COLLOQUIUM MATHEMATICUM

VOL. 74 1997 NO. 1

ON LOWER SEMICONTINUITY OF MULTIPLE INTEGRALS

BY

AGNIESZKA KAŁAMAJSKA (WARSZAWA)

We give a new short proof of the Morrey–Acerbi–Fusco–Marcellini Theorem on lower semicontinuity of the variational functional $\int_\Omega F(x,u,\nabla u)\,dx$. The proofs are based on arguments from the theory of Young measures.

1. Introduction and statement of results. Let Ω be a bounded open domain in $\mathbb{R}^n$. Define the functional

$$I(u)=\int_\Omega F(x,u,\nabla u)\,dx \qquad \text{for } u\in W^{1,p}(\Omega,\mathbb{R}^m). \tag{1}$$

Such functionals are related to questions of nonlinear elasticity and Skyrme's model for meson fields and have been investigated by many authors (see e.g. [1], [2], [4], [6], [10]–[17], [19], [20], [22], [23]).

We give a short proof of the following theorem due to Morrey, Acerbi, Fusco, and Marcellini (see [22], [1], [19]; the definition of quasiconvexity is given in Section 2).

THEOREM 1.1. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded domain*, $1\le p\le\infty$, *and let $F:\Omega\times\mathbb{R}^m\times\mathbb{R}^m_n\to[0,\infty]$ satisfy*

(i) *$F(x,s,\lambda)$ is a Carathéodory function* (*i.e. measurable in $x\in\Omega$ and continuous in $(s,\lambda)\in\mathbb{R}^m\times\mathbb{R}^m_n$*),

(ii) *there exists a Carathéodory function $E(\cdot,\cdot)$ such that*, *for almost every x and all (s,λ), $|F(x,s,\lambda)|\le E(x,s)g(\lambda)$ if $p=\infty$, for some continuous function g, and $|F(x,s,\lambda)|\le E(x,s)(1+|\lambda|^p)$ if $p<\infty$,*

(iii) *for almost every x and all s, the mapping $\lambda\mapsto F(x,s,\lambda)$ is quasiconvex.*

If $u^j\to u$ in $L^p(\Omega,\mathbb{R}^m)$ and $\nabla u^j\rightharpoonup\nabla u$ in $L^p(\Omega,\mathbb{R}^m_n)$ as $j\to\infty$ ($\nabla u^j\overset{*}{\rightharpoonup}\nabla u$ in $L^\infty(\Omega,\mathbb{R}^m_n)$ if $p=\infty$) then the functional (1) *satisfies*

$$I(u)\le\liminf_{j\to\infty} I(u^j). \tag{2}$$

1991 *Mathematics Subject Classification*: Primary 49J45.

Our proof of Theorem 1.1 is based on the theory of Young measures, and will be obtained as an easy consequence of the following Jensen-type inequalities for Young measures.

THEOREM 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be any bounded domain and $1 \le p \le \infty$. Suppose that $\{\nu_x\}_{x\in\Omega}$ is the Young measure* (see Definition 3.2) *generated by the sequence ∇u^j where $u^j \in W^{1,p}(\Omega, \mathbb{R}^m)$, and $\nabla u^j \rightharpoonup \nabla u$ in $L^p(\Omega, \mathbb{R}^m_n)$ as $j \to \infty$ ($\nabla u^j \overset{*}{\rightharpoonup} \nabla u$ in $L^\infty(\Omega, \mathbb{R}^m_n)$ if $p = \infty$). If $F : \Omega \times \mathbb{R}^m_n \to [-\infty, \infty]$ satisfies*

(i) *$F(x, \lambda)$ is a Carathéodory function (for $x \in \Omega$, $\lambda \in \mathbb{R}^m_n$),*

(ii) *there exists a mesurable function $E(\cdot)$ such that, for almost every x and all λ, $|F(x, \lambda)| \le E(x)g(\lambda)$ if $p = \infty$, for some continuous function g, and $|F(x, \lambda)| \le E(x)(1 + |\lambda|^p)$ if $p < \infty$,*

(iii) *the mapping $\lambda \mapsto F(x, \lambda)$ is quasiconvex for almost every x,*

then the following Jensen-type inequality is satisfied for almost every $x \in \Omega$:

$$F\Big(x, \int_{\mathbb{R}^m_n} \lambda\, \nu_x(d\lambda)\Big) \le \int_{\mathbb{R}^m_n} F(x, \lambda)\, \nu_x(d\lambda) \tag{3}$$

and $\nabla u(x) = \int_{\mathbb{R}^m_n} \lambda\, \nu_x(d\lambda)$.

It is known that Theorems 1.1 and 1.2 are equivalent (see e.g. [6], [15]–[17]), but as far as I know a direct proof of Theorem 1.2 is missing in the literature. The known proof of Theorem 1.2 requires Theorem 1.1, or its slightly less general version due to Acerbi and Fusco [1]. Theorem 1.1 in the formulation given here was obtained by Marcellini [19]. He did not use Young measures, but the proof was rather long. We want to show that a direct application of Young measures is a useful tool and can abbreviate the already known reasonings.

2. Preliminaries and notation. We use standard notation for the well known function spaces $W^{1,p}(\Omega)$ (Sobolev space), $C_0(\mathbb{R}^l)$ (continuous functions vanishing at infinity), $C(\Omega)$ (continuous functions), $\mathrm{Lip}(\Omega)$ (Lipschitz functions), and $\mathcal{M}(\Omega)$ (Radon measures). If $f \in C(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$, then (f, μ) will stand for $\int_\Omega f(\lambda)\, \mu(d\lambda)$. We write $⨍_A f\, dx$ for $|A|^{-1} \int_A f\, dx$. We denote by $Q(x, r)$ the cube with center x and edges of length r. If x_n, x are elements of a Banach space then we denote by $x_n \to x$ the strong (norm) convergence, by $x_n \rightharpoonup x$ the weak convergence and by $x_n \overset{*}{\rightharpoonup} x$ the weak $*$ convergence. By C we denote the general constant, which can change even in the same line.

The following theorem is well known and has many extensions (see e.g. [9, Theorem 13], [18], [21], [7], [8]).

THEOREM 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $1 \le p < \infty$. Then for any $u \in W^{1,p}(\Omega)$ and any $\lambda > 0$ there exists a closed set $F_\lambda \subseteq \Omega$ and a mapping $u_\lambda \in \mathrm{Lip}(\Omega)$ such that*

(i) $\lambda^p |\Omega \setminus F_\lambda| \to 0$ *as* $\lambda \to \infty$,
(ii) $\nabla u = \nabla u_\lambda$ *for almost every* $x \in F_\lambda$,
(iii) $|\nabla u_\lambda(x)| < C\lambda$ *for almost every* $x \in \Omega$, *with* C *independent of* x *and* λ,
(iv) $\|\nabla u - \nabla u_\lambda\|_{L^p(\Omega)} \to 0$ *as* $\lambda \to \infty$.

We recall the fundamental theorem of Young (see [3]).

THEOREM 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a measurable bounded set. Assume that $u^j : \Omega \to \mathbb{R}^m$, $j = 1, 2, \dots$, is a sequence of measurable functions satisfying the following tightness condition*:

$$\sup_j |\{x \in \Omega : |u^j(x)| \ge k\}| \to 0 \qquad \text{as } k \to \infty.$$

Then there exists a subsequence $\{u^k\}$ and a family $\{\nu_x\}_{x\in\Omega}$ of probability measures, $\nu_x \in \mathcal{M}(\mathbb{R}^m)$, such that

(i) *for every $f \in C_0(\mathbb{R}^m)$ the function $x \mapsto (f, \nu_x)$ is measurable,*
(ii) *if $K \subseteq \mathbb{R}^n$ is a closed set, and for every j and almost every x, $u^j(x) \in K$, then* $\operatorname{supp} \nu_x \subseteq K$ *for almost every* x,
(iii) *if $A \subseteq \Omega$ is measurable and $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfies*

- *f is a Carathéodory function,*
- *the sequence $\{f(x, u^k(x))\}$ is sequentially weakly relatively compact in $L^1(A)$,*

then $\{f(x, u^k(x))\}$ converges weakly in $L^1(A)$ to $\overline{f}$ given by

$$\overline{f}(x) = \int_{\mathbb{R}^m} f(x, \lambda)\, \nu_x(d\lambda).$$

DEFINITION 2.1. We say that u^j *generates the Young measure* $\{\nu_x\}_{x\in\Omega}$ if $\{\nu_x\}_{x\in\Omega}$ satisfies (i) and for any $f \in C_0(\mathbb{R}^m)$, $f(u^j) \overset{*}{\rightharpoonup} \overline{f} = (f, \nu_x)$ in $L^\infty(\Omega)$.

The following useful fact is a generalization of that given in [6, Lemma 2.2]. Although this form is not required for our needs, for completeness, and to show some particular techniques, we try to give a possibly general formulation and include a detailed proof.

THEOREM 2.3. *Suppose that $u^j = (w^j, v^j) : \Omega \to \mathbb{R}^m \times \mathbb{R}^k$ satisfy the tightness condition and generate the Young measure $\{\mu_x\}_{x\in\Omega}$. Suppose further that $w^j \to w$ in measure and that $\{v^j\}_{j\in\mathbb{N}}$ generates the Young measure $\{\nu_x\}_{x\in\Omega}$. Then for almost every $x \in \Omega$ we have $\mu_x = \delta_{w(x)} \otimes \nu_x$, which means that for any $f \in C_0(\mathbb{R}^m \times \mathbb{R}^k)$ and almost every $x \in \Omega$,*

$$\int_{\mathbb{R}^m \times \mathbb{R}^k} f(s,\lambda)\,\mu_x(ds,d\lambda) = \int_{\mathbb{R}^k} f(w(x),\lambda)\,\nu_x(d\lambda). \tag{4}$$

If f is a Carathéodory function on $\Omega \times (\mathbb{R}^m \times \mathbb{R}^k)$ and $|f(z,s,\lambda)| \le C(z)g(s,\lambda)$ with some measurable function C and a continuous function g such that for almost all $x \in \Omega$,

$$\int_{\mathbb{R}^m \times \mathbb{R}^k} g(s,\lambda)\,\mu_x(ds,d\lambda) < \infty \quad \textit{and} \quad \int_{\mathbb{R}^k} g(w(x),\lambda)\,\nu_x(d\lambda) < \infty, \tag{5}$$

then for almost every $x \in \Omega$,

$$\int_{\mathbb{R}^m \times \mathbb{R}^k} f(x,s,\lambda)\,\mu_x(ds,d\lambda) = \int_{\mathbb{R}^k} f(x,w(x),\lambda)\,\nu_x(d\lambda). \tag{6}$$

Proof. Let $f \in C_0(\mathbb{R}^m \times \mathbb{R}^k)$ and set $h^j = f(w^j,v^j) - f(w,v^j)$. Since f is uniformly continuous, it follows that $h^j \to 0$ in measure, and moreover $|h^j| \le 2\|f\|_{L^\infty(\mathbb{R}^m \times \mathbb{R}^k)}$. Thus, by the Lebesgue Dominated Convergence Theorem we have $h^j \to 0$ strongly in $L^1(\Omega)$, while on the other hand, by Theorem 2.2 it converges to $(f,\mu_x) - (f(w(x),\cdot),\nu_x)$ weakly in $L^1(\Omega)$. Hence $(f,\mu_x) - (f(w(x),\cdot),\nu_x) = 0$ almost everywhere, from which (4) follows. To prove (6) we consider three cases: 1) f does not depend on x, 2) $C(z) \le K < \infty$ and f is continuous on $\Omega \times \mathbb{R}^m \times \mathbb{R}^k$, and 3) the general case.

In the first case define $\phi : [0,\infty) \to \mathbb{R}$ by $\phi(t) = 1$ on $[0,1]$, $\phi(t) = -t+2$ on $[1,2]$ and $\phi(t) = 0$ for $t > 2$. Since $f^j(s,\lambda) = f(s,\lambda)\phi(|(s,\lambda)|/j) \in C_0(\mathbb{R}^m \times \mathbb{R}^k)$, it follows that the formula

$$\int_{\mathbb{R}^m \times \mathbb{R}^k} f^j(s,\lambda)\,\mu_x(ds,d\lambda) = \int_{\mathbb{R}^k} f^j(w(x),\lambda)\,\nu_x(d\lambda) \tag{7}$$

holds everywhere on a set $\Omega(j)$ of full measure. In particular, (7) holds everywhere on the set $\widetilde{\Omega} = \bigcap_j \Omega(j)$, which is still of full measure. We can assume additionally that (5) holds for all $x \in \widetilde{\Omega}$. Since $|f^j| \le |f|$, we can let j tend to infinity, apply the Lebesgue Dominated Convergence Theorem, and verify that (6) holds everywhere on $\widetilde{\Omega}$.

In the second case choose a dense countable subset $\{B^j\} \subseteq \Omega$ and consider the functions $F^j(s,\lambda) = f(B^j,s,\lambda)$. Since by Case 1 the equality (6) is satisfied with $f = F^j$ on a set $\Omega_1(j)$ of full measure we see that

$$\int_{\mathbb{R}^m \times \mathbb{R}^k} f(B^j,s,\lambda)\,\mu_x(ds,d\lambda) = \int_{\mathbb{R}^k} f(B^j,w(x),\lambda)\,\nu_x(d\lambda)$$

on the set $\Omega_1 = \bigcap_j \Omega_1(j)$, which is still of full measure and does not depend on j. Take an arbitrary $x \in \Omega_1$ and a sequence $B^{j_k} \to x$ as $k \to \infty$. Now it suffices to check that by the Lebesgue Dominated Convergence Theorem the left hand side of the equality tends to $\int_{\mathbb{R}^m \times \mathbb{R}^k} f(x,s,\lambda)\,\mu_x(ds,d\lambda)$, while the right hand side tends to $\int_{\mathbb{R}^k} f(x,w(x),\lambda)\,\nu_x(d\lambda)$.

Finally, in the last case we use the Scorza Dragoni Theorem and Lusin Theorem (see e.g. [13]) and bite off sets Ω_ε of arbitrarily small measure such that f is continuous on $(\Omega \setminus \Omega_\varepsilon) \times \mathbb{R}^m \times \mathbb{R}^k$ and C is bounded on $\Omega \setminus \Omega_\varepsilon$. Thus (6) is satisfied almost everywhere on $\Omega \setminus \Omega_\varepsilon$, and hence it is satisfied almost everywhere on Ω. ■

Let us state Chacon's Biting Lemma (see e.g. [5]).

THEOREM 2.4 [Biting Lemma]. *Let $\Omega \subseteq \mathbb{R}^n$, $|\Omega| < \infty$ and suppose that $\{f^j\}$ is a bounded sequence in $L^1(\Omega)$. Then there exists a subsequence $\{f^\nu\}$, a function $f \in L^1(\Omega)$ and a decreasing family of measurable sets E_k such that $|E_k| \to 0$ as $k \to \infty$ and for any k,*

$$f^\nu \rightharpoonup f \quad \text{in } L^1(\Omega \setminus E_k) \text{ as } \nu \to \infty.$$

DEFINITION 2.2 (see e.g. [24]). We will say that $\{f^j\}$ *converges to f in the biting sense* ($f^j \xrightarrow{b} f$) whenever there is a set E of arbitrarily small measure such that $f^j \rightharpoonup f$ in $L^1(\Omega \setminus E)$.

Finally, we recall the known definition of quasiconvexity.

DEFINITION 2.3. The function $F : \mathbb{R}^m_n \to \mathbb{R}$ is *quasiconvex* if for any $A \in \mathbb{R}^m_n$, any cube $Q \subseteq \mathbb{R}^n$ and any $\phi \in C_0^\infty(Q, \mathbb{R}^m)$,

$$F(A) \le \fint_Q F(A + \nabla\phi)\, dx. \tag{8}$$

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.2. We can assume that Ω is a ball (if $h_1 \le h_2$ almost everywhere on every ball $B \subseteq \Omega$ then $h_1 \le h_2$ almost everywhere on Ω). If we take $f(\lambda) = \lambda$, $f : \mathbb{R}^m_n \to \mathbb{R}^m_n$ and apply the Young Theorem to every coordinate of $f(\nabla u^j)$ we immediately derive $\int_{\mathbb{R}^m_n} \lambda\, \nu_x(d\lambda) = \nabla u(x)$, for almost every x.

We distinguish the following cases: 1) $F = F(\lambda)$ and $p = \infty$, 2) $F = F(\lambda)$ and $1 \le p < \infty$, and 3) the general case.

Case 1. Let $x \in \Omega$ and $r > 0$ be such that $Q(x,r) \subseteq \Omega$. Take $0 < \sigma < r$ and choose $\phi_\sigma \in C_0^\infty(Q(x,r))$, $\phi_\sigma \equiv 1$ on $Q(x, r-\sigma)$. By standard arguments the function $w^j_\sigma = \phi_\sigma(u^j - u)$ can be substituted in (8). That gives for arbitrary $A \in \mathbb{R}^m_n$,

$$F(A) \le \fint_{Q(x,r)} F(A + \nabla\phi_\sigma \cdot (u^j - u) + \phi_\sigma(\nabla u^j - \nabla u))\, dy = I(x, r, \sigma, j).$$

Since $\{F(A + \nabla w^j_\sigma)\}_j$ is relatively compact in $L^1(\Omega)$, by the Young Theorem applied to the sequence $(u^j - u, \nabla u^j)$ and by Theorem 2.3,

$$I(x,r,\sigma,j)\to \fint_{Q(x,r)}\int_{\mathbb{R}^m_n} F(A+\phi_\sigma(y)(\lambda-\nabla u(y)))\,\nu_y(d\lambda)\,dy = I(x,r,\sigma)$$

as $j\to\infty$ and ν_y is supported on a bounded set. Hence, if we apply the Lebesgue Dominated Convergence Theorem and let $\sigma\to 0$, we see that

$$I(x,r,\sigma)\to \fint_{Q(x,r)}\int_{\mathbb{R}^m_n} F(A+\lambda-\nabla u(y))\,\nu_y(d\lambda)\,dy.$$

By the Lebesgue Differentiation Theorem for any $A\in\mathbb{R}^m_n$ there is a set $\Omega(A)\subseteq\Omega$ such that $|\Omega\setminus\Omega(A)|=0$, and for each $x\in\Omega(A)$,

$$F(A)\le \int_{\mathbb{R}^m_n} F(A+\lambda-\nabla u(x))\,\nu_x(d\lambda). \tag{9}$$

We can additionally assume that $|\nabla u(x)|<\infty$ for every $x\in\Omega(A)$. Let $\{A^j\}$ be a countable dense subset in $\mathbb{R}^m_n$. Since $\Omega_1=\bigcap_j\Omega(A^j)$ is still of full measure in Ω, for every $x\in\Omega_1$ the inequality (9) is satisfied with $A=A^j$, for arbitrary j. Take $x\in\Omega_1$ and let $A^{j_k}\to\nabla u(x)$ as $k\to\infty$. Now it suffices to note that $F(A^{j_k})\to F(\nabla u(x))$, and

$$\int_{\mathbb{R}^m_n} F(A^{j_k}-\nabla u(x)+\lambda)\,\nu_x(d\lambda)\to\int_{\mathbb{R}^m_n} F(\lambda)\,\nu_x(d\lambda).$$

Case 2. We apply Theorem 2.1 and find u_λ such that $\|\nabla u_\lambda\|_{L^\infty(\Omega)}\le C\lambda$ and $\nabla u_\lambda=\nabla u$ almost everywhere on F_λ, where $\lambda^p|\Omega\setminus F_\lambda|\to 0$ as $\lambda\to\infty$. Let $\lambda_k=k$, and let $\{\nu^k_x\}$ be the Young measure generated by a subsequence of $\{\nabla u^j_k\}_j$. Note that for any k we have $F(\lambda,\nu^k_x)\le(F,\nu^k_x)$ almost everywhere. Thus it suffices to apply the following lemma.

Lemma 3.1. *Let $f=f(\lambda)$ with $|f(\lambda)|\le C(1+|\lambda|^p)$, and $\{\nu_x\}_{x\in\Omega}$ and $\{\nu^k_x\}_{x\in\Omega}$ be as above. Then for every $\varepsilon>0$ we can find a set $E\subseteq\Omega$ such that $|E|<\varepsilon$ and $(f,\nu^k_x)\to(f,\nu_x)$ in $L^1(\Omega\setminus E)$ as $k\to\infty$.*

Proof. Take $\varepsilon>0$. According to Theorems 2.4 and 2.2 we find a set $E\subseteq\Omega$ such that $|E|<\varepsilon$ and $f(\nabla u^j)\rightharpoonup(f,\nu_x)$ weakly in $L^1(\Omega\setminus E)$. We have

$$\begin{aligned}
&\int_{\Omega\setminus E}|(f,\nu^k_x)-(f,\nu_x)|\,dx\\
&\quad=\sup_{\|\phi\|_{L^\infty(\Omega)}\le 1}\Big|\int_{\Omega\setminus E}\phi(x)((f,\nu^k_x)-(f,\nu_x))\,dx\Big|\\
&\quad=\sup_{\|\phi\|_{L^\infty(\Omega)}\le 1}\Big|\lim_{j\to\infty}\int_{\Omega\setminus E}\phi(x)(f(\nabla u^j_k)-f(\nabla u^j))\,dx\Big|\\
&\quad\le\sup_j\int_{(\Omega\setminus E)\setminus F_k}|f(\nabla u^j_k)|+\sup_j\int_{(\Omega\setminus E)\setminus F_k}|f(\nabla u^j)|\,dx\stackrel{k\to\infty}{\longrightarrow}0.
\end{aligned}$$

The convergence follows from the estimates on f and the Dunford–Pettis Theorem. ■

Case 3. We use exactly the same arguments as in the proof of Theorem 2.3, cases 2 and 3. ■

Proof of Theorem 1.1. Suppose that $\{u^j\}_{j\in\mathbb{N}}$ satisfies the assumptions of Theorem 1.1. Let $\alpha = \liminf_{j\to\infty} I(u^j)$. If $\alpha = \infty$ the assertion is satisfied. Suppose that $\alpha < \infty$. In this case the sequence $\{F(x, u^j, \nabla u^j)\}_{j\in\mathbb{N}}$ is bounded in $L^1(\Omega)$. By Theorems 2.2–2.4 we find a subsequence $\{u^l\}$ with the properties: 1) $I(u^l) \to \alpha$ as $l \to \infty$, 2) the sequence $\{\nabla u^l\}$ generates the Young measure $\{\nu_x\}_{x\in\Omega}$, 3) there exists a family $\{E_k\}$ of sets such that $|E_k| \to 0$ and $\{F(x, u^l, \nabla u^l)\}_l$ is weakly convergent in $L^1(\Omega \setminus E_k)$ to $\int_{\mathbb{R}^m_n} F(x, u(x), \lambda)\, \nu_x(d\lambda)$.

Since $F_u(x, \lambda) = F(x, u(x), \lambda)$ satisfies the assumptions of Theorem 1.2, we have $\int_{\mathbb{R}^m_n} F(x, u(x), \lambda)\, \nu_x(d\lambda) \geq F(x, u(x), \nabla u(x))$ for almost every x. Now it suffices to note that

$$\alpha = \lim_{l\to\infty} \int_{\Omega} F(x, u^l, \nabla u^l)\, dx \geq \lim_{l\to\infty} \int_{\Omega\setminus E_k} F(x, u^l, \nabla u^l)\, dx$$
$$= \int_{\Omega\setminus E_k} \int_{\mathbb{R}^m_n} F(x, u(x), \lambda)\, \nu_x(d\lambda) \geq \int_{\Omega\setminus E_k} F(x, u(x), \nabla u(x))\, dx. \ ■$$

Remark 3.1. It has been proved by Kristensen [17] that the Jensen inequalities of Theorem 1.2 can be generalized to a certain class of functions which are Borel measurable with respect to the last variable.

REFERENCES

[1] E. Acerbi and N. Fusco, *Semicontinuity problems in the calculus of variations*, Arch. Rational Mech. Anal. 86 (1984), 125–145.

[2] L. Ambrosio, *New lower semicontinuity results for integral functionals*, Rend. Accad. Naz. Sci. XL Mem. Mat. Sci. Fis. Mat. Natur. 105 (1987), 1–42.

[3] J. M. Ball, *A version of the fundamental theorem for Young measures*, in: PDE's and Continuum Models of Phase Transitions, M. Rascle, D. Serre and M. Slemrod (eds.), Lecture Notes in Phys. 344, Springer, Berlin, 1989.

[4] —, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. 63 (1977), 337–403.

[5] J. M. Ball and F. Murat, *Remarks on Chacon's Biting Lemma*, Proc. Amer. Math. Soc. 107 (1989), 655–663.

[6] J. M. Ball and K. W. Zhang, *Lower semicontinuity of multiple integrals and the Biting Lemma*, Proc. Roy. Soc. Edinburgh Sect. A 114 (1990), 367–379.

[7] B. Bojarski, *Remarks on some geometric properties of Sobolev mappings*, in: Functional Analysis and Related Topics, S. Koshi (ed.), World Scientific, Singapore, 1991, 65–76.

[8] B. Bojarski and P. Hajłasz, *Pointwise inequalities for Sobolev functions and some applications*, Studia Math. 106 (1993), 77–92.
[9] A. P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, ibid. 20 (1961), 171–225.
[10] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer, Berlin, 1989.
[11] M. Esteban, *A direct variational approach to Skyrme's model for meson fields*, Comm. Math. Phys. 105 (1986), 571–591.
[12] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, CMBS Regional Conf. Ser. in Math. 74, Amer. Math. Soc., Providence, R.I., 1990.
[13] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
[14] A. D. Ioffe, *On lower semicontinuity of integral functionals*, *I*, *II*, SIAM J. Control Optim. 15 (1977), 521–538, 991–1000.
[15] D. Kinderlehrer and P. Pedregal, *Characterisation of Young measures generated by gradients*, Arch. Rational Mech. Anal. 115 (1991), 329–365.
[16] —, —, *Gradient Young measures generated by sequences in Sobolev spaces*, J. Geom. Anal. (to appear).
[17] J. Kristensen, *Lower semicontinuity of variational integrals*, Ph.D. Thesis, Mathematical Institute, The Technical University of Denmark, 1994.
[18] F. C. Liu, *A Luzin type property of Sobolev functions*, Indiana Univ. Math. J. 26 (1977), 645–651.
[19] P. Marcellini, *Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals*, Manuscripta Math. 51 (1985), 1–28.
[20] N. G. Meyers, *Quasi-convexity and lower semi-continuity of multiple variational functionals of any order*, Trans. Amer. Math. Soc. 119 (1965), 125–149.
[21] J. Michael and W. Ziemer, *A Lusin type approximation of Sobolev functions by smooth functions*, in: Contemp. Math. 42, Amer. Math. Soc., 1985, 135–167.
[22] C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer, Berlin, 1966.
[23] M. Struwe, *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Berlin, 1990.
[24] K. Zhang, *Biting theorems for Jacobians and their applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1990), 345–365.

Institute of Mathematics
University of Warsaw
Banacha 2
00-097 Warszawa, Poland
E-mail: kalamajs@mimuw.edu.pl

Received 1 April 1996;
revised 27 November 1996