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MOMENTS OF SOME RANDOM FUNCTIONALS

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The paper deals with nonnegative stochastic processes $X(t,\omega)$ $(t\ge 0)$, not identically zero, with stationary and independent increments, right-continuous sample functions, and fulfilling the initial condition $X(0,\omega) = 0$. The main aim is to study the moments of the random functionals $\int_0^\infty f(X(\tau,\omega))\,d\tau$ for a wide class of functions f. In particular, a characterization of deterministic processes in terms of the exponential moments of these functionals is established.

1. Preliminaries and notation. We denote by $\mathcal{M}$ the set of all nonnegative bounded measures defined on Borel subsets of the half-line $\mathbb{R}_+ = [0,\infty)$, and by $\mathcal{P}$ the subset of $\mathcal{M}$ consisting of probability measures. The probability measure concentrated at the point c is denoted by δ_c. Given $s\in(-\infty,\infty)$ we denote by $\mathcal{P}_s$ the subset of $\mathcal{P}$ consisting of measures μ with finite moment $m_s(\mu)=\int_0^\infty x^s\,\mu(dx)$. Given $M\in\mathcal{M}$ by $\widehat{M}$ and $\langle M\rangle$ we denote the Laplace and the Bernstein transformation of M respectively, i.e.

$$\widehat{M}(z)=\int_0^\infty e^{-zx}\,M(dx)\quad\text{and}\quad \langle M\rangle(z)=\int_0^\infty \frac{1-e^{-zx}}{1-e^{-x}}\,M(dx)$$

for $z\ge 0$. For $x=0$ the last integrand is assumed to be z.

Let $\mu\in\mathcal{P}$. By standard calculations we get the formulae

$$m_{-s}(\mu)=\frac{1}{\Gamma(s)}\int_0^\infty \widehat{\mu}(z)z^{s-1}\,dz\quad (s>0) \tag{1.1}$$

and

$$m_q(\mu)=\frac{q}{\Gamma(1-q)}\int_0^\infty \frac{1-\widehat{\mu}(z)}{z^{1+q}}\,dz\quad (0<q<1). \tag{1.2}$$

In the sequel $\operatorname{distr}\xi$ will denote the probability distribution of a random variable ξ.

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Let $\mathcal{X}$ be the class of nonnegative stochastic processes $X = \{X(t,\omega) : t \geq 0\}$, not identically zero, with stationary and independent increments, right-continuous sample functions and fulfilling the initial condition $X(0,\omega) = 0$. It is well known that to every process X from $\mathcal{X}$ there corresponds a measure M from $\mathcal{M}$ with $M(\mathbb{R}_+) > 0$ satisfying the condition

$$p_t(z) = e^{-t\langle M\rangle(z)} \tag{1.3}$$

where $p_t = \operatorname{distr} X(t,\omega)$ $(t \geq 0)$. This uniquely determined measure M is called the *representing measure* for X. We note that each measure M from $\mathcal{M}$ with $M(\mathbb{R}_+) > 0$ is the representing measure for a process from $\mathcal{X}$.

A stochastic process X from $\mathcal{X}$ is said to be *deterministic* if $X(t,\omega) = ct$ with probability 1 for a positive constant c or, equivalently, $c\delta_0$ is the representing measure for X.

A stochastic process X from $\mathcal{X}$ with the representing measure M is said to be a *compound Poisson process* if

$$0 < c = \int_0^\infty (1-e^{-x})^{-1}\, M(dx) < \infty.$$

Setting for Borel subsets E of $\mathbb{R}_+$,

$$Q(E) = c^{-1}\int_E (1-e^{-x})^{-1} M\,(dx)$$

we have in this case $Q \in \mathcal{P}$ and

$$p_t = e^{-ct}\sum_{n=0}^{\infty}\frac{(ct)^n}{n!}\,Q^{*n} \tag{1.4}$$

where Q^{*n} for $n \geq 1$ is the nth convolution power of Q and $Q^{*0} = \delta_0$. The set of all processes X satisfying (1.4) will be denoted by $\operatorname{Poiss}(c,Q)$.

Throughout this paper π_s $(s > 0)$ will denote the exponential distribution on $\mathbb{R}_+$ with parameter s, i.e. $\pi_s(dx) = se^{-sx}dx$. We shall often refer to the following representation of processes X from $\operatorname{Poiss}(c,Q)$ ([2], Chapter IV, 2):

$$X(t,\omega) = 0 \qquad \text{for } t \in [0,\vartheta_0), \tag{1.5}$$

$$X(t,\omega) = \sum_{j=1}^{k}\xi_j \qquad \text{for } t \in \Big[\sum_{j=0}^{k-1}\vartheta_j,\ \sum_{j=0}^{k}\vartheta_j\Big) \tag{1.6}$$

for $k \geq 1$ where the random variables $\vartheta_0, \vartheta_1, \ldots, \xi_1, \xi_2, \ldots$ are independent, $\vartheta_0, \vartheta_1, \ldots$ have probability distribution π_c and $\xi_1, \xi_2, \ldots$ have probability distribution Q.

It is well known that for processes from $\mathcal{X}$ the potential

$$\varrho(E) = \int_0^\infty p_t(E)\,dt$$

is finite on bounded Borel subsets E of $\mathbb{R}_+$ ([1], Prop. 14.1). Moreover, the Laplace transform of the potential is given by the formula

$$\widehat{\varrho}(z) = \langle M \rangle^{-1}(z) \quad (z > 0) \tag{1.7}$$

where M is the representing measure of the process in question.

2. Integral functionals. Denote by $\mathcal{F}$ the set of all nonnegative, continuous, decreasing functions f defined on $\mathbb{R}_+$, not identically zero, satisfying the condition $\int_0^\infty f(x)\,dx < \infty$. Given $r \in (0,1]$ we denote by $\mathcal{F}_r$ the subset of $\mathcal{F}$ consisting of functions fulfilling the condition $\int_0^\infty f^r(x)\,dx < \infty$. Put $(T_a f)(x) = f(x+a)$. Obviously $T_a\mathcal{F} \subset \mathcal{F}$ for $a \geq 0$.

Let $X \in \mathcal{X}$. It was shown in [4] that for every $f \in \mathcal{F}$ the random functional

$$[X,f] = \int_0^\infty f(X(\tau,\omega))\,d\tau$$

is well defined. Moreover, setting $\mu_a = \operatorname{distr}[X, T_a f]$ $(a \geq 0)$ we have the equation

$$\widehat{\mu}_a(z) = 1 - z\int_0^\infty f(a+y)\widehat{\mu}_y(z)\,\varrho(dy) \tag{2.1}$$

where ϱ is the potential for the process in question ([3], Th. 2.4).

If $X \in \operatorname{Poiss}(c,Q)$, then, by (1.5) and (1.6), we have the formula

$$[X,f] = f(0)\vartheta_0 + \sum_{k=1}^\infty f(\xi_1+\ldots+\xi_k)\vartheta_k.$$

Consequently, introducing the notation

$$\mathbb{R}_+^\infty = \mathbb{R}_+\times\mathbb{R}_+\times\ldots, \quad y=(y_1,y_2,\ldots)\in\mathbb{R}_+^\infty, \quad Q^\infty(dy) = Q(dy_1)\,Q(dy_2)\ldots$$

and

$$\Phi(y,z) = (1+c^{-1}f(0)z)^{-1}\prod_{k=1}^\infty (1+c^{-1}f(y_1+\ldots+y_k)z)^{-1}$$

we get the formula

$$\widehat{\mu}_0(z) = \int_{\mathbb{R}_+} \Phi(y,z)\,Q^\infty(dy). \tag{2.2}$$

LEMMA 2.1. *Let ϱ be the potential of a process X from $\mathcal{X}$, $f \in \mathcal{F}$, $\mu_a = \mathrm{distr}[X, T_a f]$ and $0 < q < 1$. Then*

$$m_{1-q}(\mu_0) = (1-q) \int_0^\infty f(y) m_{-q}(\mu_y)\, \varrho(dy). \tag{2.3}$$

P r o o f. We have, by (2.1),

$$(1 - \widehat{\mu}_0(z)) z^{-1} = \int_0^\infty f(y) \widehat{\mu}_y(z)\, \varrho(dy).$$

Multiplying both sides of the above equation by $(1-q)\Gamma(q)^{-1} z^{q-1}$ and integrating from 0 to ∞ we get, by (1.1) and (1.2), the assertion of the lemma.

THEOREM 2.1. *For every $X \in \mathcal{X}$, $f \in \mathcal{F}$ and $s > -1$ we have* $\mathrm{distr}[X, f] \in \mathcal{P}_s$.

P r o o f. Put $\mu_a = \mathrm{distr}[X, T_a f]$ $(a \ge 0)$. By Lemma 2.2 and Corollary 2.1 in [4] we conclude that

$$m_s(\mu_0) < \infty \quad \text{for } s \ge 0. \tag{2.4}$$

Suppose that $0 < q < 1$. Observe that $T_y f \le f$ for $y \ge 0$. Consequently, $[X, T_y f] \le [X, f]$, which yields the inequality $m_{-q}(\mu_y) \ge m_{-q}(\mu_0)$. Applying (2.3) we get the inequality

$$m_{1-q}(\mu_0) \ge (1-q) m_{-q}(\mu_0) \int_0^\infty f(y)\, \varrho(dy).$$

Since, by (2.4), $m_{1-q}(\mu_0) < \infty$, we have $m_{-q}(\mu_0) < \infty$, which completes the proof.

In what follows e_a $(a > 0)$ will denote the family of exponential functions, i.e. $e_a(x) = e^{-ax}$. Obviously $e_a \in \mathcal{F}$.

THEOREM 2.2. *Let $X \in \mathcal{X}$, $a > 0$, $p_a = \mathrm{distr}\, X(a, \omega)$ and $\nu_a = \mathrm{distr}[X, e_a]$. Then $m_{-1}(\nu_a) = m_1(p_a)$.*

P r o o f. Observe that $T_y e_a = e^{-ay} e_a$, which yields the formula $[X, T_y e_a] = e^{-ay}[X, e_a]$. Consequently, by (1.7) and (2.3) with $f = e_a$, we have the formula

$$m_{1-q}(\nu_a) = (1-q) m_{-q}(\nu_a) \widehat{\varrho}((1-q)a) = (1-q) m_{-q}(\nu_a) \langle M \rangle ((1-q)a)^{-1}$$

where $0 < q < 1$ and M is the representing measure for X. Now taking into account (1.3) and letting $q \to 1$ we get our assertion.

EXAMPLE 2.1. Given $0 < \alpha < 1$ we denote by Z_α the α-stable stochastic process from $\mathcal{X}$ with $\langle M \rangle (z) = z^\alpha$. Obviously $m_1(p_a) = \infty$ for $a > 0$. If

$\nu_a = \mathrm{distr}[Z_\alpha, e_a]$, then, by Theorem 2.2, $m_{-1}(\nu_a) = \infty$. Thus $\mathrm{distr}[Z_\alpha, e_a] \notin \mathcal{P}_{-1}$, which shows that Theorem 2.1 cannot be sharpened. $\blacksquare$

EXAMPLE 2.2. Let Y_1 be a compound Poisson process from $\mathrm{Poiss}(1, \pi_1)$. Given $f \in F$ we put $\lambda = \mathrm{distr}[Y_1, f]$. It was shown in [3] (Example 3.1) that

$$\widehat{\lambda}(z) = (1 + f(0)z)^{-1} \exp\Big(- z \int_0^\infty (1 + f(u)z)^{-1} f(u)\, du\Big). \tag{2.5}$$

In particular, setting $f = e_a$ $(a > 0)$ we get $\widehat{\lambda}(z) = (1 + z)^{-1-1/a}$. Thus $\lambda(dx) = e^{-x}\ x^{1/a}\, dx$, which shows that $\mathrm{distr}[Y_1, e_a] \in P_r$ if and only if $r > -1 - 1/a$.

Given $0 < s < 1$ we put

$$f_s(x) = (1 + x^{1/s})^{-1}. \tag{2.6}$$

It is clear that $f_s \in \mathcal{F}$. Setting $\lambda_s = \mathrm{distr}[Y_1, f_s]$ we have, by (2.5),

$$\widehat{\lambda}_s(z) = (1 + z)^{-1} \exp(-c_s z(1 + z)^{s-1}) \tag{2.7}$$

where $c_s = s\pi/\sin s\pi$. By (1.1) we get $\mathrm{distr}[Y_1, f_s] \in \mathcal{P}_r$ for all $r \in \mathbb{R}$.

3. Exponential moments. Given $p > 0$ we denote by $\mathcal{A}_p$ the subset of $\mathcal{P}$ consisting of measures μ for which the exponential moment

$$n_{p,r}(\mu) = \int_0^\infty e^{rx^{-p}}\ \mu(dx)$$

is finite for some $r > 0$. Let ξ be a nonnegative random variable. It is clear that $\mathrm{distr}\,\xi \in \mathcal{A}_p$ if and only if the Laplace transform of $\mathrm{distr}\,\xi^{-p}$ can be extended to an analytic function in a neighbourhood of the origin.

LEMMA 3.1. *Let $p > 0$ and $s = p/(1 + p)$. Then $\mu \in \mathcal{A}_p$ if and only if $\int_0^\infty \widehat{\mu}(z)e^{cz^s}\, dz < \infty$ for some $c > 0$.*

Proof. Applying (1.1) we get the formula

$$n_{p,r}(\mu) = 1 + \sum_{k=1}^\infty \frac{r^k}{k!} m_{-kp}(\mu) = 1 + \int_0^\infty \widehat{\mu}(z)g(p, r, z)\, dz$$

where $g(p, r, z) = prz^{p-1}\, h(p, rz^p)$ and

$$h(p, z) = \sum_{k=0}^\infty \frac{z^k}{k!\Gamma(pk + 1 + p)}.$$

E. M. Wright proved in [5] and [6], Th. 1, that for some positive constants a_p and b_p the limit

$$\lim_{z\to\infty} h(p, z)z^{(p+1/2)/(1+p)} \exp(-b_p z^{1/(1+p)}) = a_p$$

exists. Consequently,

$$\lim_{z\to\infty} g(p,r,z)\, z^{(p+1/2)/(1+p)} \exp(-b_p r^{1/(1+p)} z^s) = p r^{1/(2+2p)} a_p,$$

which yields our assertion.

LEMMA 3.2. *Let X_1 and X_2 be processes from $\mathcal{X}$ with the representing measures M_1 and M_2 respectively. If $M_1 \ge M_2$ and* $\mathrm{distr}[X_1, f] \in \mathcal{A}_p$ *for some $f \in \mathcal{F}$ and $p > 0$, then* $\mathrm{distr}[X_2, f] \in \mathcal{A}_p$.

Proof. Setting $M_3 = M_1 - M_2$ we have $M_3 \in \mathcal{M}$. If $M_3(\mathbb{R}_+) > 0$, then M_3 is the representing measure of a process X_3 from $\mathcal{X}$. In the remaining case $M_3(\mathbb{R}_+) = 0$ we put $X_3 = 0$. Without loss of generality we may assume that the processes X_2 and X_3 are independent. Hence the process $Y = X_2 + X_3$ belongs to $\mathcal{X}$ and M_1 is its representing measure. Consequently, $\mathrm{distr}\, X_1(t,\omega) = \mathrm{distr}\, Y(t,\omega)$ for all $t \ge 0$, which yields the equality $\mathrm{distr}[X_1, f] = \mathrm{distr}[Y, f]$ for every $f \in \mathcal{F}$. Moreover, $[X_2, f]^{-p} \le [Y, f]^{-p}$, which yields the assertion of the lemma.

We are now in a position to prove the following rather unexpected result.

THEOREM 3.1. *Let $p > 0$, $s = p/(1+p)$, $X \in \mathcal{X}$ and*

$$f \in \mathcal{F}_s. \tag{3.1}$$

If $\mathrm{distr}[X, f] \in \mathcal{A}_p$, *then the process X is deterministic.*

Proof. Suppose the contrary. Then the representing measure M for X is not concentrated at the origin. Thus $M([a,\infty)) > 0$ for a certain $a > 0$. Setting $N(E) = M(E \cap [a,\infty))$ for Borel subsets E of $\mathbb{R}_+$ we get the representing measure for a process Y from $\mathcal{X}$. By Lemma 3.2, $\mathrm{distr}[Y, f] \in \mathcal{A}_p$. Observe that Y is a compound Poisson process. Assume that $Y \in \mathrm{Poiss}(q, Q)$ and put $\mu_0 = \mathrm{distr}[Y, f]$. By Lemma 3.1, we have

$$\int_0^\infty \widehat{\mu}_0(z)\, e^{cz^s}\, dz < \infty$$

for some $c > 0$. Using formula (2.2) we conclude that

$$\int_0^\infty \Phi(y,z)\, e^{cz^s}\, dz < \infty \tag{3.2}$$

for Q^∞-almost all $y \in \mathbb{R}_+^\infty$. Denote by B_1 the subset of $\mathbb{R}_+^\infty$ consisting of all y fulfilling condition (3.2). By the strong law of large numbers we infer that

$$\lim_{n\to\infty} \frac{1}{n}(y_1 + \ldots + y_n) = \int_0^\infty x\, Q(dx) \tag{3.3}$$

for Q^∞-almost all $y = (y_1, y_2, \ldots) \in \mathbb{R}_+^\infty$. Of course $0 < \int_0^\infty x\, Q(dx) \le \infty$. Denote by B_2 the subset of $\mathbb{R}_+^\infty$ consisting of all y fulfilling condition (3.3).

Let $u = (u_1, u_2, \ldots) \in B_1 \cap B_2$. Given $0 < b < \int_0^\infty x\, Q(dx)$ we can find an index $k_0 > 1$ such that

$$\frac{1}{k}(u_1 + \ldots + u_k) > b \quad \text{for } k \geq k_0. \tag{3.4}$$

Since $f \in \mathcal{F}_s$ we may also assume that

$$s^{-1} q^{-s} \sum_{k=k_0}^{\infty} f^s(kb) < c/2. \tag{3.5}$$

Put

$$\Psi_1(z) = \bigcap_{k=k_0}^{\infty} (1 + q^{-1} f(kb) z)^{-1},$$

$$\Psi_2(z) = (1 + q^{-1} f(0) z)^{-1} \bigcap_{k=1}^{k_0-1} (1 + q^{-1} f(u_1 + \ldots + u_k) z)^{-1}.$$

Since the function f is decreasing we conclude, by (3.4), that $f(u_1 + \ldots \ldots + u_k) \leq f(kb)$ for $k \geq k_0$. Consequently, $\Phi(u, z) \geq \Psi_1(z)\Psi_2(z)$. Applying the inequality $1 + y \leq \exp s^{-1} y^s$ $(y \geq 0,\ 0 < s < 1)$ we get, by (3.5),

$$\Psi_1(z) \geq \exp\Big(-s^{-1} q^{-s} z^s \sum_{k=k_0}^{\infty} f^s(kb)\Big) \geq \exp(-cz^s/2).$$

This yields, by (3.2), $\int_0^\infty e^{cz^s/2} \Psi_2(z) dz < \infty$, which is a contradiction. The theorem is thus proved.

We note that condition (3.1) of the above theorem is essential. In fact, taking the process Y_1 from Example 2.2 and the function f_s $(0 < s < 1)$ defined by formula (2.6) we infer that $f_s \in \mathcal{F}_r$ for $r > s$ and $f_s \notin \mathcal{F}_s$. Using (2.7) and Lemma 3.1 we conclude that $\mathrm{distr}[Y_1, f_s] \in \mathcal{A}_p$ where $s = p/(1+p)$.

REFERENCES

[1] C. Berg and G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Springer, Berlin, 1975.

[2] I. I. Gikhman and A. V. Skorokhod, *Theory of Random Processes*, Vol. II, Nauka, Moscow, 1973 (in Russian).

[3] K. Urbanik, *Functionals on transient stochastic processes with independent increments*, Studia Math. 103 (1992), 299–315.

[4] —, *Stability of stochastic processes defined by integral functionals*, ibid. 103 (1992), 225–238.

[5] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, J. London Math. Soc. 10 (1935), 286–293.

[6] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc. 46 (1940), 389–408.

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