

*TRANSFERENCE THEORY ON  
HARDY AND SOBOLEV SPACES*

BY

MARIA J. CARRO AND JAVIER SORIA (BARCELONA)

We show that the transference method of Coifman and Weiss can be extended to Hardy and Sobolev spaces. As an application we obtain the de Leeuw restriction theorems for multipliers.

**1. Introduction.** In 1977, R. Coifman and G. Weiss (see [CW1]) proved the transference theorem in the setting of  $L^p$  spaces for  $1 \leq p \leq \infty$ . As a first application of this result, they were able to show the classical theorem of K. de Leeuw [D] on restriction of multipliers; namely, if  $m$  is a nice function such that  $m \in M_p(\mathbb{R}^N)$ , then its restriction  $(m(n))_n$  is in  $M_p(\mathbb{Z}^N)$ , with norm bounded by  $\|m\|_{M_p(\mathbb{R}^N)}$ , where for a general locally compact group  $G$ , we say that  $m \in M_p(\widehat{G})$  if its inverse Fourier transform  $K = \check{m}$  is a convolution operator on  $L^p(\widehat{G})$ , with  $\widehat{G}$  the dual group of  $G$ . In this case, the norm of this convolution operator is denoted by either  $N_p(K)$  or  $\|m\|_{M_p(G)}$ .

This theory has been widely extended by N. Asmar, E. Berkson and T. A. Gillespie in a collection of papers (see [ABG1] and [ABG2]) where they carefully study transference for maximal operators and transference of weak type inequalities.

On the other hand, L. Colzani (see [C]) proved, using direct arguments, that if  $m$  is a multiplier on  $H^p(\mathbb{R}^N)$  and  $m$  is a continuous function, then  $(m(n))_n$  is a multiplier on  $H^p(\mathbb{T}^N)$ , in the sense that the operator

$$(SP)(x) = \sum_{n=-M}^M m(n)a_n e^{2\pi i n x}$$

(with  $P$  the trigonometric polynomial  $P(x) = \sum_{n=-M}^M a_n e^{2\pi i n x}$ ) can be extended to a bounded operator on  $H^p(\mathbb{T}^N)$ .

We shall see that this is a consequence of the fact that the transference method of Coifman and Weiss can be applied to a more general class of spaces than  $L^p$ , including Hardy spaces and Sobolev spaces.

---

1991 *Mathematics Subject Classification*: 42B30, 43A15.

This work has been partially supported by the DGICYT: PB94-0879.

This paper is organized as follows: In Section 2, we give the definition of transferred space and give several examples. Section 3 contains the main result of this paper for the case  $p \geq 1$  and several applications. Section 4 is devoted to the case  $0 < p < 1$  and Section 5 to the case of maximal operators and maximal spaces.

Although the theory can be developed for amenable groups ([CW1]), we shall restrict our attention to locally compact abelian groups where our theory can go a little further and where all of our examples belong.

As usual,  $\tilde{f}(u) = f(u^{-1})$ ,  $(\tau_v f)(u) = f(uv^{-1})$ , and constants such as  $C$  may change from one occurrence to the next.

**2. Transferred space.** Let  $G$  be a locally compact abelian group and let  $L^0(G)$  denote the set of all measurable functions on  $G$ . Consider a sublinear functional  $S : \mathcal{A} \rightarrow \mathbb{C}$ , where  $\mathcal{A} \subset L^0(G)$ .

Then, for  $0 < p \leq \infty$ , we define the space  $H^p(S)$  as the completion of

$$\{f \in L^1(G) : S(\tau \cdot \tilde{f}) \in L^p(G)\}$$

with respect to the “quasi-norm”  $\|f\|_{H^p(S)} = \|S(\tau \cdot \tilde{f})\|_{L^p(G)}$ .

Consider now a  $\sigma$ -finite measure space  $(\mathcal{M}, dx)$  and let  $R$  be a representation of  $G$  on  $L^p(\mathcal{M})$  such that  $R$  is uniformly bounded (see [CW1]); that is, there exists a constant  $A$  such that, for every  $f \in L^p(\mathcal{M})$  and every  $u \in G$ ,

$$(1) \quad \|R_u f\|_{L^p(\mathcal{M})} \leq A \|f\|_{L^p(\mathcal{M})}.$$

DEFINITION 2.1. We define the *transferred space*  $H^p(S; R)$  of  $H^p(S)$  by the representation  $R$  as the completion of

$$\{f \in L^1(\mathcal{M}) : S(\tilde{R}_u f(\cdot)) \in L^p(\mathcal{M})\}$$

with respect to the “quasi-norm”  $\|f\|_{H^p(S; R)} = \|S(\tilde{R}_u f(\cdot))\|_{L^p(\mathcal{M})}$ .

Before going any further, we give some interesting examples of transferred spaces. Recall that the *transferred operator*  $T_K$  is defined by (see [CW1])

$$(T_K f)(x) = \int_G K(u)(R_{u^{-1}} f)(x) du.$$

EXAMPLES 2.2. (1) If  $S(f) = |f(e)|$ , where  $e$  is the identity element, then  $H^p(S) = L^p(G)$ , and if  $R$  is any representation of  $G$  acting on  $L^p(\mathcal{M})$ , then one can easily check that the transferred space is equal to  $L^p(\mathcal{M})$ .

(2) Consider  $G = \mathbb{R}$ ,  $\mathcal{M} = \mathbb{T}$ ,  $(R_u f)(x) = f(x - u)$  and  $S(f) = |f(0)| + |(Hf)(0)|$  where  $H$  is the Hilbert transform. Then

$$H^1(S) = \{f \in L^1(\mathbb{R}) : Hf \in L^1(\mathbb{R})\} = H^1(\mathbb{R}),$$

and, following the computations in [CW1], we find that

$$\begin{aligned} S(\tilde{R}_u f(x)) &= |f(x)| + \left| \lim_{N \rightarrow \infty} \int_{1/N \leq |u| \leq N} f(x-u) \frac{du}{u} \right| \\ &= |f(x)| + \left| \lim_{N \rightarrow \infty} \int_{1/N \leq |u| \leq 1} \pi \cot(\pi s) f(x-s) ds \right| = |f(x)| + |(Cf)(x)|, \end{aligned}$$

where  $Cf$  is the conjugate function of  $f$ . Therefore,

$$H^1(S; R) = \{f \in L^1(\mathbb{T}) : Cf \in L^1(\mathbb{T})\} = H^1(\mathbb{T}).$$

Similarly, using Miyachi's theorem (see [M]), we conclude that, for  $0 < p \leq 1$ ,  $H^p(S) = H^p(\mathbb{R})$ , and  $H^p(S; R) = H^p(\mathbb{T})$ .

(3) Consider  $G = \mathbb{R}$ ,  $\mathcal{M} = \mathbb{T}$ ,  $(R_u f)(x) = f(x-u)$  and  $S(f) = |f(0)| + |f'(0)|$ . Then

$$H^p(S) = \{f \in L^p(\mathbb{R}) : f' \in L^p(\mathbb{R})\} = W_{p,1}(\mathbb{R}),$$

and

$$H^p(S; R) = \{f \in L^p(\mathbb{T}) : f' \in L^p(\mathbb{T})\} = W_{p,1}(\mathbb{T}).$$

That is, we get Sobolev spaces. Obviously, we can also obtain  $W_{p,k}(\mathbb{R}^N)$  and  $W_{p,k}(\mathbb{T}^N)$ .

(4) Consider  $G = \mathbb{Z}$ ,  $(R_n f)(x) = f(T^n x)$  with  $T$  an ergodic transformation and  $S((a_n)_n) = |a_0| + |\sum_{n \neq 0} a_n/n|$ . Then  $H^1(S) = H^1(\mathbb{Z})$  and  $H^1(S; R)$  turns out to be an ergodic Hardy space (see [CW2] and [CT])

$$H^1(S; R) = \left\{ f \in L^1(\mathcal{M}) : \sum_n \frac{1}{n} f(T^n x) \in L^1(\mathcal{M}) \right\}.$$

(5) If  $G = \mathbb{R}$ ,  $(R_t f)(x) = \frac{w(T^t x)}{w(x)} f(T^t x)$  with  $T$  an ergodic transformation on a measure space  $\mathcal{M}$  and  $w$  a weight on  $\mathcal{M}$ , then for  $S(f) = |f(0)| + |(Hf)(0)|$ , the transferred space  $H^1(S; R)$  is the space of all functions  $F \in L^1(w)$  such that  $wF$  is in the ergodic Hardy space  $H^1$ ; this space can be considered as a weighted ergodic Hardy space.

(6) Consider  $G = \mathbb{R}^N$ ,  $\mathcal{M} = \mathbb{T}^N$ ,  $(R_u f)(x) = f(x-u)$  and  $Sf = \sup_{t>0} |\varphi_t * f(0)|$ , where  $\varphi \in \mathcal{S}(\mathbb{R}^N)$  and  $\int \varphi = 1$ . Then  $H^p(S) = H^p(\mathbb{R}^N)$  and  $H^p(S; R) = H^p(\mathbb{T}^N)$ .

(7) Let now  $G = \mathbb{R}$ ,  $\mathcal{M} = \overline{\mathbb{R}}$ , the Bohr compactification of  $\mathbb{R}$  (see [HR]), and  $(R_t f)(x) = f(x-t)$ . Then one can easily see that the transferred space of the Hardy space  $H^1(\mathbb{R})$  is the space of all functions in  $L^1(\overline{\mathbb{R}})$  such that  $\sum_{t \in \mathbb{R}} \text{sgn}(t) \hat{f}(t) e^{itx}$  is in  $L^1(\overline{\mathbb{R}})$ , which is  $H^1(\overline{\mathbb{R}})$ .

(8) Let  $G = \mathbb{R}^n$ ,  $\mathcal{M} = \mathbb{R}^m$  with  $m < n$  and let  $R$  be the natural representation defined by  $(R_{(x_1, \dots, x_n)} f)(y_1, \dots, y_m) = f(y_1 - x_1, \dots, y_m - x_m)$ .

If  $T_{R_j^n}$  is the transferred operator of the Riesz transform  $R_j^n$  ( $j = 1, \dots, n$ ) in  $\mathbb{R}^n$ , then  $T_{R_j^n} = 0$  if  $j = m + 1, \dots, n$  and  $T_{R_j^n} = R_j^m$  if  $j = 1, \dots, m$ . Therefore, the transferred space of  $H^p(\mathbb{R}^n)$  by this representation is  $H^p(\mathbb{R}^m)$  for every  $0 < p \leq 1$ .

Many other examples can be given in the setting of Triebel–Lizorkin spaces, Besov spaces, etc.

**3. Main results for  $p \geq 1$ .** Throughout this section we shall denote by  $K*$  the convolution operator with kernel  $K$ ,  $T_K$  the transferred operator,  $H^p(S)$  will be denoted by  $H^p(K)$  and the transferred space  $H^p(S; R)$  by  $H^p(T_K)$ , whenever  $Sf = K * f$ .

*Case of a finite family of kernels and  $p \geq 1$ .* Denote by  $H^p(\{K_i\}_{i=1, \dots, n})$  the completion of

$$\{f \in L^1(G) : K_i * f \in L^p(G), \forall i = 1, \dots, n\}$$

under the norm  $\sum_i \|K_i * f\|_p$ , and similarly for  $H^p(\{T_{K_i}\}_{i=1, \dots, n})$ .

**THEOREM 3.1.** *Let  $G$  be a locally compact abelian group and let  $1 \leq p$ . Let  $K$ ,  $\{K_i^1\}_{i=1, \dots, n}$  and  $\{K_j^2\}_{j=1, \dots, m}$  be a collection of functions in  $L^1(G)$  and assume that*

$$K* : H^p(\{K_i^1\}_{i=1, \dots, n}) \rightarrow H^p(\{K_j^2\}_{j=1, \dots, m})$$

*has the property that there exist positive constants  $\{A_i\}_i$  such that*

$$\sum_{j=1}^m \|K_j^2 * K * f\|_p \leq \sum_{i=1}^n A_i \|K_i^1 * f\|_p.$$

*Then the transferred operator*

$$T_K : H^p(\{T_{K_i^1}\}_i) \rightarrow H^p(\{T_{K_j^2}\}_j)$$

*is bounded, with*

$$\sum_{j=1}^m \|T_{K_j^2} T_K f\|_p \leq BA^2 \sum_{i=1}^n A_i \|T_{K_i^1} f\|_p,$$

*where  $A$  is as in (1) and  $B$  depends only on  $n$  and  $m$ .*

**PROOF.** We prove this for  $m = 1$ . The proof for  $m > 1$  is similar.

We first recall that since  $K \in L^1(G)$ , it is known (see [CW1]) that, for every  $v \in G$  and every  $f \in L^p(\mathcal{M})$ ,

$$(2) \quad (R_v T_K f)(x) = \int_G K(u) (R_{vu^{-1}} f)(x) du = (T_K R_v f)(x), \quad \text{a.e. } x \in \mathcal{M}.$$

Also, using the same idea, one can easily see that  $T_K T_{K_2} = T_{K * K_2}$  and therefore, we can assume without loss of generality that  $H^p(K_2) = L^p(G)$  and  $H^p(T_{K_2}) = L^p(\mathcal{M})$ .

Now, since  $K$  and  $K_i^1 = K_i$  are in  $L^1(G)$  we can approximate them by functions in  $L^1(G)$  with compact support and hence standard arguments show that for every  $\varepsilon > 0$  we can find functions  $K_n, K_{i,n}$  in  $L^1(G)$  with compact support such that

$$\|K_n * f\|_p \leq \sum_{i=1}^n A_i \|K_{i,n} * f\|_p + \varepsilon \|f\|_p.$$

Therefore, we can assume without loss of generality that  $K$  and  $K_i$  are compactly supported functions in  $L^1(G)$ .

Let  $f \in L^p(\mathcal{M})$ . By (1), we have

$$\|T_K f\|_p = \|R_{v^{-1}} R_v T_K f\|_p \leq A \|R_v T_K f\|_p.$$

Now, as in [CW1], we consider a compact set  $C$  such that the identity element  $e$  is in  $C$ ,  $\text{supp } K \subset C$  and  $\text{supp } K_i \subset C$  for every  $i = 1, \dots, n$ . Also, take a neighborhood  $V$  of  $e$  such that

$$(3) \quad \frac{\mu(VC^{-1}C)}{\mu(V)} \leq 1 + \frac{\varepsilon}{\max(1, (A\|f\|_p\|K_i\|_1)^p)}.$$

Now, by (2),

$$\begin{aligned} \|T_K f\|_p^p &\leq \frac{A^p}{\mu(V)} \int_V \|R_v T_K f\|_p^p dv \\ &= \frac{A^p}{\mu(V)} \int_V \int_{\mathcal{M}} \left| \int_G K(u) (R_{vu^{-1}} f)(x) du \right|^p dx dv \\ &= \frac{A^p}{\mu(V)} \int_V \int_{\mathcal{M}} \left| \int_G K(u) \chi_{VC^{-1}}(vu^{-1}) (R_{vu^{-1}} f)(x) du \right|^p dx dv \\ &\leq \frac{A^p}{\mu(V)} \int_{\mathcal{M}} \left[ \int_G \left| \int_G K(u) \chi_{VC^{-1}}(vu^{-1}) (R_{vu^{-1}} f)(x) du \right|^p dv \right] dx \\ &\leq \frac{A^p B}{\mu(V)} \int_{\mathcal{M}} \sum_i A_i^p \|\chi_{VC^{-1}}(R \cdot f)(x)\|_{H^p(K_i)}^p dx, \end{aligned}$$

where the last inequality follows by applying the hypothesis to the function

$$h_x(u) = \chi_{VC^{-1}}(u) (R_u f)(x).$$

The last step is to show that, for every  $i$ ,

$$(4) \quad \int_{\mathcal{M}} \|\chi_{VC^{-1}}(R \cdot f)(x)\|_{H^p(K_i)}^p dx \leq A^p \mu(V) \|f\|_{H^p(T_{K_i})}^p + \varepsilon \mu(V),$$

from which we can easily deduce the theorem.

To see this, we observe that  $\|\chi_{VC^{-1}}(R \cdot f)(x)\|_{H^p(K_i)} = \|K_i * h_x\|_{L^p(G)}$ .

Now,

$$\begin{aligned}
& \int_{\mathcal{M}} \|K_i * h_x\|_{L^p(G)}^p dx \\
&= \int_{\mathcal{M}} \left[ \int_G \left| \int_G K_i(u) \chi_{VC^{-1}}(vu^{-1})(R_{vu^{-1}}f)(x) du \right|^p dv \right] dx \\
&= \int_{\mathcal{M}} \left[ \int_V \left| \int_G K_i(u) \chi_{VC^{-1}}(vu^{-1})(R_{vu^{-1}}f)(x) du \right|^p dv \right] dx \\
&\quad + \int_{\mathcal{M}} \left[ \int_{VC^{-1}C \setminus V} \left| \int_G K_i(u) \chi_{VC^{-1}}(vu^{-1})(R_{vu^{-1}}f)(x) du \right|^p dv \right] dx \\
&= I + II,
\end{aligned}$$

where the last equality follows since  $V \subset VC^{-1}C$ .

Let us first estimate  $I$ : since  $u \in C$  and  $v \in V$ , we have  $vu^{-1} \in VC^{-1}$  and therefore

$$\begin{aligned}
I &= \int_{\mathcal{M}} \left[ \int_V \left| \int_G K_i(u)(R_{vu^{-1}}f)(x) du \right|^p dv \right] dx \\
&= \int_V \left[ \int_{\mathcal{M}} \left| \int_G K_i(u)(R_{u^{-1}}R_v f)(x) du \right|^p dx \right] dv \\
&\leq \int_V \|T_{K_i} R_v f\|_p^p dv = \int_V \|R_v T_{K_i} f\|_p^p dv \leq A^p \mu(V) \|T_{K_i} f\|_p^p.
\end{aligned}$$

To estimate  $II$ , we proceed as follows:

$$\begin{aligned}
II &= \int_{\mathcal{M}} \left[ \int_{VC^{-1}V \setminus V} \left| \int_G K_i(u) \chi_{VC^{-1}}(vu^{-1})(R_{vu^{-1}}f)(x) du \right|^p dv \right] dx \\
&\leq \|K_i\|_1^{p-1} \int_{\mathcal{M}} \left[ \int_{VC^{-1}V \setminus V} |K_i(u)| \cdot |(R_{vu^{-1}}f)(x)|^p du dv \right] dx \\
&\leq \|K_i\|_1^{p-1} \int_G |K_i(u)| \int_{VC^{-1}V \setminus V} \|R_{vu^{-1}}f\|_p^p dv du \\
&\leq A^p \|f\|_p^p \|K_i\|_1^p \mu(VC^{-1}V \setminus V).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_{\mathcal{M}} \|\chi_{VC^{-1}}(R.f)(x)\|_{H^p(K_i)}^p dx \\
&\leq A^p \mu(V) \|T_{K_i} f\|_p^p + A^p \|f\|_p^p \|K_i\|_1^p \mu(VC^{-1}C \setminus V).
\end{aligned}$$

Now, since, for every  $i$ ,

$$\mu(VC^{-1}C \setminus V) = \mu(VC^{-1}C) - \mu(V) \leq \frac{\varepsilon \mu(V)}{(A \|f\|_p \|K_i\|_1)^p},$$

we obtain

$$\int_{\mathcal{M}} \|\chi_{VC^{-1}}(R.f)(x)\|_{H^p(K_i)}^p dx \leq A^p \mu(V) \|f\|_{H^p(T_{K_i})}^p + \varepsilon \mu(V),$$

as desired. ■

**Remark 3.2.** We observe that, as it happens in the transference theorem of [CW1], the above theorem is not only a boundedness result, but the important thing is the norm of the transferred operator.

*Applications.* We now apply the previous results to the setting of Sobolev and Hardy spaces.

**A. Sobolev spaces.** Let  $K$  be a function in  $L^1(G)$  such that

$$K* : H^p(K_1) \rightarrow L^p(G),$$

with norm  $N_p(K)$ , where  $K_1$  is not, in general, in  $L^1$ .

Assume that there exists an approximation of the identity  $\varphi_n$  such that  $\varphi_n \in L^1(G)$  and  $K_1 * \varphi_n$  is a function in  $L^1(G)$ . Then, if we apply the boundedness hypothesis to the function  $f * \varphi_n$  we get

$$\|(K * \varphi_n) * f\|_p \leq N_p(K) \|(K_1 * \varphi_n) * f\|_p,$$

where the kernels  $K * \varphi_n$  and  $K_1 * \varphi_n$  are functions in  $L^1(G)$  and hence we can transfer to deduce that

$$T_{K*\varphi_n} : H^p(T_{K_1*\varphi_n}) \rightarrow L^p(\mathcal{M})$$

is bounded with norm less than or equal to  $A^2 N_p(K)$ . The boundedness of  $T_K$  from  $H^p(T_{K_1})$  into  $L^p(\mathcal{M})$  can be deduced, in the case of Sobolev spaces, by a limit process, since  $T_{K_1*\varphi_n} f$  converges to  $T_{K_1} f$  in the  $L^p(\mathcal{M})$  norm, for every  $p \geq 1$ .

**THEOREM A.1.** *Let  $1 \leq p$  and  $r, s \in \mathbb{N}$ . If  $m \in L_{\text{loc}}^\infty$  is a normalized function such that  $K = \tilde{m}$  has the property that*

$$K* : W_{p,r}(\mathbb{R}^N) \rightarrow W_{p,s}(\mathbb{R}^N)$$

*is a bounded operator with norm  $N_p(K)$ , then the transferred operator*

$$T_K : W_{p,r}(\mathbb{T}^N) \rightarrow W_{p,s}(\mathbb{T}^N)$$

*is given by  $T_K(\sum a_n e^{2\pi i n x}) = \sum_n m(n) a_n e^{2\pi i n x}$  and is a bounded operator with norm less than or equal to  $C N_p(K)$ , with  $C$  only depending on  $s$  and  $r$ .*

**Proof.** We prove this in the case  $s = 0$ . The case  $s \in \mathbb{N}$  is similar.

Take  $\varphi_n(\xi) = n^N \varphi(n\xi)$  where  $\widehat{\varphi} \in \mathcal{D}(\mathbb{R}^N)$  is such that  $\int \varphi = 1$  and  $\varphi \geq 0$ . In this case  $K_i = \delta_0^{(i)}$  for  $|i| \leq r$ , and hence,  $K_i * \varphi_n = \varphi_n^{(i)} \in L^1(\mathbb{R}^N)$ . Therefore we get the result in the case  $K \in L^1(\mathbb{R}^N)$ .

Now, for the general case we proceed as in Lemma 3.5 of [CW1]. Since  $m$  is normalized and  $m \in L_{\text{loc}}^\infty$ , we see that  $m_n = (K * \varphi_n)^\wedge$  is also normalized,

$m_n \in L^\infty$  and we can find a sequence  $(m_n^k)_k$  such that  $m_n^k(\xi) \rightarrow m_n(\xi)$  for every  $\xi \in \mathbb{R}^N$  and, if  $K_n^k = \check{m}_n^k$ , then  $K_n^k \in L^1$ , and  $N_p(K_n^k) \leq N_p(K)$ . Also,  $m_n(\xi) \rightarrow m(\xi)$  for every  $\xi \in \mathbb{R}^N$ . From this, we deduce that  $T_K = \lim_{n,k} T_{K_n^k}$  and since  $K_n^k$  satisfies the right hypothesis we obtain the desired result. ■

Similarly, in the context of the Bohr compactification  $\overline{\mathbb{R}^N}$  of  $\mathbb{R}^N$ , we get the following result:

**THEOREM A.2.** *Let  $1 \leq p$  and  $r, s \in \mathbb{N}$ . If  $m \in L_{\text{loc}}^\infty$  is a normalized function such that for  $K = \check{m}$  the operator*

$$K* : W_{p,r}(\mathbb{R}^N) \rightarrow W_{p,s}(\mathbb{R}^N)$$

*is bounded with norm  $N_p(K)$ , then the transferred operator*

$$T_K : W_{p,r}(\overline{\mathbb{R}^N}) \rightarrow W_{p,s}(\overline{\mathbb{R}^N})$$

*is given by  $T_K(\sum a_t e^{2\pi i t x}) = \sum_t m(t) a_t e^{2\pi i t x}$  and is bounded with norm less than or equal to  $C_{r,s} N_p(K)$ .*

**THEOREM A.3.** *Let  $1 \leq p, r, s \in \mathbb{N}$ . If  $m \in L_{\text{loc}}^\infty$  is a normalized function such that for  $K = \check{m}$  the operator*

$$K* : W_{p,r}(\mathbb{R}^N) \rightarrow W_{p,s}(\mathbb{R}^N)$$

*is bounded with norm  $N_p(K)$ , and  $\overline{K}$  is a convolution kernel on  $\mathbb{R}^M$  with  $M < N$  and  $\widehat{\overline{K}}(x) = m(\overline{x}, 0)$  where  $x = (\overline{x}, x^*) \in \mathbb{R}^M \times \mathbb{R}^{N-M}$ , then the operator*

$$\overline{K}* : W_{p,r}(\mathbb{R}^M) \rightarrow W_{p,s}(\mathbb{R}^M)$$

*is bounded with norm less than or equal to  $C_{r,s} N_p(K)$ .*

**PROOF.** Observe that  $W_{p,r}(\mathbb{R}^M)$  is the transferred space of  $W_{p,r}(\mathbb{R}^N)$  under the representation of Example 2.2 (8), and argue as in Theorem A.1. ■

**B. Hardy spaces ( $p = 1$ ).** Now assume that  $K$  is a function in  $L^1(\mathbb{R}^N)$  such that

$$(5) \quad K* : H^1(\mathbb{R}^N) \rightarrow H^1(\mathbb{R}^N)$$

is bounded with norm  $N_1(K)$ . The previous argument cannot be applied to this case because we cannot find an approximation of the identity  $\varphi_n$  such that  $H\varphi$  is in  $L^1$  and  $\int \varphi = 1$ . However, we obtain the following result (see [C]).

**THEOREM B.1.** *If  $K$  is such that  $\widehat{K} = m$  is a normalized function and*

$$K* : H^1(\mathbb{R}^N) \rightarrow H^1(\mathbb{R}^N)$$

*is bounded with norm  $N_1(K)$ , then the transferred operator*

$$T_K \left( \sum a_n e^{2\pi i n x} \right) = \sum a_n m(n) e^{2\pi i n x}$$



can be extended to a bounded operator from  $H^1(\mathbb{T}^N)$  into  $H^1(\mathbb{T}^N)$  with norm less than or equal to  $N_1(K)$ .

*Proof.* First assume that  $K \in L^1$  and  $N = 1$  (a similar proof works for  $N > 1$ ).

Let  $P$  be a trigonometric polynomial of degree  $j$  such that  $P(0) = 0$ . Let  $\phi \in H^1(\mathbb{R})$  be such that  $\widehat{\phi}(n) = 1$  for every  $0 < |n| \leq j$ . Then both  $K * \phi$  and  $H\phi$  are functions in  $L^1(\mathbb{R})$  and therefore

$$\|T_{K*\phi}P\|_{H^1(\mathbb{T})} \leq N_1(K)(\|T_\phi P\|_1 + \|T_{H\phi}P\|_1).$$

Since  $T_{K*\phi} = T_K T_\phi$ ,  $T_\phi P = P$  and  $T_{H\phi} = T_H T_\phi$ , we obtain the desired result.

Finally, every convolution kernel on  $H^1(\mathbb{R})$  is also a convolution kernel on  $L^2(\mathbb{R})$  and therefore  $m \in L^\infty(\mathbb{R})$ . Moreover,  $\|m\|_\infty \leq N_1(K)$ . Hence, if  $a(x) = 1$ , then  $(T_K)a(x) = m(0)$  and thus

$$\|T_K a\|_{H^1(\mathbb{T})} = |m(0)| \leq \|m\|_\infty \leq N_1(K).$$

To consider the general case  $K \notin L^1$ , we need the following technical lemma.

**LEMMA.** *If  $K*$  is a convolution operator on  $H^1(\mathbb{R}^N)$  with norm  $N_1(K)$ , then there exists a sequence  $(K_n)_n$  of compactly supported functions in  $L^1(\mathbb{R}^N)$  such that  $m_n(\xi) = \widehat{K}_n(\xi) \rightarrow m(\xi)$  for every  $\xi \in \mathbb{R}^N$  and  $N_1(K_n) \leq N_1(K)$ .*

*Proof.* We prove this for  $N = 1$ . The general case is similar. First, we know that  $m$  is a continuous function on  $\mathbb{R} \setminus \{0\}$ . Let  $\varphi \in \mathcal{S}(\mathbb{R})$  with compact support and  $\varphi(\xi) = 1$  for every  $\xi \in [-1, 1]$ . Define  $\varphi^k(x) = \varphi(x/k)$  and  $\varphi_k(x) = \varphi(kx)$ . Set  $m_k(x) = m(x)\varphi^k(x)(1 - \varphi_k(x))$ . Then  $m_k(x) \rightarrow m(x)$  as  $k \rightarrow \infty$  for every  $x \neq 0$ , and  $m_k$  is a multiplier on  $H^1(\mathbb{R})$  with norm less than or equal to  $CN_1(K)$ , with  $C$  only depending on  $\varphi$ .

Choose  $\Psi \in \mathcal{S}(\mathbb{R})$  with compact support such that  $\Psi(0) = 1$  and set  $\Psi^n(\xi) = \Psi(\xi/n)$ . Let  $\phi_n(\xi) = e^{-2\pi i \xi} \Psi^n(\xi)$ , and consider

$$m_{n,k}(x) = \int_{-\infty}^{\infty} \frac{x}{s} \widehat{\phi}_n\left(\frac{x}{s}\right) m_k(s) \frac{ds}{s} = \int_{-\infty}^{\infty} t \widehat{\phi}_n(t) m_k\left(\frac{x}{t}\right) \frac{dt}{t}.$$

Then

$$\begin{aligned} m_{n,k}(x) - m_k(x) &= \int_{-\infty}^{\infty} t \widehat{\phi}_n(t) \left( m_k\left(\frac{x}{t}\right) - m_k(x) \right) \frac{dt}{t} \\ &= \int_{-\infty}^{1-\delta} + \int_{1-\delta}^{1+\delta} + \int_{1+\delta}^{\infty}, \end{aligned}$$

with  $\delta$  to be chosen.

Now, using the decay of  $\widehat{\phi}_n$  we obtain

$$\left| \int_{-\infty}^{1-\delta} t \widehat{\phi}_n(t) \left( m_k \left( \frac{x}{t} \right) - m_k(x) \right) \frac{dt}{t} \right| \leq C \|m\|_\infty \frac{1}{n^{M-1}} \int_{-\infty}^{1-\delta} \frac{1}{|t-1|^M} dt,$$

and the above expression converges to zero whenever  $M$  is large enough and  $n$  tends to infinity. Similarly for  $\int_{1+\delta}^{\infty}$ . For the second term we use the continuity of  $m_k$  to deduce that given  $\varepsilon$  there exists  $\delta$  such that for every  $t \in (1-\delta, 1+\delta)$ ,  $|m_k(x/t) - m_k(x)| \leq \varepsilon$  and hence

$$\left| \int_{1-\delta}^{1+\delta} t \widehat{\phi}_n(t) \left( m_k \left( \frac{x}{t} \right) - m_k(x) \right) \frac{dt}{t} \right| \leq C\varepsilon \int_{-\infty}^{\infty} |\widehat{\phi}_n(t)| dt = C\varepsilon.$$

Now, since

$$K_{n,k}(x) = \widetilde{m}_{n,k}(x) = \int_{-\infty}^{\infty} \phi'_n(sx) m_k(s) ds,$$

$\phi_n$  has compact support and  $m_k(s) = 0$  in a neighborhood of zero and for  $s$  large enough, we infer that  $K_{n,k}$  has compact support and obviously is in  $L^1(\mathbb{R})$ . Finally,

$$\begin{aligned} \|K_{n,k} * f\|_1 &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} m_{n,k}(\xi) \widehat{f}(\xi) e^{-2\pi i x \xi} d\xi \right| dx \\ &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} t \widehat{\phi}_n(t) m_k(\xi/t) \frac{dt}{t} \right] \widehat{f}(\xi) e^{-2\pi i x \xi} d\xi \right| dx \\ &= \int_{\mathbb{R}} \left| \widehat{\phi}_n(t) \left[ \int_{\mathbb{R}} m_k(\xi/t) \widehat{f}(\xi) e^{-2\pi i x \xi} d\xi \right] \right| dt dx \\ &\leq \int_{\mathbb{R}} |\widehat{\phi}_n(t)| \int_{\mathbb{R}} \left| \int_{\mathbb{R}} m_k(\xi/t) \widehat{f}(\xi) e^{-2\pi i x \xi} d\xi \right| dx dt \\ &\leq \int_{\mathbb{R}} |\widehat{\phi}_n(t)| \int_{\mathbb{R}} \left| \int_{\mathbb{R}} m_k(y) t \widehat{f}(ty) e^{-2\pi i x ty} dy \right| dx dt \\ &\leq N_1(K_k) \|f\|_{H^1(\mathbb{R})} \leq CN_1(K) \|f\|_{H^1(\mathbb{R})}, \end{aligned}$$

and hence,  $K_{n,k}$  satisfies the lemma. ■

The proof of Theorem B.1 now follows by standard approximation arguments. ■

Similarly, we get

**THEOREM B.2.** *If  $m$  is a normalized function such that for  $K = \widetilde{m}$  the operator*

$$K* : H^1(\mathbb{R}^N) \rightarrow H^1(\mathbb{R}^N)$$

is bounded with norm  $N_1(K)$ , then the operator

$$T_K : H^1(\overline{\mathbb{R}^N}) \rightarrow H^1(\overline{\mathbb{R}^N})$$

defined by  $T_K(\sum a_t e^{2\pi i t x}) = \sum_t m(t) a_t e^{2\pi i t x}$  is bounded with norm less than or equal to  $N_1(K)$ .

**THEOREM B.3.** *If  $m$  is a normalized function such that for  $K = \check{m}$  the operator*

$$K* : H^1(\mathbb{R}^N) \rightarrow H^1(\mathbb{R}^N)$$

*is bounded with norm  $N_1(K)$ , and  $\overline{K}$  is a convolution kernel on  $\mathbb{R}^M$  with  $M < N$  and  $\widehat{\overline{K}}(x) = m(\overline{x}, 0)$  where  $x = (\overline{x}, x^*) \in \mathbb{R}^M \times \mathbb{R}^{N-M}$ , then the operator*

$$\overline{K}* : H^1(\mathbb{R}^M) \rightarrow H^1(\mathbb{R}^M)$$

*is bounded with norm less than or equal to  $N_1(K)$ .*

If we want to use the techniques of Theorem B.1 to cover the case of Example 2.2(4), that is, to transfer the boundedness of a convolution operator from  $H^1(\mathbb{R})$  to an ergodic Hardy space  $H^1(\mathcal{M})$ , we observe that, in general, it is not the case that, for every  $f$  in a dense set of  $H^1(\mathcal{M})$ , there exists  $\varphi \in H^1(\mathbb{R})$  such that  $T_\varphi f = f$  with  $T_\varphi$  the transference operator of the convolution operator  $\varphi*$ . Therefore, we can only show that, if  $N_1(K)$  is the norm of the convolution operator  $K*$  in  $H^1(\mathbb{R})$ , then, for every  $\varphi \in H^1(\mathbb{R})$ ,

$$\|T_{K*\varphi} f\|_{H^1(\mathcal{M})} \leq N_1(K)(\|T_\varphi f\|_{L^1(\mathcal{M})} + \|T_\varphi T_H f\|_{L^1(\mathcal{M})}),$$

where  $T_H$  is the transference operator of the Hilbert transform. From this, we can deduce that if  $m = \check{K}$  has compact support away from zero, then

$$\|T_K f\|_{H^1(\mathcal{M})} \leq N_1(K)\|\varphi\|_1(\|f\|_{L^1(\mathcal{M})} + \|T_H f\|_{L^1(\mathcal{M})}),$$

since, in this case, there exists  $\varphi \in H^1(\mathbb{R})$  such that  $K*\varphi = K$ .

**4. Case of a finite family of kernels and  $0 < p < 1$ .** Now consider a  $\sigma$ -finite measure space  $(\mathcal{M}, dx)$  and let  $R$  be a representation of  $G$  on  $L^p(\mathcal{M})$  and on  $L^1(\mathcal{M})$  such that  $R$  is uniformly bounded; that is, there exist constants  $A$  and  $B$  such that, for every  $f \in L^p(\mathcal{M})$  and every  $u \in G$ ,

$$\|R_u f\|_{L^p(\mathcal{M})} \leq A\|f\|_{L^p(\mathcal{M})},$$

and, for every  $f \in L^1(\mathcal{M})$ ,

$$\|R_u f\|_{L^1(\mathcal{M})} \leq B\|f\|_{L^1(\mathcal{M})}.$$

Under this last condition, the transferred operator  $T_K$  is well defined in a dense subset of the transferred space.

We observe that in this case the boundedness of  $T_K$  is not trivial even in the case of  $K \in L^1$  with compact support since the Minkowski integral inequality does not hold.

This section is organized as follows: first we prove the transference theorem if one of the following conditions holds:

- (a)  $G$  is compact.
- (b)  $G$  is discrete.
- (c)  $\mathcal{M}$  is of finite measure.

Then, if  $G$  and  $\mathcal{M}$  are either  $\mathbb{R}^N$ ,  $\mathbb{Z}^m$  or  $\mathbb{T}^k$ , we can transfer as in the following diagram:

$$\mathbb{Z}^m \rightarrow \mathbb{R}^N \leftrightarrow \mathbb{T}^k \leftrightarrow \mathbb{Z}^m$$

and hence it remains to transfer from  $\mathbb{R}^N$  to  $\mathbb{Z}^m$ , or more generally from  $\mathbb{R}^N$  to any measure space  $\mathcal{M}$ .

The next step will be to show that under some conditions on the representation we can transfer from  $\mathbb{R}^N$  to any measure space  $\mathcal{M}$  either via the factorization

$$\mathbb{R}^N \rightarrow \mathbb{T}^k \rightarrow \mathcal{M}$$

and/or using the dilation structure of the group  $\mathbb{R}^N$ .

**THEOREM 4.1.** *Let  $G$  be either a compact or a discrete abelian group, or let  $\mathcal{M}$  be of finite measure, and let  $0 < p < 1$ . Let  $K$ ,  $\{K_i^1\}_{i=1,\dots,n}$  and  $\{K_j^2\}_{j=1,\dots,m}$  be a collection of functions in  $L^1(G)$  with compact support and assume that*

$$K * : H^p(\{K_i^1\}_{i=1,\dots,n}) \rightarrow H^p(\{K_j^2\}_{j=1,\dots,m})$$

*has the property that there exist positive constants  $\{A_i\}$  such that*

$$\sum_{j=1}^m \|K_j^2 * K * f\|_p \leq \sum_{i=1}^n A_i \|K_i^1 * f\|_p.$$

*Then the transferred operator*

$$T_K : H^p(\{T_{K_i^1}\}_i) \rightarrow H^p(\{T_{K_j^2}\}_j)$$

*is bounded, with*

$$\sum_{j=1}^m \|T_{K_j^2} T_K f\|_p \leq D A^2 \sum_{i=1}^n A_i \|T_{K_i^1} f\|_p,$$

*where  $A$  is as in (1) and  $D$  depends only on  $n$  and  $m$ .*

**Proof.** As in Theorem 3.1, we prove this for  $m = 1$ .

(a) Assume first that  $G$  is compact. Then we proceed as in Theorem 3.1 but, in this case, we can take  $V = G$  and then the term  $II$  is zero.

(b) If  $G$  is discrete, and we argue as in Theorem 3.1, it remains to show that  $II/\mu(V)$  can be made small enough. Now, since  $p < 1$ ,

$$\begin{aligned} II &= \int_{\mathcal{M}} \left[ \int_{VC^{-1}V \setminus V} \int_G |K_1(u) \chi_{VC^{-1}}(vu^{-1})(R_{vu^{-1}}f)(x) du|^p dv \right] dx \\ &\leq \int_{\mathcal{M}} \left[ \int_{VC^{-1}C \setminus V} \int_G |K_1(u)|^p |(R_{vu^{-1}}f)(x)|^p du dx \right] dv \\ &\leq \int_G |K_1(u)|^p \int_{VC^{-1}C \setminus V} \|R_{vu^{-1}}f\|_p^p dv du \\ &\leq A^p \|f\|_p^p \|K_1\|_1^p \mu(C)^{1/(1/p)'} \mu(VC^{-1}C \setminus V), \end{aligned}$$

and hence we can choose  $V$  in such a way that  $II/\mu(V)$  is arbitrarily small.

(c) If  $\mathcal{M}$  is of finite measure, and we assume that  $R$  acts on  $L^1(\mathcal{M})$ , then

$$\begin{aligned} II &\leq (m(\mathcal{M})\mu(VC^{-1}C \setminus V))^{1/(1/p)'} \\ &\quad \times \left[ \int_{\mathcal{M}} \left[ \int_{VC^{-1}C \setminus V} \int_G |K_1(u) \chi_{VC^{-1}}(vu^{-1})(R_{vu^{-1}}f)(x)| du dv \right] dx \right]^p \\ &\leq (m(\mathcal{M})\mu(VC^{-1}C \setminus V))^{1/(1/p)'} \mu(VC^{-1}V \setminus V)^p \|K_1\|_1^p \|R_{vu^{-1}}f\|_1^p \\ &\leq m(\mathcal{M})\mu(VC^{-1}C \setminus V) \|K_1\|_1^p B^p \|f\|_1^p, \end{aligned}$$

and this expression converges to zero on choosing  $V$  appropriately. ■

*Transference from  $\mathbb{R}^N$  to  $\mathcal{M}$ .* Let us now consider the case of transference from  $\mathbb{R}^N$  to a general measure space  $\mathcal{M}$ . Let  $R$  be a representation from  $\mathbb{R}^N$  into  $L^p(\mathcal{M})$ . Assume that one of the following two conditions hold:

(i) For every  $f$  in a dense subset of  $H^p(\{T_{K_i}\}_i)$ , there exists  $M > 0$  such that  $R_M f = f$ . Then, if we define  $(R_\theta^M f)(x) = (R_{M\theta} f)(x)$  for  $\theta \in [-1/2, 1/2]^N = \mathbb{T}^N$ , we find that  $R^M$  is a uniformly bounded representation of  $\mathbb{T}^N$  in  $L^p(\mathcal{M})$ .

If  $(R_M f)(x) = f(S_M x)$ , then  $M$  may also depend on  $f$  and  $x$ .

(ii) For every  $f$  in a dense subset of  $H^p(\{T_{K_i}\}_i)$ , there exist  $C > 0$  and  $M_0 > 0$  such that, for every  $M \geq M_0$ ,

$$\int_{\mathcal{M}} \left( \frac{1}{M^N} \int_{(-M, M)^N} |(R_u f)(x)| du \right)^p dx \leq C.$$

In the first case, consider a kernel  $K \in L^1(\mathbb{R}^N)$  and set  $K_M(x) = M^{-N} K(x/M)$ . Let

$$\tilde{K}_M(\theta) = \sum_{m \in \mathbb{Z}^N} K_M(\theta + m)$$

be the periodic extension. Then for the transferred operator we have

$$\begin{aligned}
(T_{\widetilde{K}_M}^{R^M} f)(x) &= \int_{\mathbb{T}^N} \widetilde{K}_M(\theta) (R_{-\theta}^M f)(x) d\theta \\
&= \int_{\mathbb{T}^N} M^N \sum_m K(M\theta + Mm) (R_{-M\theta} f)(x) d\theta \\
&= \int_{[-M/2, M/2]^N} \sum_m K(u + Mm) (R_{-u} f)(x) du \\
&= \int_{[-M/2, M/2]^N} K(u) (R_{-u} f)(x) du \\
&\quad + \int_{[-M/2, M/2]^N} \sum_{m \neq 0} K(u + Mm) (R_{-u} f)(x) du \\
&= I_M + II_M.
\end{aligned}$$

Now, since the representation  $R$  acts on  $L^1(\mathcal{M})$  we see that, by the Minkowski integral inequality,

$$\begin{aligned}
\|II_M\|_1 &\leq A \|f\|_1 \int_{[-M/2, M/2]^N} \sum_{m \neq 0} |K(u + Mm)| du \\
&= A \|f\|_1 \int_{|u| \geq M/2} |K(u)| du,
\end{aligned}$$

and therefore  $\|II_M\|_1$  converges to zero as  $M$  tends to infinity. Therefore, there exists a subsequence  $M_k$  such that  $II_{M_k}$  converges to zero almost everywhere. Since  $I_M$  converges to the transferred operator  $T_K^R$ , we get

$$(T_K^R f)(x) = \lim_k (T_{\widetilde{K}_{M_k}}^{R^{M_k}} f)(x).$$

From this, we can deduce the following result,

**THEOREM 4.2.** *Let  $G = \mathbb{R}^N$  and let  $\mathcal{M}$  be a  $\sigma$ -finite measure space. Let  $0 < p < 1$ . Let  $K$ ,  $\{K_i^1\}_{i=1, \dots, n}$  and  $\{K_j^2\}_{j=1, \dots, m}$  be a collection of functions in  $L^1(G)$  with compact support and assume that*

$$K * : H^p(\{K_i^1\}_{i=1, \dots, n}) \rightarrow H^p(\{K_j^2\}_{j=1, \dots, m})$$

*has the property that there exist positive constants  $\{A_i\}_i$  such that*

$$\sum_{j=1}^m \|K_j^2 * K * f\|_p \leq \sum_{i=1}^n A_i \|K_i^1 * f\|_p.$$

*If the representation  $R$  satisfies condition (i) then the transferred operator*

$$T_K : H^p(\{T_{K_i^1}\}_i) \rightarrow H^p(\{T_{K_j^2}\}_j)$$

is bounded, with

$$\sum_{j=1}^m \|T_{K_j^2} T_K f\|_p \leq CA \sum_{i=1}^n A_i \|T_{K_i^1} f\|_p,$$

where  $A$  is as in (1) and  $C$  depends only on  $n$  and  $m$ .

Proof. As always, take  $m = 1$  and  $H^p(\{K_j^2\}_{j=1,\dots,m}) = L^p$ . Using the dilation structure of  $\mathbb{R}^N$  we see that, for every  $M > 0$ ,

$$\|K_M * f\|_p \leq \sum_{i=1}^n A_i \|(K_i^1)_M * f\|_p.$$

Since  $\mathbb{T}^N$  is a measure space of finite measure, we can apply Theorem 4.1 to deduce that we can transfer the boundedness of  $K_M *$  to  $L^p(\mathbb{T}^N)$  via the natural representation  $(S_u f)(\theta) = f(\theta - u)$ . Hence, for every  $M$ ,  $T_{K_M}^S$  is a bounded operator with

$$\|T_{K_M}^S f\|_p \leq CA^2 \sum_i A_i \|T_{(K_i^1)_M} f\|_p.$$

But, since  $K$  and  $K_i^1$  have compact support, for  $M$  large enough we have  $\tilde{K}_M(x) = K_M(x)$  for every  $x \in \mathbb{T}^N$  and similarly for the kernels  $K_i^1$ . Now, since  $T_{\tilde{K}_M}^S = \tilde{K}_M *$ , we see that if we take  $M$  large enough such that this condition holds and also that  $R_M f = f$ , we get

$$\begin{aligned} \|T_K f\|_p^p &= \left\| \int_{\mathbb{R}^N} K(y)(R_{-y} f)(\cdot) dy \right\|_p^p = \left\| \int_{\mathbb{T}^N} K_M(y)(R_{-y}^M f)(\cdot) dy \right\|_p^p \\ &= \left\| \int_{\mathbb{T}^N} \tilde{K}_M(y)(R_{-y}^M f)(\cdot) dy \right\|_p^p \leq \sum_i A_i^p \left\| \int_{\mathbb{T}^N} (K_i^1)_M(y)(R_{-y}^M f)(\cdot) dy \right\|_p^p \\ &= \sum_i A_i^p \left\| \int_{\mathbb{R}^N} K_i^1(y)(R_{-y} f)(\cdot) dy \right\|_p^p. \blacksquare \end{aligned}$$

**THEOREM 4.3.** *Under the hypothesis of Theorem 4.2, if the representation  $R$  satisfies condition (ii), then the transferred operator*

$$T_K : H^p(\{T_{K_i^1}\}_i) \rightarrow H^p(\{T_{K_j^2}\}_j)$$

is bounded, with

$$\sum_{j=1}^m \|T_{K_j^2} T_K f\|_p \leq CA^2 \sum_{i=1}^n A_i \|T_{K_i^1} f\|_p,$$

where  $A$  is as in (1) and  $C$  depends only on  $n$  and  $m$ .

Proof. We follow the same steps as for Theorem 3.1.

Let  $f \in L^p(\mathcal{M})$ . Take  $M$  large enough such that the supports of the functions  $K_M$  and  $(K_i^1)_M = (K_i)_M$  are contained in  $(-\varepsilon, \varepsilon)^N$  for  $\varepsilon > 0$  and

(ii) holds. Then, if  $V = (-1, 1)^N$ , we get

$$\begin{aligned}
& \|T_K f\|_p^p \\
&= \left\| \int_{\mathbb{R}^N} K(y)(R_{-y}f)(\cdot) dy \right\|_p^p = \left\| \int_{(-\varepsilon, \varepsilon)^N} K_M(y)(R_{-y}^M f)(\cdot) dy \right\|_p^p \\
&\leq \frac{A^p}{\mu(V)} \int_V \int_{\mathcal{M}} \left| \int_{(-\varepsilon, \varepsilon)^N} K_M(y)(R_{v-y}^M f)(x) dy \right|^p dx dv \\
&= \frac{A^p}{\mu(V)} \int_V \int_{\mathcal{M}} \left| \int_{(-\varepsilon, \varepsilon)^N} K_M(y) \chi_{(-1-\varepsilon, 1+\varepsilon)^N}(v-y)(R_{v-y}^M f)(x) dy \right|^p dx dv \\
&\leq \frac{A^p}{\mu(V)} \int_{\mathcal{M}} \left[ \int_{(-1, 1)^N} \left| \int_{(-\varepsilon, \varepsilon)^N} K_M(y) \chi_{(-1-\varepsilon, 1+\varepsilon)^N}(v-y)(R_{v-y}^M f)(x) dy \right|^p dv \right] dx \\
&\leq \frac{A^p C}{\mu(V)} \int_{\mathcal{M}} \sum_i A_i^p \|\chi_{(-1-\varepsilon, 1+\varepsilon)^N}(R_{\cdot}^M f)(x)\|_{H^p((K_i)_M)}^p dx.
\end{aligned}$$

Now, if we take  $h_x(y) = \chi_{(-1-\varepsilon, 1+\varepsilon)^N}(y)(R_y^M f)(x)$  and  $V_\varepsilon = (-1 - 2\varepsilon, 1 + 2\varepsilon)^N \setminus (-1, 1)^N$ , we get

$$\begin{aligned}
& \int_{\mathcal{M}} \|(K_i)_M * h_x\|_{L^p(\mathbb{R}^N)}^p dx \\
&= \int_{\mathcal{M}} \left[ \int_{\mathbb{R}^N} \left| \int_{(-\varepsilon, \varepsilon)^N} (K_i)_M(y) \chi_{(-1-\varepsilon, 1+\varepsilon)^N}(v-y)(R_{v-y}^M f)(x) dy \right|^p dv \right] dx \\
&\leq \int_{\mathcal{M}} \left[ \int_{(-1, 1)^N} \left| \int_{(-\varepsilon, \varepsilon)^N} (K_i)_M(y)(R_{v-y}^M f)(x) dy \right|^p dv \right] dx \\
&\quad + \int_{\mathcal{M}} \left[ \int_{V_\varepsilon} \left| \int_{(-\varepsilon, \varepsilon)^N} |(K_i)_M(y)| \cdot |(R_{v-y}^M f)(x)| dy \right|^p dv \right] dx \\
&= I + II.
\end{aligned}$$

To estimate  $I$  we proceed as in Theorem 3.1, and for the second term,

$$\begin{aligned}
II &\leq |V_\varepsilon|^{1-p} \int_{\mathcal{M}} \left[ \int_{V_\varepsilon} \int_{(-\varepsilon, \varepsilon)^N} |(K_i)_M(y)| \cdot |(R_{v-y}^M f)(x)| dy dv \right]^p dx \\
&\leq C\varepsilon^N \|K_i\|_1^p \int_{\mathcal{M}} \left( \int_{(-2, 2)^N} |(R_y^M f)(x)| dy \right)^p dx \\
&= C\varepsilon^N \|K_i\|_1^p \int_{\mathcal{M}} \left( \frac{1}{M^N} \int_{(-2M, 2M)^N} |(R_y f)(x)| dy \right)^p dx \leq C(f) \|K_i\|_1^p \varepsilon^N.
\end{aligned}$$

Letting  $\varepsilon$  tend to zero, we are done. ■



For the examples, it will be very convenient to get rid of the hypothesis of  $K$  being with compact support. Because of the lack of the Minkowski integral inequality, we cannot argue as in the case  $p \geq 1$ . However, we are going to show that whenever  $\mathcal{M}$  is of finite measure we do not need that condition on  $K$ . Then we shall prove that under certain conditions on the representation we can restrict ourselves to this case.

Assume then that  $\mathcal{M}$  is of finite measure. Then, if  $K_n$  is a sequence of functions in  $L^1(G)$  such that  $K_n$  has compact support and  $K_n$  converges to  $K$  in the  $L^1$  norm, we have

$$\begin{aligned} \|T_K f\|_p^p &\leq \|T_{K-K_n} f\|_p^p + \|T_{K_n} f\|_p^p \leq D \|T_{K-K_n} f\|_1^p + \|T_{K_n} f\|_p^p \\ &\leq D\varepsilon \|f\|_1 + \|T_{K_n} f\|_p^p. \end{aligned}$$

Now,

$$\begin{aligned} \|T_{K_n} f\|_p^p &\leq D \frac{1}{\mu(V)} \int_V \int_{\mathcal{M}} \left| \int_G K_n(u) \chi_{VC^{-1}}(vu^{-1})(R_{vu^{-1}} f)(x) du \right|^p dx dv \\ &\leq D \frac{1}{\mu(V)} \left[ \int_{\mathcal{M}V} \int_G \left| \int_G (K_n - K)(u) \chi_{VC^{-1}}(vu^{-1})(R_{vu^{-1}} f)(x) du \right|^p dv dx \right. \\ &\quad \left. + \int_{\mathcal{M}V} \int_G \left| \int_G K(u) \chi_{VC^{-1}}(vu^{-1})(R_{vu^{-1}} f)(x) du \right|^p dv dx \right] \\ &\leq D \frac{1}{\mu(V)} \mu(V)^{1-p} \|K_n - K\|_1^p \left( \int_{\mathcal{M}} \left| \int_{VC^{-1}} (R_u f)(x) du \right| dx \right)^p \\ &\quad + D \frac{1}{\mu(V)} \sum_i A_i^p \int_{\mathcal{M}G} \int_G \left| \int_G K_i(u) \chi_{VC^{-1}}(vu^{-1})(R_{vu^{-1}} f)(x) du \right|^p dv dx. \end{aligned}$$

Following the ideas in Theorem 3.1 and using the fact that, by density, we can consider  $f \in L^1(\mathcal{M})$ , we get the result by letting  $\varepsilon$  tend to zero.

**DEFINITION 4.4.** We say that  $R$  acts locally on  $L^p(\mathcal{M})$  if the following condition holds: Given a compact set  $C$ , and given  $\varepsilon > 0$ , there exists  $V$  such that  $\mu(VC^{-1}) \leq (1 + \varepsilon)\mu(V)$  and, for every finite family  $\{K_i\}_i$  of kernels in  $L^1$ , there exists a positive constant  $B$  such that, given any measurable set  $M$  in  $\mathcal{M}$  of finite measure and given any  $u \in G$ , there exists a measurable set  $M_u$  such that  $\|R_u f\|_{L^p(M)} \leq B \|f\|_{L^p(M_u)}$  for every  $f$  in a dense subset of  $H^p(T_{K_i})$  and, for every neighborhood  $V$  of the identity there exists another measurable set  $M_V$  such that  $M_v \subset M_V$  for every  $v \in V$  and

$$\frac{|M_V|^{1-p} \mu(VC^{-1}C \setminus V)}{\mu(V)} \leq \varepsilon.$$

In this case, we can reduce ourselves to the case of  $\mathcal{M}$  of finite measure

and therefore we do not need the hypothesis of  $K$  being of compact support. To see this we just have to start computing  $\|T_K f\|_{L^p(M)}$  for any  $M$  of finite measure. Then

$$\begin{aligned} \|T_K f\|_{L^p(M)}^p &\leq A^p \frac{1}{\mu(V)} \int_V \|R_v T_K f\|_{L^p(M_{v^{-1}})}^p dv \\ &\leq A^p \frac{1}{\mu(V)} \int_V \|R_v T_K f\|_{L^p(M_V)}^p dv, \end{aligned}$$

and the rest of the proof follows as usual.

One can easily check that if  $R$  is the representation of Example 2.2(8), then  $R$  acts locally on  $L^p(\mathbb{R}^m)$ , and therefore, we can transfer from  $\mathbb{R}^N$  to  $\mathbb{R}^m$  ( $m < N$ ) with  $0 < p < 1$ .

**C. Hardy spaces** ( $p < 1$ ). Let  $K$  be such that  $m = \check{K}$  is a normalized function with  $m(0) = 0$ . Assume that

$$\|K * f\|_p \leq C \|Hf\|_p,$$

for some  $p < 1$ . As in Theorem B.1, let  $P$  be a trigonometric polynomial of degree  $j$  such that  $P(0) = 0$ . Let  $\phi \in H^1(\mathbb{R}^N)$  be such that  $\hat{\phi}(n) = 1$  for every  $0 < |n| \leq j$ . Take  $\phi_n$  converging to  $\phi$  in the  $L^1$  norm and such that  $H\phi_n$  has compact support for every  $n$ . Then  $K * \phi_n$  and  $H\phi_n$  are functions in  $L^1(\mathbb{R}^N)$  and the latter has compact support. Therefore

$$\|T_{K * \phi_n} P\|_{H^p(\mathbb{T}^N)} \leq C \|T_{H\phi_n} P\|_p.$$

But, since  $m$  is normalized, we have  $T_{K * \phi_n} = T_K T_{\phi_n}$ . Taking the limit as  $n \rightarrow \infty$  and using the fact that  $T_\phi P = P$ , we get the following result:

**PROPOSITION C.1.** *Let  $K$  be such that  $m = \check{K}$  is a normalized function with  $m(0) = 0$ . If  $\|K * f\|_p \leq C \|Hf\|_p$ , then*

$$\left\| \sum m(n) a_n e^{2\pi i n x} \right\|_{L^p(\mathbb{T})} \leq C \left\| \sum \operatorname{sgn}(n) a_n e^{2\pi i n x} \right\|_{L^p(\mathbb{T})}.$$

**5. Maximal operators and maximal spaces.** In this section, we consider the case where the operator  $S$  is determined by an infinite collection of kernels  $K_i$  in  $L^1(G)$  with compact support; namely  $Sf = \sup_i |K_i * f(0)|$ . In this case, we write  $H^p(S) = H^p(\{K_i\}_i)$ .

Hence, if we have two collections of functions satisfying the above conditions,  $\{K_i^1\}_i$  and  $\{K_j^2\}_j$ , and  $K \in L^1(G)$  has the property that the convolution operator

$$K * : H^p(\{K_i^1\}_i) \rightarrow H^p(\{K_j^2\}_j)$$

is bounded with norm less than or equal to  $N_p(K)$ , then the maximal operator

$$\sup_j |K_j^2 * | : H^p(\{K_i^1\}_i) \rightarrow L^p(G)$$

is bounded with norm less than or equal to  $N_p(K)$ .

Therefore, we can reduce ourselves to the case of a maximal operator acting on a maximal space. Obviously this maximal space can be  $L^p(G)$  and then our case will include the maximal transference of [ABG1]. For that reason, throughout this section we consider only representations such that

- (i)  $R_u$  is separation-preserving for every  $u \in G$ ,
- (ii) there exists  $B$  such that  $\|R_u f\|_{H^p(\{T_{K_i^1}\}_i)} \leq B \|f\|_{H^p(\{T_{K_i^1}\}_i)}$  for every  $u \in G$  and every  $f \in L^p(\mathcal{M})$ , and
- (iii) if  $0 < p < 1$ , then the representation  $R$  also acts into  $L^1(\mathcal{M})$ .

**THEOREM 5.1.** *Let  $G$  be a compact abelian group and let  $0 < p < \infty$ . Let  $K_j^2$  and  $K_i^1$  be two collections of functions in  $L^1(G)$  and let  $N_p(K)$  be the norm of the convolution operator*

$$\sup_j |K_j^2 * | : H^p(\{K_i^1\}_i) \rightarrow L^p(G).$$

If  $R$  is a representation from  $G$  into  $L^p(\mathcal{M})$  satisfying (i)–(iii), then the transferred operator

$$\sup_j |T_{K_j^2}| : H^p(\{T_{K_i^1}\}_i) \rightarrow L^p(\mathcal{M})$$

is bounded, with norm less than or equal to  $ABN_p(K)$ , where  $A$  is as in (1) and  $B$  as in (ii).

**Proof.** Since  $R$  is separation-preserving, we get (see [ABG1])

$$R_v(\sup_j |(T_{K_j^2} f)(x)|) \leq \sup_j |(T_{K_j^2} R_v f)(x)|,$$

and therefore

$$\begin{aligned} \|\sup_j |T_{K_j^2} f|\|_p^p &\leq A^p \int_G \|\sup_j |T_{K_j^2} R_v f|\|_p^p dv \\ &= A^p \int_G \int_{\mathcal{M}} \sup_j \left| \int_G K_j^2(u) (R_{vu^{-1}} f)(x) du \right|^p dx dv \\ &\leq A^p \int_{\mathcal{M}} \left[ \int_G \sup_j \left| \int_G K_j^2(u) (R_{vu^{-1}} f)(x) du \right|^p dv \right] dx \\ &\leq (AN_p(K))^p \int_{\mathcal{M}} \|(R.f)(x)\|_{H^p(\{K_i^1\}_i)}^p dx, \end{aligned}$$

where the last inequality follows by applying the hypothesis to the function

$$h_x(u) = (R_u f)(x).$$

The last step is to show that

$$\int_{\mathcal{M}} \|(R.f)(x)\|_{H^p(\{K_i^1\}_i)}^p dx \leq B^p \|f\|_{H^p(\{T_{K_i^1}\}_i)}^p.$$

Now,

$$\begin{aligned} \int_{\mathcal{M}} \|(R.f)(x)\|_{H^p(\{K_i^1\}_i)}^p dx &= \int_{\mathcal{M}} \left[ \int_G \sup_i \left| \int_G K_i^1(u) (R_{vu^{-1}} f)(x) du \right|^p dv \right] dx \\ &\leq \int_G \left[ \int_{\mathcal{M}} \sup_i \left| \int_G K_i^1(u) (R_{u^{-1}} R_v f)(x) du \right|^p dx \right] dv \\ &= \int_G \|R_v f\|_{H^p(\{T_{K_i^1}\}_i)}^p dv \leq B^p \|f\|_{H^p(\{T_{K_i^1}\}_i)}^p, \end{aligned}$$

where the last inequality follows by (ii). ■

If the group  $G$  is not compact, the proof is not so clear. Moreover, the natural extension of Theorem 3.1 does not work in general since condition (4) fails. However, we can formulate a quite general result that will be useful for our purpose.

**THEOREM 5.2.** *Let  $G$  be a locally compact abelian group and let  $0 < p < \infty$ . Let  $K_j^2$  and  $K_i^1$  be two collections of compactly supported functions in  $L^1(G)$  and let  $N_p(K)$  be the norm of the convolution operator*

$$\sup_j |K_j^2 * \cdot| : H^p(\{K_i^1\}_i) \rightarrow L^p(G).$$

*Let  $R$  be a representation from  $G$  into  $L^p(\mathcal{M})$  satisfying (i)–(iii). Let  $f \in H^p(\{T_{K_i^1}\}_i)$  satisfy the following condition: there exists  $B > 0$  so that, for every compact  $E \subset G$  large enough, there exists  $\varphi_E$  such that  $\varphi_E(u) = 1$  for every  $u \in E$ , and*

$$(6) \quad \int_{\mathcal{M}} \|\varphi_E(R.f)(x)\|_{H^p(\{K_i^1\}_i)}^p dx \leq B^p \mu(E) \|f\|_{H^p(\{T_{K_i^1}\}_i)}^p.$$

*Then*

$$\|\sup_j |T_{K_j^2} f|\|_p \leq BA^2 \|f\|_{H^p(\{T_{K_i^1}\}_i)},$$

*with  $B$  as in (6) and  $A$  as in (1).*

**Proof.** First, by Fatou's lemma, it is enough to estimate the norm  $\|\sup_{j=1, \dots, N} |T_{K_j^2} f|\|_p$ . Consider  $C$  such that  $\text{supp } K_j^2 \subset C$  for every  $j = 1, \dots, N$ . Then we can adapt the proof of Theorem 3.1 quite easily to get

$$\begin{aligned}
 & \left\| \sup_{j=1, \dots, N} |T_{K_j^2} f| \right\|_p \\
 & \leq \frac{A^p}{\mu(V)} \int_V \left\| \sup_{j=1, \dots, N} |T_{K_j^2} R_v f| \right\|_{H^p(\{T_{K_i^1}\}_i)}^p dv \\
 & = \frac{A^p}{\mu(V)} \int_V \int_{\mathcal{M}} \sup_{j=1, \dots, N} \left| \int_G K_j^2(u) (R_{vu^{-1}} f)(x) du \right|^p dx dv \\
 & = \frac{A^p}{\mu(V)} \int_V \int_{\mathcal{M}} \sup_{j=1, \dots, N} \left| \int_G K_j^2(u) \varphi_{VC^{-1}}(vu^{-1}) (R_{vu^{-1}} f)(x) du \right|^p dx dv \\
 & \leq \frac{A^p}{\mu(V)} \int_{\mathcal{M}} \left[ \int_G \sup_j \left| \int_G K_j^2(u) \varphi_{VC^{-1}}(vu^{-1}) (R_{vu^{-1}} f)(x) du \right|^p dv \right] dx \\
 & \leq \frac{(AN_p(K))^p}{\mu(V)} \int_{\mathcal{M}} \|\varphi_{VC^{-1}}(R \cdot f)(x)\|_{H^p(\{K_i^1\}_i)}^p dx,
 \end{aligned}$$

where the last inequality follows by applying the hypothesis to the function

$$h_x(u) = \varphi_{VC^{-1}}(u)(R_u f)(x).$$

The last step is to show that

$$\int_{\mathcal{M}} \|\varphi_{VC^{-1}}(R \cdot f)(x)\|_{H^p(\{K_i^1\}_i)}^p dx \leq B^p A^p \mu(V) (1 + \varepsilon) \|f\|_{H^p(\{T_{K_i^1}\}_i)}^p,$$

but this follows by (6) and the choice of  $V$  such that  $\mu(VC^{-1})/\mu(V) \leq 1 + \varepsilon$ . ■

As in Sections 2 and 4, if  $p \geq 1$  or if  $\mathcal{M}$  is of finite measure (or it can be reduced to this case) and  $p < 1$ , we do not need the hypothesis on the support of  $K_j^2$  but we do need it for the support of  $K_i^1$  (see also [ABG1]).

**D. Maximal spaces and maximal operators.** We start with the result of [C] we mentioned in the introduction.

**THEOREM D.1.** *Let  $0 < p < 1$ . If  $K$  is such that  $\widehat{K} = m$  is a normalized function and the operator*

$$K* : H^p(\mathbb{R}^N) \rightarrow H^p(\mathbb{R}^N)$$

*is bounded with norm  $N_p(K)$ , then the operator*

$$T_K \left( \sum a_n e^{2\pi i x} \right) = \sum a_n m(n) e^{2\pi i x}$$

*with  $m = \widehat{K}$  can be extended to a bounded operator from  $H^p(\mathbb{T}^N)$  into  $H^p(\mathbb{T}^N)$  with norm less than or equal to  $N_p(K)$ .*

**Proof.** As for the case  $p = 1$ , first assume that  $K \in L^1$ .

Since  $K*$  is a convolution kernel in  $L^2$ , we see by interpolation that  $K*$  is also a convolution kernel in  $H^1$ .

Now, since we are transferring to a measure space of finite measure, we do not need any condition on the support of  $K$  but, if we want to apply Theorem 4.1, we need to have that restriction on the kernels that define the space  $H^p(\mathbb{R}^N)$ . Since these kernels do not have compact supports, we are forced to use Theorem 5.2.

Take  $a$  to be an atom in  $H^p(\mathbb{T}^N)$ . Then either  $a = 1$  or  $a$  is a  $(p, q)$ -atom. For the first case, we proceed as in Theorem B.1, since

$$\|T_K a\|_{H^p(\mathbb{T}^N)} = |m(0)| \leq \|m\|_\infty \leq N_p(K),$$

and for a general atom we observe that if  $\mathbb{T}^N = (-1, 1)^N$ , then the function  $\chi_{(-M, M)^N} a$  is an atom in  $H^p(\mathbb{R}^N)$  and

$$\|\chi_{(-M, M)^N} a\|_{H^p(\mathbb{R}^N)} \leq CM^N \|a\|_{H^p(\mathbb{T}^N)}$$

and therefore condition (6) holds. Hence

$$\|T_K a\|_{H^p(\mathbb{T}^N)} \leq N_p(K) \|a\|_{H^p(\mathbb{T}^N)}.$$

For the general case of a normalized multiplier we argue as in Theorem B.1. Take a trigonometric polynomial  $P$  in  $H^p(\mathbb{T}^N)$  with  $P(0) = 0$  and degree  $j$  and let  $\phi$  be as in B.1. Then  $K * \phi$  is in  $L^1$  and since  $T_\phi P = P$  and  $P$  satisfies condition (6), we can apply Theorem 5.2 to obtain the result. ■

Similarly, we can obtain the analogue to Theorem B.3 for  $0 < p < 1$  observing that if  $a \in H^p(\mathbb{R})$  is a  $(p, \infty)$ -atom and for every compact  $E \subset \mathbb{R}^2$  we define  $\varphi_E(u_1, u_2) = 1$  whenever  $(u_1, u_1) \in E$  and  $\varphi_E(u_1, u_2) = 0$  if  $(u_1, u_1) \in \tilde{E}$ , where  $\tilde{E} = E + (0, n)$  is such that  $E \cap \tilde{E} = \emptyset$ , then  $\varphi_E(u_1, u_2)a(x - u_1)$  is a  $(p, \infty)$ -atom of  $H^p(\mathbb{R}^2)$  satisfying condition (6).

A final application we want to mention is the following: Assume that we have two equivalent norms in a fixed space  $H^p$ . Then we can try to transfer the identity operator to obtain two equivalent norms in the transferred space. We illustrate this situation with the following example.

Let  $\varphi \in L^1$  with compact support and consider the atomic and maximal versions of the space  $H^p(\mathbb{R}^N)$  which of course are equivalent. Then, if  $H^p(\mathbb{R}^N)$  denotes the atomic space, the operator

$$\sup_t |\varphi_t * \cdot| : H^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$$

is bounded.

Now, since the atomic version of  $H^p(\mathbb{T}^N)$  satisfies condition (6) we can transfer this maximal operator to obtain

$$\|\sup_t |\tilde{\varphi}_t * F|\|_{L^p(\mathbb{T}^N)} \leq C \|F\|_{H^p(\mathbb{T}^N)},$$

which is a well-known result.

## REFERENCES

- [ABG1] N. Asmar, E. Berkson and T. A. Gillespie, *Transference of strong type maximal inequalities by separation-preserving representations*, Amer. J. Math. 113 (1991), 47–74.
- [ABG2] —, —, —, *Transference of weak type maximal inequalities by distributionally bounded representations*, Quart. J. Math. Oxford 43 (1992), 259–282.
- [CT] R. Caballero and A. de la Torre, *An atomic theory for ergodic  $H^p$  spaces*, Studia Math. 82 (1985), 39–59.
- [CW1] R. Coifman and G. Weiss, *Transference Methods in Analysis*, CBMS Regional Conf. Ser. in Math. 31, Amer. Math. Soc., 1977.
- [CW2] —, —, *Maximal functions and  $H^p$  spaces defined by ergodic transformations*, Proc. Nat. Acad. Sci. U.S.A. 70 (1973), 1761–1763.
- [C] L. Colzani, *Fourier transform of distributions in Hardy spaces*, Boll. Un. Mat. Ital. A (6) 1 (1982), 403–410.
- [D] K. de Leeuw, *On  $L_p$  multipliers*, Ann. of Math. 81 (1965), 364–379.
- [HR] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. I, Springer, 1963.
- [M] A. Miyachi, *On some Fourier multipliers for  $H^p(\mathbb{R}^n)$* , J. Fac. Sci. Univ. Tokyo 27 (1980), 157–179.

Departament de Matemàtica Aplicada i Anàlisi  
Universitat de Barcelona  
08071 Barcelona, Spain  
E-mail: carro@cerber.mat.ub.es  
soria@cerber.mat.ub.es

*Received 9 April 1996;  
revised 27 August 1996 and 7 November 1996*