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THE SPECTRAL MAPPING THEOREM FOR THE ESSENTIAL APPROXIMATE POINT SPECTRUM

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1. Introduction and preliminaries. Let X be an infinite-dimensional complex Banach space and denote the set of bounded linear operators on X by $\mathcal{B}(X)$. $\mathcal{K}(X)$ denotes the ideal of compact operators on X. Let $\sigma(T)$ and $\varrho(T)$ denote, respectively, the spectrum and the resolvent set of an element T of $\mathcal{B}(X)$. The set of those operators T of $\mathcal{B}(X)$ for which the range $T(X)$ is closed and $\alpha(T)$, the dimension of the null space $N(T)$ of T, is finite is denoted by $\Phi_+(X)$. Set

$$\Phi_-(X) = \{T \in \mathcal{B}(X) : \beta(T) \text{ is finite}\},$$

where $\beta(T)$ is the codimension of $T(X)$. Observe that $T(X)$ is closed if $T \in \Phi_-(X)$ ([3], Satz 55.4). Operators in $\Phi_+(X) \cup \Phi_-(X)$ are called *semi-Fredholm operators*. For such an operator T we define the index of T by $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. An operator T is called a *Fredholm operator* if $T \in \Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. Let $\Phi_+^-(X)$ denote the set of those operators T in $\Phi_+(X)$ for which $\mathrm{ind}(T) \le 0$.

For an operator T in $\mathcal{B}(X)$ we will use the following notations:

$$\begin{aligned} \Phi(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi(X)\}, \\ \Sigma(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is semi-Fredholm}\}, \\ \Sigma_+(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_+(X)\} \end{aligned}$$

and

$$\mathcal{H}(T) = \{f : \Delta(f) \to \mathbb{C} : \Delta(f) \text{ is open},\ \sigma(T) \subseteq \Delta(f),\ f \text{ is holomorphic}\}.$$

It is well known that $\Phi(T)$, $\Sigma(T)$ and $\Sigma_+(T)$ are open [3], §82. For $f \in \mathcal{H}(T)$, the operator $f(T)$ is defined by the well-known analytic calculus (see [3]).

Let $T \in \mathcal{B}(X)$. We write $\sigma_{\mathrm{e}}(T)$ for Schechter's *essential spectrum* of T

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(see [11]), i.e.,

$$\sigma_{\mathrm{e}}(T) = \bigcap_{K\in\mathcal{K}(X)} \sigma(T+K).$$

This essential spectrum has the following properties:

1. $\mathbb{C}\setminus\sigma_{\mathrm{e}}(T) = \{\lambda\in\Phi(T) : \operatorname{ind}(\lambda I - T) = 0\}$ ([3], Satz 107.3).

2. $\sigma_{\mathrm{e}}(f(T)) \subseteq f(\sigma_{\mathrm{e}}(T))$ for each $f\in\mathcal{H}(T)$, and this inclusion may be proper (see [2] and [6]; see also [12], where the above inclusion is shown in the context of Fredholm elements in Banach algebras).

3. If $f\in\mathcal{H}(T)$ is univalent, then $\sigma_{\mathrm{e}}(f(T)) = f(\sigma_{\mathrm{e}}(T))$ (see [6], Remark 1 in Section 3).

In [12] we have introduced (in a more general context) the following class of operators:

$$\mathcal{S}(X) = \{T\in\mathcal{B}(X) : \operatorname{ind}(\lambda I - T) \le 0 \text{ for all } \lambda\in\Phi(T) \text{ or } \operatorname{ind}(\lambda I - T)\ge 0 \text{ for all } \lambda\in\Phi(T)\}.$$

We have shown in [12] that

$$(*)\qquad T\in\mathcal{S}(X) \Leftrightarrow \sigma_{\mathrm{e}}(f(T)) = f(\sigma_{\mathrm{e}}(T)) \text{ for all } f\in\mathcal{H}(T).$$

Thus $(*)$ is a generalization of Theorem 1 in [5].

Let $\sigma_{\mathrm{ap}}(T)$ denote the *approximate point spectrum* of $T\in\mathcal{B}(X)$, i.e.,

$$\sigma_{\mathrm{ap}}(T) = \{\lambda\in\mathbb{C} : \inf_{\|x\|=1} \|(\lambda I - T)x\| = 0\}.$$

The *essential approximate point spectrum* $\sigma_{\mathrm{eap}}(T)$ of T was introduced by V. Rakočević in [8] as follows:

$$\sigma_{\mathrm{eap}}(T) = \bigcap_{K\in\mathcal{K}(X)} \sigma_{\mathrm{ap}}(T+K)$$

(see also [9] and [10]).

Set further

$$\mathcal{S}_+(X) = \{T\in\mathcal{B}(X) : \operatorname{ind}(\lambda I - T) \le 0 \text{ for all } \lambda\in\Sigma_+(T) \text{ or } \operatorname{ind}(\lambda I - T)\ge 0 \text{ for all } \lambda\in\Sigma_+(T)\}.$$

Clearly we have $\mathcal{S}_+(X)\subseteq\mathcal{S}(X)$.

The aim of the paper is to show the following result:

$$(**)\qquad T\in\mathcal{S}_+(X) \Leftrightarrow \sigma_{\mathrm{eap}}(f(T)) = f(\sigma_{\mathrm{eap}}(T)) \text{ for all } f\in\mathcal{H}(T).$$

The first part of the following proposition is probably known. According to C. Pearcy [7], this result has already appeared in a preprint *Fredholm operators* by P. R. Halmos in 1967. For the convenience of the reader we shall include a proof.

PROPOSITION 1. (1) *If* $T, S \in \Phi_+(X)$ [*resp.* $\in \Phi_-(X)$] *then* $TS \in \Phi_+(X)$ [*resp.* $\in \Phi_- X)$], *and*

$$\operatorname{ind}(TS) = \operatorname{ind}(T) + \operatorname{ind}(S).$$

(2) *If* $T, S \in \mathcal{B}(X)$, $TS \in \Phi_+(X)$ [*resp.* $\in \Phi_-(X)$] *then* $S \in \Phi_+(X)$ [*resp.* $T \in \Phi_-(X)$].

Proof. (1) It suffices to consider the case where $T, S \in \Phi_+(X)$ (because of [3], Satz 82.1).

Case 1: $T, S \in \Phi(X)$. Then, by [3], §71, $TS \in \Phi(X)$ and $\operatorname{ind}(TS) = \operatorname{ind}(T) + \operatorname{ind}(S)$.

Case 2: $T \notin \Phi(X)$ or $S \notin \Phi(X)$. Then $\beta(T) = \infty$ or $\beta(S) = \infty$. Use [3], Aufgabe 82.2,4, to get $TS \in \Phi_+(X)$ and $\beta(TS) = \infty$. Hence

$$\operatorname{ind}(TS) = -\infty = \operatorname{ind}(T) + \operatorname{ind}(S).$$

(2) See [3], Aufgabe 82.3,4. ∎

2. Properties of $\sigma_{\mathrm{eap}}(T)$. We begin with some properties of $\sigma_{\mathrm{eap}}(T)$ due to V. Rakočević:

PROPOSITION 2. *Let* $T \in \mathcal{B}(X)$.

(1) $\partial\sigma_{\mathrm{e}}(T) \subseteq \sigma_{\mathrm{eap}}(T)$ (*where* $\partial\sigma_{\mathrm{e}}(T)$ *denotes the boundary of* $\sigma_{\mathrm{e}}(T)$).
(2) $\sigma_{\mathrm{eap}}(T) \neq \emptyset$.
(3) $\lambda \notin \sigma_{\mathrm{eap}}(T) \Leftrightarrow \lambda I - T \in \Phi_+(X)$ *and* $\operatorname{ind}(\lambda I - T) \leq 0$.
(4) $\sigma_{\mathrm{eap}}(T)$ *is compact*, $\sigma_{\mathrm{eap}}(T) \subseteq \sigma(T)$.

Proof. For (1), (2), see [8], Theorem 1. For (3), see [8], Lemmata 1 and 2. (4) is clear. ∎

PROPOSITION 3. *Let* $T \in \mathcal{B}(X)$ *and let* λ_0 *be a boundary point of* $\sigma(T)$. *If* $\lambda_0 \in \Sigma(T)$ *then* λ_0 *is an isolated point of* $\sigma(T)$.

Proof. Theorem 3 of [4] shows the existence of $\delta > 0$ such that $\lambda \in \Sigma(T)$ for $|\lambda - \lambda_0| < \delta$, $\alpha(\lambda I - T)$ is a constant for $0 < |\lambda - \lambda_0| < \delta$ and $\beta(\lambda I - T)$ is a constant for $0 < |\lambda - \lambda_0| < \delta$. Take $\mu_0 \in \varrho(T)$ with $0 < |\mu_0 - \lambda_0| < \delta$. Then $\alpha(\mu_0 I - T) = \beta(\mu_0 I - T) = 0$, thus $\alpha(\lambda I - T) = \beta(\lambda I - T) = 0$ for $0 < |\lambda - \lambda_0| < \delta$. This shows that $\lambda \in \varrho(T)$ for $0 < |\lambda - \lambda_0| < \delta$. ∎

PROPOSITION 4. *Let* $T \in \mathcal{B}(X)$ *and* $h \in \mathcal{H}(T)$. *If* h *has no zeroes in* $\sigma_{\mathrm{eap}}(T)$ *then* h *has at most a finite number of zeroes in* $\sigma(T)$.

Proof. Assume that the number of zeroes of h in $\sigma(T)$ is infinite. Then there is $z_0 \in \sigma(T)$ such that z_0 is an accumulation point of the zeroes of h in $\sigma(T)$. Denote by C the connected component of $\sigma(T)$ which contains z_0 and by K the connected component of $\Delta(h)$ which contains z_0 (where $\Delta(h)$ is the open set of the definition of h). It follows that $C \subseteq K$ and $h \equiv 0$ on K. Let $\lambda_0 \in \partial C$. Then $h(\lambda_0) = 0$. Since h does not vanish on $\sigma_{\mathrm{eap}}(T)$, we have $\lambda_0 \notin \sigma_{\mathrm{eap}}(T)$ and therefore $\lambda_0 \in \Sigma(T)$. Since C is a connected component of $\sigma(T)$, we also have $\lambda_0 \in \partial\sigma(T)$. By Proposition 3 we see that λ_0 is an isolated point of $\sigma(T)$. Thus $C = \{\lambda_0\}$. Hence we get $z_0 = \lambda_0$, a contradiction, since z_0 is an accumulation point of $\sigma(T)$. ■

PROPOSITION 5. *Let (T_n) be a sequence in $\mathcal{B}(X)$ converging to $T \in \mathcal{B}(X)$ in the operator norm. If $V \subseteq \mathbb{C}$ is open and $0 \in V$, then there exists $n_0 \in \mathbb{N}$ such that*

$$\sigma_{\mathrm{eap}}(T_n) \subseteq \sigma_{\mathrm{eap}}(T) + V \qquad \text{for all } n \geq n_0.$$

Proof. Assume not. Then by passing to a subsequence (if necessary) it may be assumed that for each n there exists $\lambda_n \in \sigma_{\mathrm{eap}}(T_n)$ such that $\lambda_n \notin \sigma_{\mathrm{eap}}(T) + V$. Since (λ_n) is bounded, we may assume (if necessary pass to a subsequence) that $\lim_{n\to\infty} \lambda_n = \lambda_0$. This gives $\lambda_0 \notin \sigma_{\mathrm{eap}}(T) + V$, hence $\lambda_0 \notin \sigma_{\mathrm{eap}}(T)$. Thus $\lambda_0 I - T \in \Phi_+^-(X)$ (Proposition 2(3)). Since $\Phi_+^-(X)$ is an open multiplicative semigroup (see [3], § 82) and $\lambda_n I - T_n \to \lambda_0 I - T$ $(n \to \infty)$, we get some $N \in \mathbb{N}$ such that $\lambda_n I - T_n \in \Phi_+^-(X)$ for all $n \geq N$. Use again Proposition 2(3) to derive $\lambda_n \notin \sigma_{\mathrm{eap}}(T_n)$ for each $n \geq N$, a contradiction. ■

3. Spectral mapping theorem for $\sigma_{\mathrm{eap}}(T)$. The following result is due to V. Rakočević ([10], Theorem 3.3). For the convenience of the reader we give a (slightly simpler) proof.

THEOREM 1. *Let $T \in \mathcal{B}(X)$ and $f \in \mathcal{H}(T)$. Then*

$$\sigma_{\mathrm{eap}}(f(T)) \subseteq f(\sigma_{\mathrm{eap}}(T)).$$

Proof. Let $\mu \notin f(\sigma_{\mathrm{eap}}(T))$ and put $h(\lambda) = \mu - f(\lambda)$. Then h has no zeroes in $\sigma_{\mathrm{eap}}(T)$. Applying Proposition 4 we conclude that h has at most a finite number of zeroes in $\sigma(T)$.

Case 1: h has no zeroes in $\sigma(T)$. Then $h(T) = \mu I - f(T)$ is invertible, thus $\mu \notin \sigma_{\mathrm{eap}}(f(T))$.

Case 2: h has finitely many zeroes in $\sigma(T)$. Let $\lambda_1, \ldots, \lambda_k$ be those zeroes. Then there exist $n_1, \ldots, n_k \in \mathbb{N}$ and $g \in \mathcal{H}(T)$ such that

$$h(\lambda) = g(\lambda) \prod_{j=1}^{k} (\lambda_j - \lambda)^{n_j}, \qquad g(T) \text{ is invertible,}$$

and

$$h(T) = g(T) \prod_{j=1}^{k} (\lambda_j I - T)^{n_j}.$$

Since $\lambda_1, \ldots, \lambda_k \notin \sigma_{\mathrm{eap}}(T)$ we get

$$\lambda_j I - T \in \Phi_+(X) \quad \text{and} \quad \mathrm{ind}(\lambda_j I - T) \leq 0 \quad (j = 1, \ldots, k).$$

Use Proposition 1(1) to derive $h(T) \in \Phi_+(X)$ and

$$\operatorname{ind}(h(T)) = \underbrace{\operatorname{ind}(g(T))}_{=0} + \sum_{j=1}^{k} n_j \underbrace{\operatorname{ind}(\lambda_j I - T)}_{\le 0} \le 0.$$

Thus $\mu I - f(T) = h(T) \in \Phi_+^-(X)$ and therefore $\mu \notin \sigma_{\mathrm{eap}}(f(T))$. ∎

Example 4.2 in [9] shows that the inclusion in Theorem 1 may be proper.

In the first section of this paper we introduced the following class of operators:

$$\mathcal{S}_+(X) = \{T \in \mathcal{B}(X) : \operatorname{ind}(\lambda I - T) \le 0 \text{ for all } \lambda \in \Sigma_+(T) \\ \text{or } \operatorname{ind}(\lambda I - T) \ge 0 \text{ for all } \lambda \in \Sigma_+(T)\}.$$

PROPOSITION 6. *Let $T \in \mathcal{S}_+(X)$ and let r be a rational function in $\mathcal{H}(T)$. Then*

$$\sigma_{\mathrm{eap}}(r(T)) = r(\sigma_{\mathrm{eap}}(T)).$$

Proof. By Theorem 1 we only have to show $r(\sigma_{\mathrm{eap}}(T)) \subseteq \sigma_{\mathrm{eap}}(r(T))$. Let $r = p/q$, where p and q are polynomials and q has no zeroes in $\sigma(T)$. Hence $q(T)$ is invertible. Let $\mu \notin \sigma_{\mathrm{eap}}(r(T))$, thus, by Proposition 2(3),

$$\mu I - r(T) \in \Phi_+(X) \quad \text{and} \quad \operatorname{ind}(\mu I - r(T)) \le 0.$$

Put $h(\lambda) = \mu - r(\lambda)$, thus $h(\lambda) = (\mu q(\lambda) - p(\lambda))/q(\lambda)$. There exist $\mu_1, \ldots, \mu_k$, $\alpha \in \mathbb{C}$ such that

$$h(\lambda) = \alpha \frac{(\mu_1 - \lambda) \ldots (\mu_k - \lambda)}{q(\lambda)}.$$

This gives $q(T)h(T) = \alpha(\mu_1 I - T) \ldots (\mu_k I - T)$. Since $q(T)h(T) \in \Phi_+(X)$, Proposition 1(2) shows that

$$\mu_j I - T \in \Phi_+(X) \qquad \text{for } j = 1, \ldots, k.$$

Furthermore, by Proposition 1(1), we have

$$\sum_{j=1}^{k} \operatorname{ind}(\mu_j I - T) = \operatorname{ind}(q(T)h(T)) = \underbrace{\operatorname{ind}(q(T))}_{=0} + \operatorname{ind}(h(T))$$
$$= \operatorname{ind}(h(T)) = \operatorname{ind}(\mu I - r(T)) \le 0.$$

Case 1: $\operatorname{ind}(\lambda I - T) \le 0$ for all $\lambda \in \Sigma_+(T)$. Since $\mu_j \in \Sigma_+(T)$ for $j = 1, \ldots, k$, we derive $\operatorname{ind}(\mu_j I - T) \le 0$ for $j = 1, \ldots, k$, hence $\mu_j I - T \in \Phi_+^-(X)$ $(j = 1, \ldots, k)$ and therefore, by Proposition 2(3),

$$\mu_j \notin \sigma_{\mathrm{eap}}(T) \qquad \text{for } j = 1, \ldots, k.$$

This gives $\mu \notin r(\sigma_{\mathrm{eap}}(T))$.

Case 2: $\operatorname{ind}(\lambda I - T) \geq 0$ for all $\lambda \in \Sigma_+(T)$. Then $\operatorname{ind}(\mu_j I - T) \geq 0$ $(j = 1, \ldots, k)$ and therefore

$$0 \leq \sum_{j=1}^{k} \operatorname{ind}(\mu_j I - T) = \operatorname{ind}(\mu I - r(T)) \leq 0.$$

This shows that $\operatorname{ind}(\mu_j I - T) = 0$ for $j = 1, \ldots, k$. Thus $\mu_j \notin \sigma_{\mathrm{eap}}(T)$ $(j = 1, \ldots, k)$ and hence $\mu \notin r(\sigma_{\mathrm{eap}}(T))$. ∎

Now we are in a position to state the main result of this paper:

THEOREM 2. *If* $T \in \mathcal{B}(X)$ *then*

$$T \in \mathcal{S}_+(X) \Leftrightarrow \sigma_{\mathrm{eap}}(f(T)) = f(\sigma_{\mathrm{eap}}(T)) \text{ for all } f \in \mathcal{H}(T).$$

Proof. "$\Rightarrow$". The inclusion "$\subseteq$" follows from Theorem 1. Let $\Delta(f)$ denote the (open) set of the definition of f. Corollary 6.6 of [1] shows the existence of a sequence (r_n) of rational functions such that (r_n) converges to f uniformly on compact subsets of $\Delta(f)$. Thus $\|r_n(T) - f(T)\| \to 0$ $(n \to \infty)$ ([3], Aufgabe 99.1). Let V be an open set in $\mathbb{C}$ containing the origin. By Proposition 5 and the uniform convergence on $\sigma_{\mathrm{eap}}(T)$, there exists $n_0 \in \mathbb{N}$ such that

$$f(\sigma_{\mathrm{eap}}(T)) \subseteq r_n(\sigma_{\mathrm{eap}}(T)) + V$$

and

$$\sigma_{\mathrm{eap}}(r_n(T)) \subseteq \sigma_{\mathrm{eap}}(f(T)) + V$$

for all $n \geq n_0$. Proposition 6 gives

$$r_n(\sigma_{\mathrm{eap}}(T)) = \sigma_{\mathrm{eap}}(r_n(T)) \quad \text{for all } n \in \mathbb{N},$$

thus

$$f(\sigma_{\mathrm{eap}}(T)) \subseteq \sigma_{\mathrm{eap}}(r_{n_0}(T)) + V \subseteq \sigma_{\mathrm{eap}}(f(T)) + V + V.$$

Since V was an arbitrary neighbourhood of 0, we get

$$f(\sigma_{\mathrm{eap}}(T)) \subseteq \sigma_{\mathrm{eap}}(f(T)).$$

"$\Leftarrow$". Assume to the contrary that $T \notin \mathcal{S}_+(X)$. Then there are $\lambda_1, \lambda_2 \in \Sigma_+(T)$ with

$$\operatorname{ind}(\lambda_1 I - T) > 0 \quad \text{and} \quad \operatorname{ind}(\lambda_2 I - T) < 0.$$

It follows that $\beta(\lambda_1 I - T) < \infty$, hence $\lambda_1 I - T \in \Phi(X)$ and thus $k := \operatorname{ind}(\lambda_1 I - T) \in \mathbb{N}$.

Case 1: $\lambda_2 I - T \in \Phi(X)$. Put $m := -\operatorname{ind}(\lambda_2 I - T)$, thus $m \in \mathbb{N}$. Define the function $f \in \mathcal{H}(T)$ by $f(\lambda) = (\lambda_1 - \lambda)^m (\lambda_2 - \lambda)^k$. Then $f(T) \in \Phi(X)$ and $\operatorname{ind}(f(T)) = mk + k(-m) = 0$, thus $0 \notin \sigma_{\mathrm{eap}}(f(T))$. Since $\lambda_1 I - T \notin \Phi_+^-(X)$ we see by Proposition 2(3) that $\lambda_1 \in \sigma_{\mathrm{eap}}(T)$ and therefore $0 = f(\lambda_1) \in f(\sigma_{\mathrm{eap}}(T))$, a contradiction.

Case 2: $\lambda_2 I - T \notin \Phi(X)$. Then $\beta(\lambda_2 I - T) = \infty$ and $\operatorname{ind}(\lambda_2 I - T) = -\infty$. Put $f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)$. It follows from Proposition 1(1) that $f(T) \in \Phi_+(X)$ and that

$$\operatorname{ind}(f(T)) = k - \infty = -\infty,$$

thus $0 \notin \sigma_{\mathrm{eap}}(f(T))$. As in Case 1 we have $0 = f(\lambda_1) \in f(\sigma_{\mathrm{eap}}(T))$, a contradiction. ■

4. The essential defect spectrum. For $T \in \mathcal{B}(X)$ the *defect spectrum* $\sigma_\delta(T)$ is defined by

$$\sigma_\delta(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}.$$

We define the *essential defect spectrum* $\sigma_{\mathrm{e}\delta}(T)$ of T by

$$\sigma_{\mathrm{e}\delta}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma_\delta(T + K).$$

We let X^* designate the conjugate space of X and T^* the adjoint of $T \in \mathcal{B}(X)$.

PROPOSITION 7. *Let* $T \in \mathcal{B}(X)$.

(1) $\lambda \notin \sigma_{\mathrm{e}\delta}(T) \Leftrightarrow \lambda I - T \in \Phi_-(X)$ *and* $\operatorname{ind}(\lambda I - T) \geq 0$.
(2) $\sigma_{\mathrm{e}\delta}(T) = \sigma_{\mathrm{eap}}(T^*)$.
(3) $\sigma_{\mathrm{e}\delta}(T) \neq \emptyset$.

Proof. (1) "$\Rightarrow$". If $\lambda \notin \sigma_{\mathrm{e}\delta}(T)$ then there is $K \in \mathcal{K}(X)$ such that $\lambda \notin \sigma_\delta(T + K)$, thus $\lambda I - T - K$ is surjective, hence $\lambda I - T - K \in \Phi_-(X)$ and $\operatorname{ind}(\lambda I - T - K) = \alpha(\lambda I - T - K) \geq 0$. Satz 82.5 of [3] shows then that $\lambda I - T \in \Phi_-(X)$ and $\operatorname{ind}(\lambda I - T) = \operatorname{ind}(\lambda I - T - K) \geq 0$.

"$\Leftarrow$". If $\lambda I - T \in \Phi_-(X)$ and $\operatorname{ind}(\lambda I - T) \geq 0$ then, by [13], Theorem 3.13, there are $U_1, U_2 \in \mathcal{B}(X)$ such that

$$\lambda I - T = U_1 + U_2, \quad U_2 \in \mathcal{K}(X), \quad U_1(X) = X.$$

Thus $\lambda I - (T + U_2)$ is surjective and therefore $\lambda \notin \sigma_\delta(T + U_2)$. This gives $\lambda \notin \sigma_{\mathrm{e}\delta}(T)$.

(2) Use (1), Proposition 2(3) and [3], Satz 82.1, to get

$$\begin{aligned}\lambda \notin \sigma_{\mathrm{e}\delta}(T) &\Leftrightarrow \lambda I^* - T^* \in \Phi_+(X^*) \text{ and } \operatorname{ind}(\lambda I^* - T^*) \leq 0\\ &\Leftrightarrow \lambda \notin \sigma_{\mathrm{eap}}(T^*).\end{aligned}$$

(3) This follows from (2) and Proposition 2(2). ■

THEOREM 3. *For* $T \in \mathcal{B}(X)$ *and* $f \in \mathcal{H}(T)$ *we have*

$$\sigma_{\mathrm{e}\delta}(f(T)) \subseteq f(\sigma_{\mathrm{e}\delta}(T)).$$

Proof. We have

$$\begin{aligned}\sigma_{e\delta}(f(T)) &= \sigma_{eap}((f(T))^*) && \text{(by Proposition 7(2))}\\ &= \sigma_{eap}(f(T^*))\\ &\subseteq f(\sigma_{eap}(T^*)) && \text{(by Theorem 1)}\\ &= f(\sigma_{e\delta}(T)) && \text{(by Proposition 7(2)).} \ \blacksquare\end{aligned}$$

For our final result in this section, which is dual to Theorem 2, we need the following definitions. For T in $\mathcal{B}(X)$ set $\Sigma_-(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_-(X)\}$. The class $\mathcal{S}_-(X)$ of operators is defined by

$$\mathcal{S}_-(X) = \{T \in \mathcal{B}(X) : \operatorname{ind}(\lambda I - T) \ge 0 \text{ for all } \lambda \in \Sigma_-(T) \text{ or } \operatorname{ind}(\lambda I - T) \le 0 \text{ for all } \lambda \in \Sigma_-(T)\}.$$

It follows from [3], Satz 82.1, that $\Sigma(T) = \Sigma(T^*)$, $\Sigma_+(T) = \Sigma_-(T^*)$, $\Sigma_-(T) = \Sigma_+(T^*)$ and that

$$\operatorname{ind}(\lambda I - T) = -\operatorname{ind}(\lambda I^* - T^*) \quad \text{for all } \lambda \in \Sigma(T).$$

This gives

$$T \in \mathcal{S}_-(X) \Leftrightarrow T^* \in \mathcal{S}_+(X^*), \qquad T \in \mathcal{S}_+(X) \Leftrightarrow T^* \in \mathcal{S}_-(X^*).$$

As an immediate consequence of Theorem 2 and Proposition 7 we get

THEOREM 4. *Let $T \in \mathcal{B}(X)$. Then*

$$T \in \mathcal{S}_-(X) \Leftrightarrow f(\sigma_{e\delta}(T)) = \sigma_{e\delta}(f(T)) \text{ for all } f \in \mathcal{H}(T).$$

5. Schechter's essential spectrum. In this final section we return to $\sigma_e(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma(T+K)$. Recall that $\lambda \notin \sigma_e(T)$ if and only if $\lambda \in \Phi(T)$ and $\operatorname{ind}(\lambda I - T) = 0$. We have mentioned in Section 1 that the following result holds.

THEOREM 5. *Let $T \in \mathcal{B}(X)$.*

(1) *$\sigma_e(f(T)) \subseteq f(\sigma_e(T))$ for each $f \in \mathcal{H}(T)$.*
(2) *$T \in \mathcal{S}(X) \Leftrightarrow \sigma_e(f(T)) = f(\sigma_e(T))$ for all $f \in \mathcal{H}(T)$.*

The aim of this section is to prove Theorem 5 with the aid of the results of the previous sections of this paper.

PROPOSITION 8. *For $T \in \mathcal{B}(X)$ we have*:

(1) *$\sigma_e(T) = \sigma_{eap}(T) \cup \sigma_{e\delta}(T)$.*
(2) *$\mathcal{S}(X) = \mathcal{S}_+(X) \cup \mathcal{S}_-(X)$.*

Proof. (1) Use Propositions 2(3) and 7(1).

(2) The inclusion $\mathcal{S}_+(X) \cup \mathcal{S}_-(X) \subseteq \mathcal{S}(X)$ is clear. Let $T \in \mathcal{S}(X)$ and assume $T \notin \mathcal{S}_+(X) \cup \mathcal{S}_-(X)$. Then there are $\lambda_1, \lambda_2 \in \Sigma_+(T)$ and $\lambda_3, \lambda_4 \in$

$\Sigma_-(T)$ such that $\operatorname{ind}(\lambda_1 I-T)>0$, $\operatorname{ind}(\lambda_2 I-T)<0$, $\operatorname{ind}(\lambda_3 I-T)>0$ and $\operatorname{ind}(\lambda_4 I-T)<0$. This gives $\beta(\lambda_1 I-T)<\infty$ and $\alpha(\lambda_4 I-T)<\infty$, hence $\lambda_1,\lambda_4\in\Phi(T)$. Since $T\in\mathcal{S}(X)$ and $\operatorname{ind}(\lambda_1 I-T)>0$, $\operatorname{ind}(\lambda_4 I-T)<0$, we have a contradiction. ∎

Proof of Theorem 5. (1) Use Proposition 8(1), Theorem 1 and Theorem 3 to derive

$$\begin{aligned}\sigma_{\mathrm{e}}(f(T)) &= \sigma_{\mathrm{eap}}(f(T))\cup\sigma_{\mathrm{e}\delta}(f(T))\subseteq f(\sigma_{\mathrm{eap}}(T))\cup f(\sigma_{\mathrm{e}\delta}(T))\\ &= f(\sigma_{\mathrm{eap}}(T)\cup\sigma_{\mathrm{e}\delta}(T)) = f(\sigma_{\mathrm{e}}(T)).\end{aligned}$$

(2) "⇒". Let $T\in\mathcal{S}(X)$ and $f\in\mathcal{H}(T)$. We only have to show that $f(\sigma_{\mathrm{e}}(T))\subseteq\sigma_{\mathrm{e}}(f(T))$. Let $\mu\notin\sigma_{\mathrm{e}}(f(T))=\sigma_{\mathrm{eap}}(f(T))\cup\sigma_{\mathrm{e}\delta}(f(T))$. Put $h:=\mu-f$. Assume that there are $\lambda_1\in\sigma_{\mathrm{eap}}(T)$ and $\lambda_2\in\sigma_{\mathrm{e}\delta}(T)$ such that $h(\lambda_1)=h(\lambda_2)=0$. It follows that $\mu\in f(\sigma_{\mathrm{eap}}(T))$ and $\mu\in f(\sigma_{\mathrm{e}\delta}(T))$. If $T\in\mathcal{S}_+(X)$ then we see by Theorem 2 that $\mu\in\sigma_{\mathrm{eap}}(f(T))\subseteq\sigma_{\mathrm{e}}(f(T))$, a contradiction. Similarly we get a contradiction if $T\in\mathcal{S}_-(X)$. Hence we have shown that h does not vanish on $\sigma_{\mathrm{eap}}(T)$ or h does not vanish on $\sigma_{\mathrm{e}\delta}(T)$. It suffices to consider the case $h(\lambda)\neq0$ for each $\lambda\in\sigma_{\mathrm{eap}}(T)$ (since $\sigma_{\mathrm{e}\delta}(T)=\sigma_{\mathrm{eap}}(T^*)$ the other case can be treated in the same manner). By Proposition 4, h has at most a finite number of zeroes in $\sigma(T)$.

Case 1: h has no zeroes in $\sigma(T)$. Then $\mu\notin\sigma(f(T))=f(\sigma(T))$. This gives $\mu\notin f(\sigma_{\mathrm{e}}(T))$.

Case 2: There are $\mu_1,\ldots,\mu_k\in\sigma(T)$ and $g\in\mathcal{H}(T)$ such that $h(\lambda)=g(\lambda)\prod_{j=1}^k(\mu_j-\lambda)$ and $g(\lambda)\neq0$ for $\lambda\in\sigma(T)$. Then we get

$$h(T)=g(T)\prod_{j=1}^k(\mu_j I-T),\qquad g(T)\text{ is invertible.}$$

Since $\mu\notin\sigma_{\mathrm{e}}(f(T))$ we see that $h(T)\in\Phi(X)$ and $\operatorname{ind}(h(T))=0$. Now use Proposition 1 to derive

$$\mu_j I-T\in\Phi(X)\qquad\text{for } j=1,\ldots,k$$

and

$$\sum_{j=1}^k\operatorname{ind}(\mu_j I-T)=\operatorname{ind}(h(T))=0.$$

Since $T\in\mathcal{S}(X)$ it follows that $\operatorname{ind}(\mu_j I-T)=0$ $(j=1,\ldots,k)$. Thus we have $\mu_j\notin\sigma_{\mathrm{e}}(T)$ $(j=1,\ldots,n)$, hence $\mu\notin f(\sigma_{\mathrm{e}}(T))$.

"⇐". Assume to the contrary that $T\notin\mathcal{S}(X)$. Then there are $\lambda_1,\lambda_2\in\Phi(T)$ with $k:=\operatorname{ind}(\lambda_1 I-T)>0$ and $m:=-\operatorname{ind}(\lambda_2 I-T)>0$. Put $f(\lambda)=(\lambda_1-\lambda)^m(\lambda_2-\lambda)^k$. We get $f(T)\in\Phi(X)$, $\operatorname{ind}(f(T))=0$, $0\notin\sigma_{\mathrm{e}}(f(T))$ but $0=f(\lambda_1)=f(\lambda_2)\in f(\sigma_{\mathrm{e}}(T))$. This contradiction completes the proof. ∎

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