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EXTREME POINTS OF THE CLOSED UNIT BALL IN C^*-ALGEBRAS

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In this short note we give a short and elementary proof of a characterization of those extreme points of the closed unit ball in C^*-algebras which are unitary. The result was originally proved by G. K. Pedersen using some methods from the theory of approximation by invertible elements.

Let $\mathcal{A}$ be a C^*-algebra with unit 1, i.e. a unital Banach algebra $\mathcal{A}$ with involution * fulfilling the so called C^*-condition $\|a^*a\| = \|a\|^2$. R. V. Kadison proved the following characterization of the extreme points of the closed unit ball:

THEOREM ([Kad, Pe1]). *Let $\mathcal{A}$ be a unital C^*-algebra. Then the extreme points of the closed unit ball of $\mathcal{A}$ are precisely those elements v of $\mathcal{A}$ for which $(1-v^*v)\mathcal{A}(1-vv^*) = \{0\}$. In this case v is a partial isometry*, i.e. *v^*v and vv^* are projections.*

Clearly, all unitary elements are extreme points (where unitary means $u^*u = uu^* = 1$). We use the above result to give a short proof of the characterization of those extreme points which are unitary. This characterization is due to G. K. Pedersen who uses techniques from the theory of approximation by invertible and unitary elements.

PROPOSITION ([Pe2]). *Let $\mathcal{A}$ be a (unital) C^*-algebra, and suppose v is an extreme point of the closed unit ball. Then v is unitary if and only if* $\operatorname{dist}(v, \mathcal{A}^{-1}) < 1$. *(Here, $\mathcal{A}^{-1}$ denotes the set of all invertible elements of $\mathcal{A}$.)*

Proof. The implication (1)⇒(2) is trivial since u is invertible. For (2)⇒(1) suppose that $\operatorname{dist}(v, \mathcal{A}^{-1}) < 1$, i.e. there exists an invertible element $a \in \mathcal{A}$ with $\|a - v\| < 1$. Then

$$\begin{aligned}\|v^*va^{-1}(1-vv^*)\| &= \|v^*va^{-1}(1-vv^*) - v^*(1-vv^*)\| \\ &\le \|v^*\| \cdot \|v-a\| \cdot \|a^{-1}(1-vv^*)\|.\end{aligned}$$

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If $vv^* \neq 1$ then this implies $\|v^*va^{-1}(1-vv^*)\| < \|a^{-1}(1-vv^*)\|$, i.e. $v^*va^{-1}(1-vv^*) \neq a^{-1}(1-vv^*)$. Hence, $0 \neq (1-v^*v)a^{-1}(1-vv^*) \in (1-v^*v)\mathcal{A}(1-vv^*)$, contradicting $v \in \mathrm{ex}(\mathcal{A})_1$. Thus, we get $vv^* = 1$, and similarly $v^*v = 1$, i.e. v is a unitary. ■

As an immediate corollary one gets the following:

COROLLARY. *Let $\mathcal{A}$ be a unital C^*-algebra. If the invertible elements are dense in $\mathcal{A}$ then the extreme points of the closed unit ball are precisely the unitary elements.*

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