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HADAMARD'S MULTIPLICATION THEOREM— RECENT DEVELOPMENTS

BY

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Dedicated to Professor Czesław Ryll-Nardzewski

Introduction. This paper is an extension of a talk given at the conference at Wierzba on the occasion of the 70th anniversary of Prof. Ryll-Nardzewski. It surveys some new developments concerning the Hadamard product of holomorphic functions of one complex variable. Throughout the paper we assume that G_1 and G_2 are domains in $\mathbb{C}$ containing 0. Let $f : G_1 \to \mathbb{C}$ and $g : G_2 \to \mathbb{C}$ be holomorphic functions. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are the Taylor series then the *Hadamard product* of f and g is defined by $f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. In 1899 Jacques Hadamard published his famous multiplication theorem stating that $f * g$ extends to a holomorphic function on a domain G_3 which is the complement of the set $G_1^c \cdot G_2^c$. A rigorous proof of this general result (without the assumption in [12] that G_1, G_2 are simply connected) was recently given by J. Müller, whereas in [3] and [13] only starlike domains have been considered. The most general approach to Hadamard's multiplication theorem leads to the definition of a coefficient multiplier given in [10, 17]: Let G_1, G_2 be domains containing 0. A power series $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is a *coefficient multiplier* if $g * f \in H(G_2)$ for all $f \in H(G_1)$, i.e., $T_g(f) = g * f$ defines a linear mapping $T_g : H(G_1) \to H(G_2)$. In the first section we give a proof of a result stated in [17], namely that a power series $g(u) := \sum_{n=0}^{\infty} b_n u^n$ is a coefficient multiplier if and only if for every $w \in G_1^c$ the power series g has a holomorphic extension to the domain $\frac{1}{w} G_2$. For the case $G := G_1 = G_2$ one infers that $H(G)$ is always a module (with respect to Hadamard multiplication) over the algebra $H(\widehat{G})$, where $\widehat{G}$ is given by $\bigcup_{w \in G^c} \frac{1}{w} G$. A domain G of $\mathbb{C}$ containing 0 is called *admissible* if for all $f, g \in H(G)$ the Hadamard product $f * g$ extends to a (unique) function of $H(G)$, i.e., $H(G)$ is a com-

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mutative algebra. It follows that G is admissible iff G^c is a multiplicative semigroup.

The results of Section 1 lead to a natural embedding of $H(\widehat{G})$ into the multiplier algebra. In Section 2 it is shown that for a simply connected domain this embedding is an isomorphism if and only if G is α-starlike. Recall that a domain G is *α-starlike* (with respect to 0 and a given real number α) if $\{t^{1+i\alpha} \cdot g : t \in [0,1],\ g \in G\} \subset G$. This characterization is related to a result of Arakelyan stating that G is α-starlike if and only if G is a domain of efficient summability.

In the third section we give a survey of the algebraic properties of $H(G)$ which have been investigated by a number of authors [1, 6, 8, 9, 18, 22, 27]. The fourth and last section is devoted to the question when two algebras $H(G_1)$ and $H(G_2)$ or their multiplier algebras are algebraically isomorphic. Surprisingly, this is indeed the case if and only if G_1 is equal to G_2.

Let us introduce some notations. The set of all multipliers $T : H(G_1) \to H(G_2)$ is denoted by $M(H(G_1), H(G_2))$. In the case of $G = G_1 = G_2$ we just write $M(H(G))$. The interior of a set K is denoted by $\operatorname{int}(K)$. The distance of a point z from G^c is given by $\operatorname{dist}(z, G^c) := \inf\{|z-w| : w \in G^c\}$. If γ is a path its trace is denoted by $\operatorname{sp}(\gamma) := \{\gamma(t) : t \in [a,b]\}$. If Γ is a cycle the index $n(\Gamma, z)$ is defined by

$$n(\Gamma, z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\xi - z}\, d\xi.$$

By $\mathbb{D}$ we denote the open unit ball. More generally, $\mathbb{D}_r$ denotes the open ball with center 0 and radius $r > 0$.

1. Hadamard's multiplication theorem. Let G be a domain containing 0. Then $H(G)$ is a Fréchet space, i.e. a completely metrizable locally convex vector space where the (semi)norms are given by $|f|_K := \sup_{z\in K} |f(z)|$ for an arbitrary compact subset K of G. The (continuous) functionals $\delta_n : H(G) \to \mathbb{C}$ defined by $\delta_n(f) := a_n$ (where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ locally) are called the *Dirac functionals*. Coefficient multipliers can be characterized in the following way (see [24]).

1.1. Theorem. *Let $T : H(G_1) \to H(G_2)$ be a linear operator. Then the following statements are equivalent*:

(a) *T is a coefficient multiplier.*

(b) *$\delta_n \circ T = b_n \delta_n$ for all $n \in \mathbb{N}_0$ and suitable $b_n \in \mathbb{C}$.*

(c) *T is continuous and $T(f * \exp) = T(f) * \exp$ for all $f \in H(G_1)$.*

(d) *There exist $b_n \in \mathbb{C}$, $n \in \mathbb{N}_0$, such that $T(f)(z) = \sum_{n=0}^{\infty} b_n a_n z^n$ in a neighborhood of zero for all $f \in H(G_1)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$.*

(e) *$T(f * z^n) = T(f) * z^n$ for all $f \in H(G_1)$ and $n \in \mathbb{N}_0$.*

We are going to prove a generalized version of Hadamard's multiplication theorem which was already stated in [17] (for $D := G_1^{\mathrm{c}}$ under the redundant assumption $1 \notin G_1$). It seems that the proof in [17] has a serious gap depending on an incorrect use of the monodromy theorem.

1.2. THEOREM. *Let D be a dense subset of G_1^{c}. A power series $\sum_{n=0}^{\infty} b_n u^n$ induces a coefficient multiplier if and only if the function $g(u) := \sum_{n=0}^{\infty} b_n u^n$ possesses a holomorphic extension on $\frac{1}{w} G_2$ for all $w \in D$.*

Proof. Suppose that $T(f) := g * f$ defines a linear map between $H(G_1)$ and $H(G_2)$. Since $\gamma_w(z) := \frac{w}{w-z} \in H(G_1)$ we can define $g_w(u) := T(\gamma_w)(wu)$ for $u \in w^{-1} G_2$, which is a domain containing zero. By Theorem 1.1(a)⇒(d) we have $T(\gamma_w)(z) = \sum_{n=0}^{\infty} b_n (\frac{z}{w})^n$, i.e., $g_w(u) = T(\gamma_w)(wu) = \sum_{n=0}^{\infty} b_n u^n$. It follows that each g_w, $w \in G_1^{\mathrm{c}}$, is a holomorphic extension of $\sum_{n=0}^{\infty} b_n u^n$ on the domain $\frac{1}{w} G_2$.

For the converse let g_w be the holomorphic extension on $\frac{1}{w} G_2$ ($w \in G_1^{\mathrm{c}}$) of $g(u) = \sum_{n=0}^{\infty} b_n u^n$. Roughly speaking, we want to define a linear map $T : H(G_1) \to H(G_2)$ by the Parseval integral representation

$$T(f)(z) = \frac{1}{2\pi i} \int_{\Gamma} g\left(\frac{z}{t}\right) f(t) \, \frac{dt}{t}, \tag{1}$$

where Γ is a cycle in G_1 very near to G_1^{c} and z varies in a given compact subset K of G_2. The main obstacle is the fact that we do not have a function g, i.e., that $g(\frac{z}{t}) := g_w(\frac{z}{t})$ is not uniquely defined. This difficulty is solved by decomposing Γ into small line segments Γ_i which are contained in a suitable $\frac{1}{w_i} G_2$. We proceed to the proof: Let $\widetilde{w}_0 \in G_1^{\mathrm{c}}$ be an element such that $|\widetilde{w}_0| = \inf\{|w| : w \in G_1^{\mathrm{c}}\}$ and let $w_0 \in D$ with $|w_0/\widetilde{w}_0| < 2$. For $\delta > 0$ we define $B_\delta := \{z \in \mathbb{C} : |z| < \delta\}$. Clearly, there exists $1 > \delta_2 > 0$ such that $B_{\delta_2} \subset \frac{1}{w_0} G_2$, and there exists $\frac{1}{2} > \delta_1 > 0$ such that $B_{2\delta_1} \subset G_1$. Let K be a compact connected subset of G_2 containing 0 as an interior point and let $r > 1$ be so large that $\frac{|z|}{r-1} < \delta_2 < 1$ for all $z \in K$, in particular $K \subset G_2 \cap B_r$. By continuity of the map $(\lambda, z) \to \lambda z$ there exists $\varepsilon > 0$ such that $\lambda \cdot z \in G_2 \cap B_r$ for all $z \in K$ and $\lambda \in B_\varepsilon(1) := \{z \in \mathbb{C} : |z - 1| < \varepsilon\}$. We now construct a cycle Γ "very near" to G_1^{c}. Choose $0 < \eta < 1$ so small that $\eta < \varepsilon \cdot \delta_1$ and $B_{\delta_1} \subset L := \{y \in G_1 \cap B_r : \operatorname{dist}(y, (G_1 \cap B_r)^{\mathrm{c}}) \geq \frac{\eta}{3}\}$. By Satz 3.3 in [11, p. 112] there exists a cycle Γ in $(G_1 \cap B_r) \setminus L$ such that $n(\Gamma, y) = 1$ for all $y \in L$ and $n(\Gamma, y) = 0$ for all $y \in (G_1 \cap B_r)^{\mathrm{c}}$. Clearly, we have $|t| \geq \delta_1$ for all $t \in \operatorname{sp}(\Gamma)$. Moreover, Γ is composed by finitely many polygons consisting of horizontal and vertical line segments Γ_i, which will be numbered by $i = 1, \ldots, n$. We can assume that the length of Γ_i is smaller than $\eta/3$. Moreover, we have $\operatorname{dist}(t, (G_1 \cap B_r)^{\mathrm{c}}) < \eta/3$ for all $t \in \operatorname{sp}(\Gamma)$ by the definition of the compact set L. We claim that for each $i = 1, \ldots, n$ there exists $w_i \in D$ with $\operatorname{sp}(\Gamma_i) \subset \frac{1}{w_i} G_2$. In the first case suppose that, for given i, there exists $t_i \in \mathrm{sp}(\Gamma_i)$ and $\widetilde{w}_i \in G_1^{\mathrm{c}}$ with $|t_i - \widetilde{w}_i| < \eta/3$. Since D is dense there exists $w_i \in D$ with $|w_i - \widetilde{w}_i| < \eta/3$. Then $|t - w_i| < \eta$ for all $t \in \mathrm{sp}(\Gamma_i)$ since Γ_i has length at most $\frac{\eta}{3}$. It follows that $\frac{w_i}{t} \in B_\varepsilon(1)$ for all $t \in \mathrm{sp}(\Gamma_i)$ since $|\frac{w_i}{t} - 1| = \frac{1}{t} \cdot |w_i - t| \le \frac{\eta}{\delta_1} < \varepsilon$. Thus we have proved that $\frac{z}{t} = \frac{w_i}{t} \cdot z \cdot \frac{1}{w_i} \in \frac{1}{w_i} G_2$ for all $t \in \mathrm{sp}(\Gamma_i)$ and for all $z \in K$. In the second case we know that there exist $t_0 \in \mathrm{sp}(\Gamma_i)$ and $w \in B_r^{\mathrm{c}}$ with $|t_0 - w| < \eta/3$. Hence $|t| \ge r - |t_0 - w| - |t - t_0| \ge r - \frac{2\eta}{3} \ge r - 1$ for all $t \in \mathrm{sp}(\Gamma_i)$. It follows that $|\frac{z}{t}| \le \frac{|z|}{r-1} < \delta_2$. In this case we have $\frac{z}{t} \in B_{\delta_2} \subset \frac{1}{w_0} G_2$. For each $i = 1, \ldots, n$ we define

$$T_i(f)(z) := \frac{1}{2\pi i} \int\limits_{\Gamma_i} g_{w_i}\left(\frac{z}{t}\right) f(t)\, \frac{dt}{t}, \tag{2}$$

which is well-defined since $\mathrm{sp}(\Gamma_i) \subset \frac{1}{w_i} G_2$ and g_{w_i} is a holomorphic function on $\frac{1}{w_i} G_2$. It follows that $T_i(f)$ is holomorphic at each point of the interior of K. Thus $T(f) := \sum_{i=1}^n T_i(f)$ is holomorphic in the interior of K. Now we compute the power series of $T(f)$ at $z = 0$: Since $0 \in \frac{1}{w_i} G_2$ for all $i = 1, \ldots, n$ there exists $\delta > 0$ with $B_\delta \subset \frac{1}{w_i} G_2$ for all $i = 1, \ldots, n$. Choose $\varepsilon_1 > 0$ so small that $|\frac{z}{t}| < \delta$ for all $t \in \mathrm{sp}(\Gamma_i)$, $i = 1, \ldots, n$ and $|z| < \varepsilon_1$. Then $g_{w_i}(\frac{z}{t})$ is given by the Taylor expansion and we obtain

$$T(f)(z) = \sum_{i=1}^{n} \frac{1}{2\pi i} \int\limits_{\Gamma_i} g_{w_i}\left(\frac{z}{t}\right) f(t) \frac{dt}{t} = \sum_{k=0}^{\infty} b_k z^k \frac{1}{2\pi i} \int\limits_{\Gamma} \left(\frac{1}{t}\right)^{k+1} f(t)\, dt. \tag{3}$$

Furthermore, Γ is a cycle in G_1 with $n(\Gamma, y) = 0$ for all $y \in G_1^{\mathrm{c}}$ and $f : G_1 \to \mathbb{C}$ is holomorphic. Cauchy's Theorem and (3) imply that $T(f)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$. It follows that $T(f)$ is an analytic continuation of $f * g$ on the component containing 0 (of the interior of K). Passing to a sequence of compact connected subsets K_n containing 0 as an interior point satisfying $G_2 = \bigcup_{n=1}^{\infty} \mathrm{int}(K_n)$ we infer that $T(f)$ defines a function on G_2. ■

In the following we discuss the consequences of Theorem 1.2: Let G_1, G_2 be domains containing 0. As already pointed out in [17] every function g holomorphic on the set

$$\widehat{G_1 G_2} := \{z \in \mathbb{C} : \exists w \in G_1^{\mathrm{c}} \text{ with } zw \in G_2\} = \bigcup_{w \in G_1^{\mathrm{c}}} w^{-1} G_2 \tag{4}$$

induces a multiplier (since g restricted to $\frac{1}{w} G_2$ is a holomorphic extension) but the converse does not hold; cf. the example in [17] or consider Theorem 2.1 below for a simply connected domain which is not α-starlike. It is easy to see that $\widehat{G_1 G_2}$ is a domain since G_2 is connected and each $\frac{1}{w} G_2$ contains 0.

1.3. THEOREM. *The map* $L : H(\widehat{G_1 G_2}) \to M(H(G_1), H(G_2))$ *defined by* $L(g)(f) = g * f$ *is a linear monomorphism. If* $\frac{1}{w_1} G_2 \cap \frac{1}{w_2} G_2$ *is connected for all* $w_1, w_2 \in G_1^c$ *then* L *is an isomorphism.*

Proof. Let $g \in H(\widehat{G_1 G_2})$ and $f \in H(G_1)$. Then g is holomorphic on each set $\frac{1}{w} G_2$ with $w_1 \in G_1^c$. By Theorem 1.2, $g * f$ is a holomorphic function on G_2. Clearly, L is linear and injective: $L(g) = 0$ implies $g * z^n = 0$ for all $n \in \mathbb{N}_0$ and therefore $g = 0$. For the surjectivity let T be a multiplier and $g(u) := \sum_{n=0}^{\infty} b_n u^n$ the induced power series. For each $w \in G_1^c$ there exists a holomorphic extension g_w on $\frac{1}{w} G$. Then $g(u) := g_w(u)$, $w \in G_1^c$, $u \in w^{-1} G_2$, is well-defined by the identity theorem and by the fact that $\frac{1}{w_1} G_2 \cap \frac{1}{w_2} G_2$ is connected. Clearly, $L(g) * f = T(f)$ for all $f \in H(G_1)$. ∎

Theorem 1.3 shows that there exists a bilinear map $* : H(\widehat{G_1 G_2}) \times H(G_1) \to H(G_2)$, $(f, g) \mapsto f * g$, for given domains G_1, G_2. Since the bilinear map is separately continuous it is continuous by Corollary 1 in [25, p. 88]. Often one wants to define a bilinear map $* : H(G_1) \times H(G_2) \to H(G_3)$ for given domains G_1, G_2 and a suitable domain G_3. Clearly, this is possible if $G_1 \supset \widehat{G_2 G_3}$. This in turn is equivalent to $G_1^c \subset \frac{1}{w} G_3^c$ for all $w \in G_2^c$. This is equivalent to the statement that $u \in G_1^c$ and $w \in G_2^c$ imply that $uw \in G_3^c$. Consequently, we have proved the sufficiency part of the following result, which is probably the most elegant form of Hadamard's multiplication theorem.

1.4. THEOREM. *There exists an extension of the Hadamard product as a bilinear map* $* : H(G_1) \times H(G_2) \to H(G_3)$ *iff* $u \in G_1^c$ *and* $w \in G_2^c$ *imply that* $uw \in G_3^c$.

Proof. For the necessity consider $f(z) = \frac{u}{u-z}$ and $g(z) = \frac{w}{w-z}$ and observe that $f * g(z) = \frac{uw}{uw-z}$. ∎

Assume now that $G = G_1 = G_2$. Instead of $\widehat{GG}$ we write $\widehat{G}$. It is an important observation due to Arakelyan (Lemma 2.1 in [2]) that $\widehat{G}^c$ is always a semigroup and therefore $H(\widehat{G})$ is an algebra. By Theorem 1.3, $H(G)$ is always a module over the ring (or algebra) $H(\widehat{G})$.

2. Approximate identities and summability methods. Let G be a domain in $\mathbb{C}$ with $0 \in G$. Then G is called a *domain of efficient summability* if there exists an infinite set I having a limit point δ_0 such that for each $\delta \in I$ there exists a sequence of complex numbers $C = (c_n(\delta))_{n \in \mathbb{N}}$ with the following two properties:

(i) The function $C_\delta(z) := \sum_{n=0}^{\infty} c_n(\delta) z^n$ converges for all $z \in \mathbb{C}$ with $|z| < R_G / r_G$, where $R_G := \sup\{|z| : z \in G\}$ and $r_G := \inf\{|w| : w \in G^c\}$.

(It follows that $C_\delta * f$ has convergence radius at least R_G; hence $C_\delta * f \in H(G)$ for all $f \in H(G)$.)

(ii) For $\delta \to \delta_0$ the function $C_\delta * f$ converges to f in the topology of compact convergence in G.

We remind that $H(G)$ is a module over the algebra $A := H(\widehat{G})$. A net $(e_j)_{j\in J}$ in A is called an *approximate identity* if $(e_j * f)_j$ converges to f for each $f \in H(G)$. The equivalence of (b), (d) and (e) in the following result is due to Arakelyan. Roughly speaking, it says that *only* α-starlike domains are domains of efficient summability. It seems that the purely topological characterizations (f) and (g) are unknown in the literature.

2.1. Theorem. *Let G be a domain containing* 0. *Then the following statements are equivalent*:

(a) *$H(G)$ possesses an approximate identity $(e_n)_{n\in\mathbb{N}}$ consisting of polynomials.*

(b) *G is a domain of efficient summability.*

(c) *$L : H(\widehat{G}) \to M(H(G))$ is an isomorphism and G is simply connected.*

(d) *$\widehat{G}$ is simply connected.*

(e) *G is α-starlike.*

(f) *There exists a path $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = 0$, $\gamma(1) = 1$ and such that $\gamma(t) \cdot g \in G$ for all $t \in [0,1]$ and $g \in G$.*

(g) *G is simply connected and $\frac{1}{w_1}G \cap \frac{1}{w_2}G$ is connected for all $w_1, w_2 \in G^{\mathrm{c}}$.*

(h) *There exists a simply connected domain $\widetilde{G}$ with $\widehat{G} \subset \widetilde{G}$ and $1 \in \widetilde{G}^{\mathrm{c}}$.*

Proof. (a)⇒(b) is obvious. For (b)⇒(c) suppose that G is not simply connected. Then there exists a non-empty compact component K in G^{c}. By [19, p. 257] there exists a closed path Γ in G with $n(\Gamma, z) = 1$ for all $z \in K$. For $w_0 \in K$ the function $\gamma_{w_0} * e_j$ has convergence radius at least R_G and therefore $\int_\Gamma \gamma_{w_0} * e_j \, d\xi = 0$. On the other hand, the last integrals converge to $\int_\Gamma \gamma_{w_0} \, d\xi \neq 0$ since $\gamma_{w_0} * e_j$ converge compactly to γ_{w_0} in G and Γ is contained in G, a contradiction. We now show that L is an isomorphism. Let T be a multiplier on $H(G)$ and $g(u) = \sum_{n=0}^\infty b_n u^n$ be the associated power series (cf. Theorem 1.2). Note that the convergence radius of g is at least 1. It suffices to show that for $u \in \widehat{G}$ of the form $u = \frac{g_1}{w_1} = \frac{g_2}{w_2}$ the value $g(u)$ is identical, i.e., that $T(\gamma_{w_1})(g_1) = T(\gamma_{w_2})(g_2)$. By assumption $\gamma_{w_i} * e_j$ converges to γ_{w_i} and therefore $T(\gamma_{w_i} * e_j)$ converges to $T(\gamma_{w_i})$. Hence it suffices to show that $T(\gamma_{w_1} * e_j)(g_1) = T(\gamma_{w_2} * e_j)(g_2)$. Let $\sum_{l=0}^\infty c_l(j) z^l$ be the power series of $T(e_j)$, which converges for all $|z| \le R_G/r_G$ since

$T(e_j) = g * e_j$. Then

$$T(\gamma_{w_1} * e_j)(g_1) = \gamma_{w_1} * T(e_j)(g_1) = \sum_{l=0}^{\infty} \frac{c_l(j)}{w_1^l} g_1^l. \tag{5}$$

Since $g_1/w_1 = g_2/w_2$ the last term equals $\gamma_{w_2} * T(e_j)(g_2) = T(\gamma_{w_2} * e_j)(g_2)$.

For (c)⇒(d) suppose that $\widehat{G}$ is not simply connected. By Lemma 1 in [9], $1 \in \widehat{G}^c$ is isolated. Hence $q_2(z) = 1/(1-z)^2$ is not invertible in $H(\widehat{G})$, since the formal inverse $f(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n$ is $\log(1/(1-z)) \notin H(\widehat{G})$. On the other hand, $f(z)$ defines a multiplier T on $H(G)$ by Theorem 1.2 and clearly it is the inverse of $L(q_2)$, a contradiction.

For (d)⇒(a) let $(p_n)_n$ be a sequence of polynomials converging to $\gamma(z) = 1/(1-z)$ (the Runge approximation theorem). In Section 1 we have seen that there exists a continuous Hadamard product $* : H(\widehat{G}) \times H(G) \to H(G)$. Hence $f * p_n$ converges to $f * \gamma = f$ in the domain G.

Hence (a) to (d) are equivalent. The implication (d)⇒(e) is due to Arakelyan: if $\widehat{G}$ is simply connected the point $1 \in \widehat{G}^c$ cannot be isolated. By Lemma 2.2 in [2] there exists $\alpha \in \mathbb{R}$ such that $L_\alpha^+ := \{t^{1+i\alpha} : t \in [1,\infty)\} \subset \widehat{G}^c$. It follows that $L_\alpha^+ \subset \frac{1}{w} G^c$ for all $w \in G^c$. It is easy to see that G is α-starlike. The implications (e)⇒(f)⇒(g) are easy and left to the reader. For (g)⇒(c) use Theorem 1.3. Further, (d)⇒(h) is trivial. For (h)⇒(a) let p_n be polynomials approximating $\gamma(z)$ on $\widetilde{G}$. Clearly, p_n approximate γ on $\widehat{G} \subset \widetilde{G}$ as well. Now proceed as in (d)⇒(a). ∎

The following result is a direct consequence of Theorem 2.1(d)⇒(e) since $\widehat{G}$ is homeomorphic to the simply connected domain G. The converse of Theorem 2.2 is not true as simple examples show.

2.2. Theorem. *Let G be a simply connected domain. If $\widehat{G}$ is equal to some $\frac{1}{w} G$ with $w \in G^c$ then G is α-starlike.*

3. The algebra $H(G)$ with the Hadamard product. In 1992 R. Brück and J. Müller started the investigation of the algebra $H(G)$ endowed with the Hadamard product. A detailed discussion for the special case of the open unit disk was already given by R. Brooks in [6, 7]. Most of the presented results can be found in [22]. Recall that a topological algebra is a B_0*-algebra* if the topology is locally convex and completely metrizable.

3.1. Theorem. *Let G be an admissible domain. Then $H(G)$ is a semisimple B_0-algebra and each multiplicative functional on $H(G)$ is continuous. It is a Fréchet algebra iff it is non-unital iff $1 \in G$.*

3.2. Theorem. *Let G be an admissible domain with $1 \notin G$. Then the set of non-invertible elements is dense in $H(G)$.*

An admissible domain G always contains the open unit disk. There are three different types of admissible domains. In the first case the number 1 is in the domain: Then G must contain the closed unit disk (otherwise $G^c \cap \{z \in \mathbb{C} : |z| = 1\}$ is either a finite subgroup or a dense subset of the unit circle, contradicting the assumption $1 \in G$). It can be shown that $H(G)$ is a so-called Q-algebra with respect to the norm given by $\|f\|_{\mathbb{N}} := \sup_{n\in\mathbb{N}_0} |a_n|$. In the second case 1 is in G^c. Then $H(G)$ possesses a unit element given by $\gamma(z) := \frac{1}{1-z}$ and we have to consider two completely different cases: first suppose that 1 is not isolated in G^c. By Lemma 1 in [9], G is α-starlike, in particular simply connected. This property is the key to very simple proofs for characterizing the closed maximal ideals of $H(G)$; cf. [9] or [22]. In particular, the multiplicative functionals are of the form δ_n for some $n \in \mathbb{N}_0$. Very interesting results and open problems concerning closed principal and finitely generated ideals in $H(G)$ can be found in [8, 9, 27].

3.3. Theorem. *Let G be an admissible simply connected domain with $1 \in G^c$ and let M be an ideal of $H(G)$. Then the following statements are equivalent*:

(a) *M is a prime ideal which is contained in a closed ideal.*
(b) *M is a closed prime ideal.*
(c) *M is a closed maximal ideal.*
(d) *There exists $n \in \mathbb{N}_0$ with $M = \ker(\delta_n)$.*

If M is a closed ideal and $B := \{n \in \mathbb{N}_0 : \delta_n(a) = 0$ for all $a \in M\}$ then $M = \bigcap_{n\in B} \ker(\delta_n) =: M_B$.

It remains to consider the case where 1 is an isolated point in G^c. This case is more involved and completely different from the previous one. First, it is clear that $A := G^c \cap \{z \in \mathbb{C} : |z| = 1\}$ is a finite subgroup of the unit circle and therefore A is the set of all kth roots of unity for a suitable $k \in \mathbb{N}$. Then $\widetilde{G} := G \cup A$ is an admissible domain containing the closed unit disk. Identifying $f \in H(\widetilde{G})$ with $f|G$ we can see $H(\widetilde{G})$ as a subalgebra of $H(G)$. By separating the singularities one obtains a topological linear isomorphism

$$T : H(G) \to H_k \oplus H(\widetilde{G}), \qquad Tf = f_1 + f_2 \tag{6}$$

(cf. [9] for details), where H_k denotes the set of all holomorphic functions $f : \widehat{\mathbb{C}} \setminus A \to \mathbb{C}$ with $f(\infty) = 0$ and $\widetilde{G} \supset G$ contains the closed unit disk. Hence the study of $H(G)$ can be reduced to the algebra H_k and the first case where the domain contains the closed unit disk. Moreover, it is easy to see that H_k and the direct sum $\bigoplus_{j=1}^k H_1$ are isomorphic topological vector spaces (see [9]). Thus investigating H_1 is the key to the general case.

Let $M(r, f) := \max_{|z|=r} |f(z)|$ be the maximum modulus of f. An entire function f is said to be of *exponential type* τ if $\limsup_{r\to\infty} \log(M(r, f))/r \leq \tau$. An equivalent definition is that for every $\varepsilon > 0$ and sufficiently large $|z|$ we have $|f(z)| \leq \exp((\tau+\varepsilon)|z|)$. Of special interest are functions of exponential type zero. We just mention the following property: *A function of exponential type zero is either constant or surjective.* (Proof: If f omits the value 0 then f is of the form $f = \exp(\varphi)$ with an entire function φ. Since f is of exponential type zero this leads to $M(r, \mathrm{Re}(\varphi)) = o(r)$. It follows that $\varphi = \text{const.}$) Clearly, f is in the algebra H_1 if and only if there exists an entire function g with $g(1/(1-z)) = f(z)$ and $g(0) = 0$. It is known that the algebra H_1 is topologically and algebraically isomorphic to the algebra E_0 of all entire functions of zero exponential type with pointwise multiplication and a suitable topology. The isomorphism is given by the Theorem of Wigert: for $f \in H_1$ there exists a unique function $\widehat{f} \in E_0$ interpolating the Taylor coefficients of f in the sense that $\widehat{f}(n) = a_n$ for all $n \in \mathbb{N}_0$. As worked out in [9] the multiplicative functionals of H_1 are given by point evaluation, i.e. $f \mapsto \widehat{f}(\alpha)$ for $\alpha \in \mathbb{C}$. In the following we indicate a quite elementary approach which shows that the interpolating function $\widehat{f} \in E_0$ is just the Gelfand transform of $f \in H_1$.

An important observation is the fact that the algebra H_1 is generated by the element $q_2 := (1-z)^{-2}$; cf. formula (7) below, where $q_n(z) := (1-z)^{-n} = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} z^k$ for $n \in \mathbb{N}$. It follows that a continuous multiplicative functional δ is determined by the value $\alpha := \delta(q_2)$ (note that $\delta(q_1) = \delta(\gamma) = 1$). For later reasons this multiplicative functional will be denoted by $\delta_{\alpha-1}$. An elementary calculation yields the equality

$$q_n = \frac{1}{n-1}\big[q_2 * q_{n-1} + (n-2)q_{n-1}\big] = \frac{1}{n-1}\big[q_2 - q_1\big] * q_{n-1} + q_{n-1} \tag{7}$$

for all $n \geq 2$. More generally, one can show that $p_\alpha := q_2 - \alpha\gamma$ is a generating element for each $\alpha \in \mathbb{C}$. The binomial coefficients are defined by

$$\binom{\beta}{n} := \beta(\beta-1)\dots(\beta-(n-1))/n!$$

and $\binom{\beta}{0} := 1$ for $\beta \in \mathbb{C}$. Then

$$\binom{\alpha+n-2}{n-1} = \alpha(\alpha+1)\dots(\alpha+(n-2))/(n-1)!.$$

Every element $f \in H_1$ is of the form $f(z) = \sum_{n=1}^{\infty} a_n q_n(z)$, where $\sum_{n=1}^{\infty} a_n z^n$ is an entire function. Define

$$\delta_\alpha(f) := \sum_{n=1}^{\infty} a_n \binom{\alpha+n-1}{n-1}. \tag{8}$$

This number exists for all $f \in H_1$ and $\alpha \in \mathbb{C}$ since $\left|\binom{\alpha+n-1}{n-1}\right| \leq (|\alpha|+1)^{n-1}$

and therefore $|\delta_\alpha(f)| \leq \sum_{n=1}^{\infty} |a_n|(|\alpha|+1)^{n-1} < \infty$. Clearly, $\delta_\alpha : H_1 \to \mathbb{C}$ defined by formula (8) is a linear functional with $\delta_\alpha(\gamma) = \delta_\alpha(q_1) = 1$. The following theorem can be proved by elementary methods (see [22]). More generally, closed ideals of H_1 have already been characterized in [18].

3.4. Theorem. *Let I be an ideal of H_1 which contains $p_\alpha := q_2 - \alpha\gamma$. Then I is generated by p_α and I is the kernel of the continuous multiplicative functional $\delta_{\alpha-1} : H_1 \to \mathbb{C}$. If ϕ is a multiplicative functional then ϕ is continuous and $\phi = \delta_{\alpha-1}$ for $\alpha := \phi(q_2)$. Hence the multiplicative functionals are exactly the functionals δ_α with $\alpha \in \mathbb{C}$.*

3.5. Corollary. *Let $f \in H_1$ with $f(z) = \sum_{n=1}^{\infty} a_n q_n$. Then the Gelfand transform $\widehat{f}$ defined by*

$$\widehat{f}(\alpha) := \delta_\alpha(f) = \sum_{n=1}^{\infty} a_n \binom{\alpha+n-1}{n-1} \tag{9}$$

is of zero exponential type and $\widehat{f}(n)$ is the nth coefficient of the Taylor expansion of $f(z)$ at $z = 0$.

3.6. Corollary. *An element $f \in H_1$ is invertible if and only if there exists $\lambda \neq 0$ with $f = \lambda\gamma$.*

Proof. "$\Rightarrow$" Suppose that f is not a scalar multiple of γ. By Corollary 3.5, $\widehat{f} : \mathbb{C} \to \mathbb{C}$ is a non-constant function of exponential type zero and therefore surjective. Hence $\widehat{f}(\alpha) = 0$ for some $\alpha \in \mathbb{C}$. So $\delta_\alpha(f) = 0$, a contradiction to the invertibility. The converse is trivial. ∎

Let G be an admissible domain with $1 \in G^{\mathrm{c}}$ and let T be the isomorphism in (6). For each $\zeta \in A_k := \{z \in \mathbb{C} : |z| = 1\} \cap G^{\mathrm{c}}$ there exists a natural continuous algebra homomorphism $T_\zeta : H_k \to H_1$ defined by

$$T_\zeta(f) = T_\zeta\Big(\sum_{j=0}^{k-1} \gamma_j * f_j\Big) := \sum_{j=0}^{k-1} \zeta^j f_j \tag{10}$$

(Lemma 2 in [9]), where $\gamma_j \in H_k$ is defined by $\gamma_j(z) = \gamma(z/\xi^j)$ for each $j = 0, \ldots, k-1$ with $\xi = \exp(2\pi i/k)$ and f is equal to $\sum_{j=0}^{k-1} \gamma_j * f_j$ with $f_j \in H_1$ (Laurent expansion). Using this decomposition of $H(G)$ it is possible to determine the set of all multiplicative functionals on $H(G)$. This leads to an invertibility criterion (proved in [9] for the case $G = D_r \setminus A_k$ with $r > 1$) for an admissible domain G with $1 \in G^{\mathrm{c}}$ isolated.

3.7. Theorem. *Let G be an admissible domain with $1 \in G^{\mathrm{c}}$ isolated. Then for each multiplicative functional $\phi : H(G) \to \mathbb{C}$ either there exists $n \in \mathbb{N}_0$ such that $\phi = \delta_n$, or there exist $\alpha \in \mathbb{C}$ and $\zeta \in A_k$ such that $\phi(f) = \delta_\alpha \circ T_\zeta(f_1)$, where $f = f_1 + f_2 \in H_k \oplus H(\widetilde{G})$.*

3.8. Theorem. *Let G be an admissible domain with $1 \in G^{c}$ isolated. For $f = f_1 + f_2 \in H_k \oplus H(\widetilde{G})$ and $f_2(z) = \sum_{n=0}^{\infty} a_n z^n$ the following statements are equivalent*:

(a) *f is invertible.*

(b) *$\phi(f) \neq 0$ for all (continuous) multiplicative functionals ϕ.*

(c) *$\delta_\alpha \circ T_\zeta(f_1) \neq 0$ for all $\alpha \in \mathbb{C}$ and $\zeta \in A_k$ and $\delta_n(f) \neq 0$ for all $n \in \mathbb{N}_0$.*

(d) *f_1 is invertible in H_k and $\delta_n(f) \neq 0$ for all $n \in \mathbb{N}_0$.*

(e) *There exist $c_0, \ldots, c_{k-1} \in \mathbb{C}$ with $f_1 = \sum_{j=0}^{k-1} c_j \gamma_j$ and $\sum_{j=0}^{k-1} c_j \zeta^j \neq 0$ for all $\zeta \in A_k$ and $a_n \neq -\sum_{j=0}^{k-1} c_j \xi^{-nj}$ for all $n \in \mathbb{N}_0$.*

Unfortunately, there is no simple invertibility criterion for admissible simply connected domains G with $1 \in G^{c}$, e.g. $\mathbb{C}_{-} := \mathbb{C} \setminus [1, \infty)$. An interesting result is the following special case of Theorem 3 of [8]:

3.9. Theorem (Brück and Müller). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{C}_{-})$ with $a_n \neq 0$ for all $n \in \mathbb{N}_0$ and let $H := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$. Then f is invertible iff there exists a function Φ on H of inner exponential type 0 with $\Phi(n) = a_n$ and a region Ω in H asymptotic in H such that Φ has no zero in Ω and $1/\Phi$ is of inner exponential type 0 on Ω.*

It would be interesting to know even in the case $\mathbb{C}_{-}$ whether a holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(G)$ is invertible provided that there exists $\delta > 0$ with $\delta \leq |a_n| \leq 1$ for all $n \in \mathbb{N}_0$. Connected to this question is the open problem posed in [9] whether the maximal ideal space of $H(G)$ is isomorphic to the Stone–Čech compactification of $\mathbb{N}_0$ (which is only proved in [6] for the case of the open unit disk).

We finish this section with some results for the non-unital case. In this case it is natural to consider the multiplier algebra. Theorem 1.1 shows that multipliers and coefficient multipliers are equivalent concepts. Multipliers can be characterized as translation-invariant operators. In contrast to classical results in harmonic analysis, invariance for only *one* non-trivial translation is already sufficient. A related result has been obtained in [17] by J. Müller for a coefficient multiplier $T : H(G_1) \to H(G_2)$.

3.10. Definition. Let G be an admissible domain. For each $w \in G^{c}$ and $f \in H(G)$ define the holomorphic function $\tau_w f$ by $\tau_w f(z) := f\left(\frac{z}{w}\right)$ (note that $\frac{z}{w} \in G$ since otherwise $\frac{z}{w} = b$ for some $b \in G^{c}$, hence $z = wb \in G^{c}$, a contradiction). Note that $\tau_w : H(G) \to H(G)$ is a linear mapping.

3.11. Theorem. *Let $G \neq \mathbb{C}$ be an admissible domain with $1 \in G$ and $T : H(G) \to H(G)$ be a linear continuous mapping. Then the following statements are equivalent*:

(a) *T is a multiplier.*
(b) *$T\tau_w = \tau_w T$ for all $w \in G^c$.*
(c) *$T\tau_w = \tau_w T$ for some $w \in G^c$ with $|w| > 1$.*
(d) *$T(f * \gamma_w) = T(f) * \gamma_w$ for all $f \in H(G)$ and for all $w \in G^c$.*
(e) *$T(f * \gamma_w) = T(f) * \gamma_w$ for all $f \in H(G)$ and for some $w \in G^c$ with $|w| > 1$.*

Recall that an *approximate identity* in a topological commutative algebra A is a net $(e_j)_{j\in J}$ such that $(ae_j)_j$ converges to a for each $a \in A$.

3.12. THEOREM. *Let G be an admissible domain with $1 \in G$. Then $H(G)$ possesses an approximate identity if and only if G is α-starlike.*

4. Permutation of power series and Hadamard isomorphisms. Let G_1, G_2 be domains containing 0. We call a linear map $\Phi : H(G_1) \to H(G_2)$ a *permutation operator* if there exists a permutation $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ such that for each function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H(G_1)$ the function $\Phi(f)$ is locally of the form

$$\Phi(f)(z) = \sum_{n=0}^{\infty} a_n z^{\varphi(n)}. \tag{11}$$

Permutation operators arise naturally in the study of isomorphisms between algebras with Hadamard multiplication (see [23], [20]). It is easy to see that permutation operators are continuous. Note that a permutation operator is always injective by the identity theorem. Mathematical intuition tells us that permutation operators should be very rare. Nonetheless, the following result is surprising:

4.1. THEOREM. *Let $\Phi : H(G_1) \to H(G_2)$ be a surjective permutation map. Then $G_1 = G_2$.*

The *proofs* of the results of this section will appear in [24]. The following theorem and Theorem 4.4 gives a complete description of bijective permutation operators:

4.2. THEOREM. *Let $\Phi : H(G_1) \to H(G_2)$ be a surjective permutation operator. Then there exists an isomorphism $\widehat{\Phi} : M(H(G_1)) \to M(H(G_2))$ which extends Φ.*

The number k_G in the next definition will be a characteristic of the domains:

4.3. DEFINITION. Let G be a domain containing 0. For $k \in \mathbb{N}$ we denote by A_k the set of all kth roots of unity. If there exists a largest natural number $k \in \mathbb{N}$ such that

$$\xi w \in G^c \quad \text{for all } \xi \in A_k,\ w \in G^c \tag{12}$$

then this number is denoted by k_G. Note that for $k = 1$ the condition is always satisfied.

Suppose that the largest number does not exist. Then we can find a sequence $(k_n)_n$ satisfying (12). Let $w_0 \in G^c$ with $|w_0| \leq |w|$ for all $w \in G^c$. Then $\{w_0\xi : \xi \in A_{k_n},\ n \in \mathbb{N}\} \subset G^c$ is dense in the circle of radius $|w_0|$. It follows that G is equal to $\{z \in \mathbb{C} : |z| < |w_0|\}$, i.e. G is an open disk. This special case has already been discussed in [23] and is completely different from the other domains; cf. Theorems 6.1, 6.2 and Theorem 2.6 in [23]. It is not very difficult to see that the number k_G is equal to the cardinality of $M := \{z \in \widehat{G}^c : |z| = 1\}$.

4.4. Theorem. *Let G_1, G_2 be domains containing* 0 *and different from* $\mathbb{D}_r$ *for all* $r > 0$. *Let* $\Psi : M(H(G_1)) \to M(H(G_2))$ *be an isomorphism. Then* $k := k_{G_1} = k_{G_2}$ *and there exist* $n_0 \in \mathbb{N}_0$ *and* $b_0, \ldots, b_{k-1} \in \mathbb{Z}$ *such that* $\psi(kn+j) = kn+b_j$ *for all* $nk+j \geq n_0$ *and for all* $j = 0, \ldots, k-1$, *where the permutation* $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ *is given by* $\Psi(z^n) = z^{\psi(n)}$ *for all* $n \in \mathbb{N}_0$.

For an α-starlike domain G the algebra of all coefficient multipliers of $H(G)$ is isomorphic to an algebra of holomorphic functions on the domain $\widehat{G}$. It is natural to ask whether different domains $\widetilde{G}$ (instead of $\widehat{G}$) may lead to better results (e.g. for more general domains G). This is not possible, as the following *uniqueness result* shows:

4.5. Theorem. *Let* G *be a domain containing* 0. *Suppose that there exists an admissible domain* $\widetilde{G} \subset \mathbb{C}$ *such that* $H(\widetilde{G})$ *is isomorphic to* $M(H(G))$. *Then* $\widetilde{G} = \widehat{G}$ *and the canonical injection* $L : H(\widehat{G}) \to M(H(G))$ *is an isomorphism.*

4.6. Theorem. *Let* G_1, G_2 *be admissible domains different from* $\mathbb{D}_r$ *for all* $r > 0$. *Suppose that* $\Phi : M(H(G_1)) \to M(H(G_2))$ *is an isomorphism. Then* $\widehat{G}_1 = \widehat{G}_2$.

4.7. Corollary. *Let* G_1, G_2 *be admissible domains such that* $H(G_1)$ *and* $H(G_2)$ *are Hadamard-isomorphic. Then* $G_1 = G_2$.

Proof. An isomorphism is a permutation operator. ∎

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