1. Introduction. Throughout, $A$ denotes a commutative Noetherian ring (with identity), $I$ denotes an ideal of $A$ and $M$ denotes a finitely generated $A$-module. We shall use $\mathbb{N}$ to denote the set of positive integers.

This paper is concerned with the theory of local cohomology introduced by A. Grothendieck [2], the theory of $d$-sequences introduced by Huneke [3] and the theory of modules of generalized fractions introduced by R. Y. Sharp and H. Zakeri [6].

In [8, Th. 2.4], Zakeri shows that the theory of $d$-sequences could be used in the theory of modules of generalized fractions. He provides a connection between local cohomology modules with respect to an ideal of $A$ generated by a $d$-sequence and modules of generalized fractions derived from a $d$-sequence. In this note, we present a generalization of this theorem. We provide a connection between local cohomology modules with respect to an arbitrary ideal $I$ of $A$ and modules of generalized fractions derived from a $d$-sequence in $I$ (Theorem 3.4). Moreover, we show that calculation of a local cohomology module with respect to an arbitrary ideal of $A$ can be reduced to calculation of a local cohomology module with respect to an ideal generated by a $d$-sequence (Lemma 3.3).

2. Preliminaries. To prove the main theorem we need the following definitions and theorems (here, $n$ denotes an element of $\mathbb{N}$).

2.1. Definition. Suppose $a_1, \ldots, a_n$ is a sequence of elements of $A$. The sequence $a_1, \ldots, a_n$ is called a $d$-sequence on $M$ if

$$(a_1, \ldots, a_i)M : Mt_{i+1}a_k = (a_1, \ldots, a_i)M : Mt_k$$

for all $i = 0, \ldots, n - 1$ and all $k \geq i + 1$.

To define a $d$-sequence $a_1, \ldots, a_n$, Huneke used this condition together with the condition that $a_1, \ldots, a_n$ form a minimal generating set for
(\(a_1, \ldots, a_n\)). In this paper, we use the above definition for \(d\)-sequences without the minimality condition.

2.2. Definition (see [7, Th. 1.1(iv)]). Suppose \(a_1, \ldots, a_n\) is a sequence of elements of \(A\). The sequence \(a_1, \ldots, a_n\) is called an absolutely superficial \(M\)-sequence if

\[
[(a_1, \ldots, a_i)M : Ma_{i+1}] \cap (a_1, \ldots, a_n)M = (a_1, \ldots, a_i)M
\]

for all \(i = 0, \ldots, n - 1\).

2.3. Proposition (see [7, p. 46]). The sequence \(a_1, \ldots, a_n \in A\) is a \(d\)-sequence on \(M\) if and only if \(a_1, \ldots, a_n\) is an absolutely superficial \(M\)-sequence.

2.4. Definition (N. V. Trung [7, p. 38]). A sequence \(a_1, \ldots, a_n\) of elements of \(A\) is called an \(I\)-filter regular \(M\)-sequence if \(a_i \not\in p\) for all \(p \in \text{Ass}(M/(a_1, \ldots, a_{i-1})M) \setminus V(I)\) (for \(i = 1, \ldots, n\)), where \(V(I)\) denotes the set of primes of \(A\) containing \(I\).

2.5. Theorem. Let \(a_1, \ldots, a_n\) be a \(d\)-sequence on \(M\). Let \(a = (a_1, \ldots, a_n)\). Then \(a_1, \ldots, a_n\) is an \(a\)-filter regular \(M\)-sequence.

Proof. This follows from Proposition 2.3 and [7, Th. 1.1(iv)].

2.6. Theorem. Let \(a_1, \ldots, a_n \in I\) be an \(I\)-filter regular \(M\)-sequence. Then, for each \(k \geq 0\), there exists an ascending sequence of integers \(k \leq m_1 \leq \ldots \leq m_n\) such that \(a_1^{m_1}, \ldots, a_n^{m_n}\) is a \(d\)-sequence on \(M\).

Proof. This follows from [7, Prop. 2.1] and Proposition 2.3.

2.7. Proposition. Let \(a_1, \ldots, a_n\) be a sequence of elements of \(A\). Then the following conditions are equivalent:

(i) \(a_1, \ldots, a_n\) is an \(I\)-filter regular \(M\)-sequence;
(ii) \(a_1/1, \ldots, a_i/1\) is a poor regular \(M_p\)-sequence in \(A_p\) for all \(p \in \text{Supp}(M) \setminus V(I)\) and \(i = 1, \ldots, n\);
(iii) \(a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}\) is an \(I\)-filter regular \(M\)-sequence for all \(\alpha_1, \ldots, \alpha_n \in \mathbb{N}\).

Proof. It is easy to see that (i) is equivalent to

\[
\text{Supp}((a_1, \ldots, a_{i-1})M : Ma_i/(a_1, \ldots, a_{i-1})M) \subseteq V(I)
\]

for all \(i = 1, \ldots, n\), and the equivalence of (i) and (ii) is an easy consequence of the above fact. The equivalence of (i) and (iii) is a consequence of elementary properties of regular sequences.

3. The results. Throughout this section, for a sequence of elements \(a_1, \ldots, a_n\) of \(A\) and \(i \in \mathbb{N}\), we set

\[
U(a)_i = \{ (a_1^{\alpha_1}, \ldots, a_i^{\alpha_i}) : \text{there exists } j \text{ with } 0 \leq j \leq i \text{ such that } a_1, \ldots, a_j \in \mathbb{N} \text{ and } a_{j+1} = \ldots = a_i = 0 \},
\]
where $a_r$ is interpreted as 1 whenever $r > n$. Then $U(a) = (U(a)_i)_{i \in \mathbb{N}}$ is a chain of triangular subsets on $A$ and we can, by [4, p. 420], construct the associated complex $C(U(a), A)$. We use $H^i(C(U(a), A))$, for $i \in \mathbb{N}$, to denote the $i$th cohomology module of the complex $C(U(a), A)$. Throughout, we shall use $H^i_i$ to denote, for $i \in \mathbb{N}$, the $i$th right derived functor of $\Gamma_i$ where $\Gamma_i(M) = \bigcup_{n \in \mathbb{N}}(0 : M^n)$ for any $A$-module $M$.

3.1. Lemma (S. Goto and K. Yamagishi [1, (6.4)]). Let $N$ be an $A$-module (not necessarily finitely generated). Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in I$ be an $I$-filter regular $N$-sequence. Then, for all $i < n$,

$$H^i_i(N) = H^i_{(a_1, \ldots, a_n)}(N).$$

Proof. Let $0 \to N \overset{d^{-1}}{\to} E^{d^1} \overset{d^1}{\to} E^{d^2} \to \ldots$ be a minimal injective resolution for $N$. Then, for all $i \in \mathbb{N}$, $E^i = \bigoplus_{p} \mu_i(p, N)E(A/p)$,

where $\mu_i(p, N)$ is the $i$th Bass number of $N$ at the prime ideal $p$ of $A$ and $E(A/p)$ is the injective envelope of $A/p$.

Let $i < n$ and $p \in \text{Supp}(N) \cap V(a_1, \ldots, a_n) \setminus V(I)$. Then, by Proposition 2.7, $\text{Ext}^i_{A_p}(A_p/pA_p, N_p) = 0$, and so, $\mu_i(p, N) = 0$. Therefore

$$\Gamma_i(E^i) = \bigoplus_{p \in \text{Supp}(N)} \mu_i(p, N)E(A/p)$$

$$= \bigoplus_{p \in \text{Supp}(N) \cap (a_1, \ldots, a_n)} \mu_i(p, N)E(A/p) = \Gamma_i_{(a_1, \ldots, a_n)}(E^i)$$

for all $i < n$. Now we have

$$\text{Ker} \Gamma_i(d^i) = \text{Ker} \Gamma_{(a_1, \ldots, a_n)}(d^i), \quad \text{Im} \Gamma_i(d^{i-1}) = \text{Im} \Gamma_{(a_1, \ldots, a_n)}(d^{i-1})$$

for all $i < n$. Therefore $H^i_i(N) = H^i_{(a_1, \ldots, a_n)}(N)$ for all $i < n$.

Now we can present the following theorem, using Lemma 3.1, the concept of filter regular sequences and [8, Th. 2.4].

3.2. Theorem. Let $M$ be a finitely generated $A$-module. Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n \in I$ be an $I$-filter regular $M$-sequence. Then, for all $i < n$,

$$H^i_i(M) \cong H^i(C(U(a), M)).$$

Proof. By Lemma 3.1, $H^i_i(M) = H^i_{(a_1, \ldots, a_n)}(M)$ for all $i < n$. By Theorem 2.6, there exist $1 \leq m_1 \leq \ldots \leq m_n$ such that $a_1^{m_1}, \ldots, a_n^{m_n}$ is a $d$-sequence on $M$ (in $I$). Now we have

$$H^i_{(a_1, \ldots, a_n)}(M) = H^i_{(a_1^{m_1}, \ldots, a_n^{m_n})}(M)$$
for all $i \in \mathbb{N} \cup \{0\}$. By [8, Th. 2.4], for all $i < n$,

$$H^i_{[a_{i}^{m_1}, \ldots, a_{i}^{m_n}]}(M) \cong H^i(C(U(b), M)),$$

where $U(b) = (U(b_i))_{i \in \mathbb{N}}$ is the chain of triangular subsets on $A$ in which, for all $i \in \mathbb{N}$,

$$U(b_i) = \{(a_i^{m_1}, \ldots, a_i^{m_n}) : \text{there exists } j \text{ with } 0 \leq j \leq i \text{ such that } 
\alpha_1, \ldots, \alpha_j \in \mathbb{N} \text{ and } \alpha_{j+1} = \ldots = \alpha_i = 0\},$$

where $a_r$ is interpreted as $1$ whenever $r > n$.

On the other hand, by using elementary properties of generalized fractions or by applying [5, Th. 2.1], one can easily see that

$$H^i(C(U(b), M)) \cong H^i(C(U(a), M))$$

for all $i \in \mathbb{N} \cup \{0\}$. Therefore, for all $i < n$,

$$H^i_{M}(M) \cong H^i(C(U(a), M)).$$

In the following lemma, we show that for any ideal $I$ of $A$ and any positive integer $n$, there exists a $d$-sequence $a_1, \ldots, a_n \in I$ such that local cohomology modules with respect to $I$ are equal to local cohomology modules with respect to $(a_1, \ldots, a_n)$.

3.3. Lemma. Let $M$ be a finitely generated $A$-module. Let $n \in \mathbb{N}$. Then there exist $a_1, \ldots, a_n \in I$ which form a $d$-sequence on $M$ and

$$H^i_{I}(M) = H^i_{(a_1, \ldots, a_n)}(M)$$

for all $i < n$.

Proof. We can find $b_1, \ldots, b_n \in I$ which form an $I$-filter regular $M$-sequence as follows. Since $I \not\subseteq \bigcup_{p \in \text{Ass}(M) \setminus V(I)} p$, there exists $b_1 \in I$ such that $b_1 \not\in p$ for all $p \in \text{Ass}(M) \setminus V(I)$. Again, since $I \not\subseteq \bigcup_{p \in \text{Ass}(M/b_1M) \setminus V(I)} p$, there exists $b_2 \in I$ such that $b_2 \not\in p$ for all $p \in \text{Ass}(M/b_1M) \setminus V(I)$. Proceeding in this way, we can find $b_1, \ldots, b_n \in I$ which form an $I$-filter regular $M$-sequence. Now, by Lemma 3.1, $H^i_{I}(M) = H^i_{(b_1, \ldots, b_n)}(M)$ for all $i < n$. On the other hand, by Theorem 2.6, there exist $1 \leq m_1 \leq \ldots \leq m_n$ such that $b_1^{m_1}, \ldots, b_n^{m_n}$ form a $d$-sequence on $M$. Let $a_i = b_i^{m_i}$ for all $1 \leq i \leq n$. Then $H^i_{I}(M) = H^i_{(a_1, \ldots, a_n)}(M)$ for all $i < n$.

Now that Lemma 3.3 has been established, we can prove the main theorem of this paper by using [8, Th. 2.4].

3.4. Theorem. Let $M$ be a finitely generated $A$-module. Let $n \in \mathbb{N}$. Then there exist $a_1, \ldots, a_n \in I$ which form a $d$-sequence on $M$ and

$$H^i_{I}(M) \cong H^i(C(U(a), M))$$

for all $i < n$. 
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