X_i)\}_{i\in I}$ is a generating system of $\mathcal{D}$ and G induces equivalences $\mathrm{Hom}^\bullet(X_i, X_j) \overset{\simeq}{\to} \mathrm{Hom}^\bullet(G(X_i), G(X_j))$ for all $X_i, X_j \in \mathcal{X}$, then G is an equivalence.

Now, if $0 \to I_1^\bullet \xrightarrow{\alpha} I_2^\bullet \xrightarrow{\beta} I_3^\bullet \to 0$ is an exact sequence in $C^{\mathrm{b}}(\mathrm{Qcoh}(\mathbb{Y}))$ with terms in $C^{\mathrm{b}}_{\mathrm{coh}(\mathbb{Y})}(\mathcal{I})$, then the complex $0 \to L'(I_1^\bullet) \xrightarrow{L'(\alpha)} L'(I_2^\bullet) \xrightarrow{L'(\beta)} L'(I_3^\bullet) \to 0$ is a pointwise split exact sequence. Interpreting the homotopy categories as Frobenius categories it follows from [7, Chapter I, Lemma 2.8] that L' and hence L is an exact functor of triangulated categories.

Case (a). $\mathbb{Y}=\mathbb{X}$ is a weighted projective line of genus one. Consider the generating system $\{\mathcal{O}, \{S_{i,j}\}_{i=1,\ldots,t,\, j=0,\ldots,p_i-1}\}$ where the $S_{i,j}$ are the simple sheaves concentrated at the exceptional points. From the exact sequences

$$0 \to \mathcal{O}(j\vec{x}_i) \to \mathcal{O}((j+1)\vec{x}_i) \to S_{i,j} \to 0$$

[5, 2.5] we conclude that the only non-vanishing Hom-spaces between the sheaves of the generating system are $\mathrm{Hom}(\mathcal{O}, S_{i,p_i-1}) = k$, $\mathrm{Ext}^1(S_{i,0}, \mathcal{O}) = k$, $\mathrm{Ext}^1(S_{i,j}, S_{i,j-1}) = k$ and $\mathrm{Hom}(X, X) = k$ for $X = \mathcal{O}$ or $S_{i,j}$. Choosing injective resolutions for $\mathcal{O}$ and $S_{i,j}$ one easily calculates that $L(\mathcal{O}) \simeq \omega^{-1}$ and $L(S_{i,j}) \simeq \mathcal{O}(-\vec{x}_i + (p_i - 1 - j)\vec{\omega})[1]$.

It follows that

$$\begin{aligned} \mathrm{Hom}(L(\mathcal{O}), L(S_{i,p_i-1})) &= k, \\ \mathrm{Hom}(L(S_{i,0}), L(\mathcal{O})[1]) &= k, \\ \mathrm{Hom}(L(S_{i,j}), L(S_{i,j-1})[1]) &= k, \\ \mathrm{Hom}(L(\mathcal{O}), L(\mathcal{O})) &= k, \\ \mathrm{Hom}(L(S_{i,j}), L(S_{i,j})) &= k, \end{aligned}$$

and the other Hom-spaces vanish. Furthermore, it is easy to check that L induces non-zero maps and therefore isomorphisms between the corresponding one-dimensional Hom-spaces. Moreover, $L(\mathcal{O})$ and the $L(S_{i,j})$ form again a generating system. Thus by Beilinson's Lemma L is an equivalence.

Case (b). $\mathbb{Y} = \mathbb{T}$ is a nonsingular elliptic curve. In this case one can proceed in the same way working with the generating system consisting of $\{\mathcal{O}, \{S_P\}_{P\in\mathbb{T}}\}$ where S_P is the simple sheaf concentrated at the point P for any $P \in \mathbb{T}$. One easily shows that $L(\mathcal{O}) \simeq \mathcal{O}$ and $L(S_P) \simeq S_P$ for each $P \in \mathbb{T}$.

Step 3. Now we show that $L = L_{\mathcal{U}}$ is an equivalence in case $\mathcal{U}$ is the τ-orbit of a simple sheaf of finite length. We proceed similarly to the previous step. In case $\mathbb{Y} = \mathbb{X}$ is a weighted projective line we have to distinguish two cases.

Case (a). $\mathbb{Y} = \mathbb{X}$ and $\mathcal{U}$ is the τ-orbit of a simple sheaf $S_{i,0}$ concentrated at an exceptional point λ_i. Choosing again injective resolutions one easily shows that $L(\mathcal{O}) = \mathcal{O}(\vec{x}_i)$, $L(S_{i,j}) = S_{i,j+1}$ and $L(S_{i',j}) = S_{i',j}$ for $i \neq i'$ and that L induces isomorphisms between the one-dimensional Hom-spaces, which do not vanish.

Case (b). $\mathbb{Y} = \mathbb{X}$ and $\mathcal{U}$ is the τ-orbit of a simple sheaf S concentrated at an ordinary point.

In this case one easily proves that $L(\mathcal{O}) \simeq \mathcal{O}(\vec{c})$ and $L(S_{i,j}) \simeq S_{i,j}$ for all i, j and proceeds as before.

Case (c). $\mathbb{Y} = \mathbb{T}$ is a nonsingular elliptic curve and $\mathcal{U}$ is a simple sheaf of finite length corresponding to a point $P \in \mathbb{T}$. Working with the same generating system as in step 2 one shows that $L(\mathcal{O}) = \mathcal{O}(P)$ and $L(S_Q) = S_Q$ for each point $Q \in \mathbb{T}$ and the assertion on L follows easily.

Step 4. In order to prove the theorem in the general case we apply the method of telescopic functors developed in [11, Section 4]. Denote by L a tubular mutation with respect to the τ-orbit of the structure sheaf and by R a quasi-inverse functor of L. Furthermore, let S be a tubular mutation with respect to the τ-orbit of $S_{t,0}$ in case $\mathbb{Y} = \mathbb{X}$ is a weighted projective line of genus one and, respectively, a tubular mutation with respect to an arbitrary simple finite length sheaf in case $\mathbb{Y} = \mathbb{T}$ is a nonsingular elliptic curve. Moreover, let S^{-1} be a quasi-inverse functor of S. Now for each $q \in \mathbb{Q}$ there is an equivalence $\Phi_{q,\infty} : \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y})) \to \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y}))$ such that the category of finite length sheaves $\mathcal{C}_\infty$ is mapped to the category $\mathcal{C}_q$ of all semistable sheaves of slope q. The functors $\Phi_{q,\infty}$ are compositions of R, S and S^{-1}; note that, in contrast to [11], here they are defined on $\mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y}))$. Now, let $\mathcal{U}_q$ be a τ-orbit of a quasi-simple sheaf of slope q. Denote by $\mathcal{U}_\infty$ the image of $\mathcal{U}_q$ under $\Phi_{q,\infty}^{-1}$ and by $L_{\mathcal{U}_\infty}$ the corresponding tubular mutation. Then the equivalence $\Phi_{q,\infty} L_{\mathcal{U}_\infty} \Phi_{q,\infty}^{-1}$ satisfies the assertion, which finishes the proof of the theorem.

2.6. The following corollary follows easily from the theorem by stability arguments. It indicates how to calculate a left mutation L with respect to the τ-orbit of the structure sheaf in the abelian category and shows in particular that in this case L coincides on indecomposable sheaves X with $0 < \mu(X) \le 1$ with the functor considered in [11]. Here μ denotes the slope of a sheaf.

COROLLARY 2.6. *Let $\mathbb{Y}$ be a weighted projective line of genus one or a nonsingular elliptic curve. Let X be an indecomposable sheaf on $\mathbb{Y}$ and $\bigoplus_{j=1}^{p} \mathrm{Hom}(\tau^j \mathcal{O}, X) \otimes \tau^j \mathcal{O} \xrightarrow{\mathrm{can}_X} X$ the canonical map.*

(a) *If $\mu(X) > 1$, then $L(X) \simeq \ker(\mathrm{can}_X)[1]$.*

(b) *If $0 < \mu(X) \le 1$, then $L(X) \simeq \mathrm{coker}(\mathrm{can}_X)$.*

(c) *If $\mu(X) = 0$, then $L(X) \simeq \tau^{-1}(X)$ provided X is in the same Auslander–Reiten component as $\mathcal{O}$, and $L(X) \simeq X$ otherwise.*

(d) *If $\mu(X) < 0$, then $L(X)$ coincides with the middle term of the universal extension of X with respect to $\bigoplus_{j=1}^{p} \tau^j \mathcal{O}$.*

2.7. Let $R : \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y})) \to \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y}))$ be a quasi-inverse functor of the left mutation L considered in 2.6. Then we have triangles

$$R(X) \to X \xrightarrow{\mathrm{can}'_X} \bigoplus_{j=1}^{p} \mathrm{Hom}^{\bullet}(X, \tau^j \mathcal{O})^* \otimes \tau^j \mathcal{O}$$

for all $X \in \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{Y}))$, where $*$ denotes the usual duality with respect to k. From Corollary 2.7 we infer

Corollary 2.7. *Let $\mathbb{Y}$ be a weighted projective line of genus one or a nonsingular elliptic curve. Let X be an indecomposable sheaf on $\mathbb{Y}$ and $X \xrightarrow{\mathrm{can}'_X} \bigoplus_{j=1}^{p} \mathrm{Hom}(X, \tau^j \mathcal{O})^* \otimes \tau^j \mathcal{O}$ the co-canonical map.*

(a) *If $\mu(X) \le -1$, then $R(X) \simeq \mathrm{coker}(\mathrm{can}'_X)[-1]$.*

(b) *If $-1 < \mu(X) < 0$, then $R(X) \simeq \ker(\mathrm{can}'_X)$.*

(c) *If $\mu(X) = 0$, then $R(X) \simeq \tau(X)$ provided X is in the same Auslander–Reiten component as $\mathcal{O}$ and $R(X) \simeq X$ otherwise.*

(d) *If $\mu(X) > 0$, then $R(X)$ coincides with the middle term of the co-universal extension of X with respect to $\bigoplus_{j=1}^{p} \tau^j \mathcal{O}$.*

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