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LOCAL COHOMOLOGY, d-SEQUENCES AND GENERALIZED FRACTIONS

BY

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1. Introduction. Throughout, A denotes a commutative Noetherian ring (with identity), I denotes an ideal of A and M denotes a finitely generated A-module. We shall use $\mathbb{N}$ to denote the set of positive integers.

This paper is concerned with the theory of local cohomology introduced by A. Grothendieck [2], the theory of d-sequences introduced by Huneke [3] and the theory of modules of generalized fractions introduced by R. Y. Sharp and H. Zakeri [6].

In [8, Th. 2.4], Zakeri shows that the theory of d-sequences could be used in the theory of modules of generalized fractions. He provides a connection between local cohomology modules with respect to an ideal of A generated by a d-sequence and modules of generalized fractions derived from a d-sequence. In this note, we present a generalization of this theorem. We provide a connection between local cohomology modules with respect to an arbitrary ideal I of A and modules of generalized fractions derived from a d-sequence in I (Theorem 3.4). Moreover, we show that calculation of a local cohomology module with respect to an arbitrary ideal of A can be reduced to calculation of a local cohomology module with respect to an ideal generated by a d-sequence (Lemma 3.3).

2. Preliminaries. To prove the main theorem we need the following definitions and theorems (here, n denotes an element of $\mathbb{N}$).

2.1. Definition. Suppose $a_1, \ldots, a_n$ is a sequence of elements of A. The sequence $a_1, \ldots, a_n$ is called a *d-sequence on M* if

$$(a_1, \ldots, a_i)M :_M a_{i+1}a_k = (a_1, \ldots, a_i)M :_M a_k$$

for all $i = 0, \ldots, n-1$ and all $k \geq i+1$.

To define a d-sequence $a_1, \ldots, a_n$, Huneke used this condition together with the condition that $a_1, \ldots, a_n$ form a minimal generating set for

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$(a_1, \dots, a_n)$. In this paper, we use the above definition for d-sequences without the minimality condition.

2.2. Definition (see [7, Th. 1.1(iv)]). Suppose $a_1, \dots, a_n$ is a sequence of elements of A. The sequence $a_1, \dots, a_n$ is called an *absolutely superficial M-sequence* if

$$[(a_1, \dots, a_i)M :_M a_{i+1}] \cap (a_1, \dots, a_n)M = (a_1, \dots, a_i)M$$

for all $i = 0, \dots, n-1$.

2.3. Proposition (see [7, p. 46]). *The sequence $a_1, \dots, a_n \in A$ is a d-sequence on M if and only if $a_1, \dots, a_n$ is an absolutely superficial M-sequence.*

2.4. Definition (N. V. Trung [7, p. 38]). A sequence $a_1, \dots, a_n$ of elements of A is called an *I-filter regular M-sequence* if $a_i \notin p$ for all $p \in \mathrm{Ass}(M/(a_1, \dots, a_{i-1})M) \setminus V(I)$ (for $i = 1, \dots, n$), where $V(I)$ denotes the set of primes of A containing I.

2.5. Theorem. *Let $a_1, \dots, a_n$ be a d-sequence on M. Let $\mathfrak{a} = (a_1, \dots, a_n)$. Then $a_1, \dots, a_n$ is an $\mathfrak{a}$-filter regular M-sequence.*

Proof. This follows from Proposition 2.3 and [7, Th. 1.1(iv)].

2.6. Theorem. *Let $a_1, \dots, a_n \in I$ be an I-filter regular M-sequence. Then, for each $k \geq 0$, there exists an ascending sequence of integers $k \leq m_1 \leq \dots \leq m_n$ such that $a_1^{m_1}, \dots, a_n^{m_n}$ is a d-sequence on M.*

Proof. This follows from [7, Prop. 2.1] and Proposition 2.3.

2.7. Proposition. *Let $a_1, \dots, a_n$ be a sequence of elements of A. Then the following conditions are equivalent*:

(i) *$a_1, \dots, a_n$ is an I-filter regular M-sequence*;

(ii) *$a_1/1, \dots, a_i/1$ is a poor regular M_p-sequence in A_p for all $p \in \mathrm{Supp}(M) \setminus V(I)$ and $i = 1, \dots, n$*;

(iii) *$a_1^{\alpha_1}, \dots, a_n^{\alpha_n}$ is an I-filter regular M-sequence for all $\alpha_1, \dots, \alpha_n \in \mathbb{N}$.*

Proof. It is easy to see that (i) is equivalent to

$$\mathrm{Supp}((a_1, \dots, a_{i-1})M :_M a_i/(a_1, \dots, a_{i-1})M) \subseteq V(I)$$

for all $i = 1, \dots, n$, and the equivalence of (i) and (ii) is an easy consequence of the above fact. The equivalence of (i) and (iii) is a consequence of elementary properties of regular sequences.

3. The results. Throughout this section, for a sequence of elements $a_1, \dots, a_n$ of A and $i \in \mathbb{N}$, we set

$$U(a)_i = \{(a_1^{\alpha_1}, \dots, a_i^{\alpha_i}) : \text{there exists } j \text{ with } 0 \leq j \leq i \text{ such that } \alpha_1, \dots, \alpha_j \in \mathbb{N} \text{ and } \alpha_{j+1} = \dots = \alpha_i = 0\},$$

where a_r is interpreted as 1 whenever $r > n$. Then $\mathcal{U}(a) = (U(a)_i)_{i\in\mathbb{N}}$ is a chain of triangular subsets on A and we can, by [4, p. 420], construct the associated complex $C(\mathcal{U}(a), M)$. We use $H^i(C(\mathcal{U}(a), M))$, for $i \in \mathbb{N}\cup\{0\}$, to denote the ith cohomology module of the complex $C(\mathcal{U}(a), M)$. Throughout, we shall use H^i_I to denote, for $i \in \mathbb{N} \cup \{0\}$, the ith right derived functor of Γ_I where $\Gamma_I(M) = \bigcup_{n\in\mathbb{N}}(0 :_M I^n)$ for any A-module M.

3.1. Lemma (S. Goto and K. Yamagishi [1, (6.4)]). *Let N be an A-module* (*not necessarily finitely generated*). *Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in I$ be an I-filter regular N-sequence. Then*, *for all $i < n$*,

$$H^i_I(N) = H^i_{(a_1,\ldots,a_n)}(N).$$

Proof. Let $0 \to N \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \ldots \to E^i \xrightarrow{d^i} \ldots$ be a minimal injective resolution for N. Then, for all $i \in \mathbb{N} \cup \{0\}$,

$$E^i = \bigoplus_{\mathfrak{p}} \mu_i(\mathfrak{p}, N)E(A/\mathfrak{p}),$$

where $\mu_i(\mathfrak{p}, N)$ is the ith Bass number of N at the prime ideal $\mathfrak{p}$ of A and $E(A/\mathfrak{p})$ is the injective envelope of $A/\mathfrak{p}$.

Let $i < n$ and $\mathfrak{p} \in \operatorname{Supp}(N) \cap V(a_1, \ldots, a_n) \setminus V(I)$. Then, by Proposition 2.7, $\operatorname{Ext}^i_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}, N_\mathfrak{p}) = 0$, and so, $\mu_i(\mathfrak{p}, N) = 0$. Therefore

$$\begin{aligned}\Gamma_I(E^i) &= \bigoplus_{\substack{\mathfrak{p}\in\operatorname{Supp}(N)\\ \mathfrak{p}\supseteq I}} \mu_i(\mathfrak{p}, N)E(A/\mathfrak{p})\\ &= \bigoplus_{\substack{\mathfrak{p}\in\operatorname{Supp}(N)\\ \mathfrak{p}\supseteq(a_1,\ldots,a_n)}} \mu_i(\mathfrak{p}, N)E(A/\mathfrak{p}) = \Gamma_{(a_1,\ldots,a_n)}(E^i)\end{aligned}$$

for all $i < n$. Now we have

$$\operatorname{Ker}\Gamma_I(d^i) = \operatorname{Ker}\Gamma_{(a_1,\ldots,a_n)}(d^i), \qquad \operatorname{Im}\Gamma_I(d^{i-1}) = \operatorname{Im}\Gamma_{(a_1,\ldots,a_n)}(d^{i-1})$$

for all $i < n$. Therefore $H^i_I(N) = H^i_{(a_1,\ldots,a_n)}(N)$ for all $i < n$.

Now we can present the following theorem, using Lemma 3.1, the concept of filter regular sequences and [8, Th. 2.4].

3.2. Theorem. *Let M be a finitely generated A-module. Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n \in I$ be an I-filter regular M-sequence. Then*, *for all $i < n$*,

$$H^i_I(M) \cong H^i(C(\mathcal{U}(a), M)).$$

Proof. By Lemma 3.1, $H^i_I(M) = H^i_{(a_1,\ldots,a_n)}(M)$ for all $i < n$. By Theorem 2.6, there exist $1 \le m_1 \le \ldots \le m_n$ such that $a_1^{m_1}, \ldots, a_n^{m_n}$ is a d-sequence on M (in I). Now we have

$$H^i_{(a_1,\ldots,a_n)}(M) = H^i_{(a_1^{m_1},\ldots,a_n^{m_n})}(M)$$

for all $i \in \mathbb{N} \cup \{0\}$. By [8, Th. 2.4], for all $i < n$,

$$H^i_{(a_1^{m_1},\ldots,a_n^{m_n})}(M) \cong H^i(C(\mathcal{U}(b), M)),$$

where $\mathcal{U}(b) = (U(b)_i)_{i\in\mathbb{N}}$ is the chain of triangular subsets on A in which, for all $i \in \mathbb{N}$,

$$U(b)_i = \{(a_1^{m_1\alpha_1}, \ldots, a_i^{m_i\alpha_i}) : \text{there exists } j \text{ with } 0 \le j \le i \text{ such that } \alpha_1, \ldots, \alpha_j \in \mathbb{N} \text{ and } \alpha_{j+1} = \ldots = \alpha_i = 0\},$$

where a_r is interpreted as 1 whenever $r > n$.

On the other hand, by using elementary properties of generalized fractions or by applying [5, Th. 2.1], one can easily see that

$$H^i(C(\mathcal{U}(b), M)) \cong H^i(C(\mathcal{U}(a), M))$$

for all $i \in \mathbb{N} \cup \{0\}$. Therefore, for all $i < n$,

$$H^i_I(M) \cong H^i(C(\mathcal{U}(a), M)).$$

In the following lemma, we show that for any ideal I of A and any positive integer n, there exists a d-sequence $a_1, \ldots, a_n \in I$ such that local cohomology modules with respect to I are equal to local cohomology modules with respect to $(a_1, \ldots, a_n)$.

3.3. Lemma. *Let M be a finitely generated A-module. Let $n \in \mathbb{N}$. Then there exist $a_1, \ldots, a_n \in I$ which form a d-sequence on M and*

$$H^i_I(M) = H^i_{(a_1,\ldots,a_n)}(M)$$

for all $i < n$.

Proof. We can find $b_1, \ldots, b_n \in I$ which form an I-filter regular M-sequence as follows. Since $I \not\subseteq \bigcup_{\mathfrak{p}\in \mathrm{Ass}(M)\setminus V(I)} \mathfrak{p}$, there exists $b_1 \in I$ such that $b_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M) \setminus V(I)$. Again, since $I \not\subseteq \bigcup_{\mathfrak{p}\in \mathrm{Ass}(M/b_1M)\setminus V(I)} \mathfrak{p}$, there exists $b_2 \in I$ such that $b_2 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M/b_1M) \setminus V(I)$. Proceeding in this way, we can find $b_1, \ldots, b_n \in I$ which form an I-filter regular M-sequence. Now, by Lemma 3.1, $H^i_I(M) = H^i_{(b_1,\ldots,b_n)}(M)$ for all $i < n$. On the other hand, by Theorem 2.6, there exist $1 \le m_1 \le \ldots \le m_n$ such that $b_1^{m_1}, \ldots, b_n^{m_n}$ form a d-sequence on M. Let $a_i = b_i^{m_i}$ for all $1 \le i \le n$. Then $H^i_I(M) = H^i_{(a_1,\ldots,a_n)}(M)$ for all $i < n$.

Now that Lemma 3.3 has been established, we can prove the main theorem of this paper by using [8, Th. 2.4].

3.4. Theorem. *Let M be a finitely generated A-module. Let $n \in \mathbb{N}$. Then there exist $a_1, \ldots, a_n \in I$ which form a d-sequence on M and*

$$H^i_I(M) \cong H^i(C(\mathcal{U}(a), M))$$

for all $i < n$.

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