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CONSTRUCTING THE DIRECTING COMPONENTS OF AN ALGEBRA

BY

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Let k be an algebraically closed field and A be a finite-dimensional k-algebra. We may assume that $A = kQ/I$, where Q is a finite connected quiver and I is an admissible ideal of the path algebra kQ; see [5]. For our considerations we may assume that Q has no oriented cycles.

Consider the category mod_A of finite-dimensional left A-modules. For each indecomposable non-projective A-module X, the Auslander–Reiten translate $\tau_A X$ is an indecomposable non-injective module; see [1, 5]. The Auslander–Reiten quiver Γ_A has as vertices representatives of the isoclasses of finite-dimensional indecomposable A-modules and as many arrows from X to Y in Γ_A as the dimension of $\mathrm{rad}_A(X,Y)/\mathrm{rad}_A^2(X,Y)$. An indecomposable A-module X is *directing* if there is no cycle of non-zero non-isomorphisms $X = X_0 \to X_1 \to \ldots \to X_s \to X_{s+1} = X$ between indecomposable modules. A component $\mathcal{C}$ of Γ_A is *directing* if all its modules are directing.

There are several important examples of directing components which have been extensively studied. Postprojective components are directing components $\mathcal{P}$ such that each module in $\mathcal{P}$ has only finitely many predecessors in the path order of $\mathcal{P}$. Algebras with such type of components are: algebras with the separation condition (in particular, tree algebras), and hereditary algebras (and more generally, tilted and quasi-tilted algebras [20, 3]). Recently, a criterion was given in [4] for the existence of postprojective components. For every tilted algebra A, the connecting component $\mathcal{C}$ of Γ_A is a directing component; see [14].

Some general properties of directing components were studied in [9], [18] and [19]. The purpose of the present work is to describe properties of directing convex components of Γ_A. Such components are standard; they are sincere if and only if the number of τ_A-orbits is the number of vertices in the quiver Q. In Section 3 we describe an inductive procedure to construct all algebras A accepting a sincere convex directing component in Γ_A. Some of our results are related to [19].

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1. Directing components

1.1. Let $A = kQ/I$ be an algebra as in the introduction. Let $Q_0 = \{1, \ldots, n\}$ be the set of vertices of Q. Denote by S_j the simple module associated to j, whose projective cover (resp. injective envelope) will be denoted by P_j (resp. I_j). The radical of P_j is $\operatorname{rad} P_j$. The Grothendieck group $K_0(A)$ is isomorphic to $\mathbb{Z}^n$; the class of a module X is its *dimension vector* $\mathbf{dim}\, X$. The Euler bilinear (non-symmetric) form $\langle -, - \rangle_A$ is defined by $\langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}_A^i(X, Y)$.

We start with recalling the following result.

THEOREM [18]. *Let $\mathcal{C}$ be a directing component of Γ_A. Then $\mathcal{C}$ has finitely many τ-orbits.* ∎

A full subquiver $\mathcal{S}$ of a directing component $\mathcal{C}$ of Γ_A is said to be a *section* in $\mathcal{C}$ if the following are satisfied:

(i) $\mathcal{S}$ is convex (= path closed) in $\mathcal{C}$ and connected;

(ii) if $X \in \mathcal{S}$, then $\tau_A X \notin \mathcal{S}$;

(iii) if $X \to Y$ is an arrow in Γ_A, $X \in \mathcal{S}$ and $Y \notin \mathcal{S}$, then Y is non-projective and $\tau_A Y \in \mathcal{S}$.

Moreover, $\mathcal{S}$ is said to be a $(+)$-*section* (resp. $(-)$-*section*) if it is a section and satisfies (iv) (resp. (iv)$'$):

(iv) there are no paths from a projective P_j or an injective I_j in $\mathcal{C}$ to any module in $\mathcal{S}$;

(iv)$'$ there are no paths from a module in $\mathcal{S}$ to a projective P_j or an injective I_j in $\mathcal{C}$.

Similar definitions have been considered before (see [11], [14], [16]).

COROLLARY. *Let $\mathcal{C}$ be a directing component of Γ_A. Then $\mathcal{C}$ is infinite if and only if $\mathcal{C}$ admits either a $(+)$-section or a $(-)$-section $\mathcal{S}$. In that situation, the underlying graph of $\mathcal{S}$ is not a Dynkin diagram.*

Proof. If $\mathcal{S}$ is a $(+)$-section in $\mathcal{C}$, then for any $X \in \mathcal{S}$, all $\tau_A^n X$, $n \geq 0$, are well-defined pairwise non-isomorphic modules. Conversely, assume that $\mathcal{C}$ is infinite. By the theorem, there is a module $X \in \mathcal{C}$ such that either all $\tau_A^n X$, $n \geq 0$, are well-defined or all $\tau_A^{-n} X$, $n \geq 0$, are well defined. Assume the first situation holds. Since $\mathcal{C}$ is directing, there is some $Y = \tau_A^{n_0} X$, $n_0 \geq 0$, such that Y has no predecessors in $\mathcal{C}$ which are projective or injective. Denote by $\Sigma(\to Y)$ the full subquiver of $\mathcal{C}$ formed by those modules Z which are predecessors of Y and such that $\tau_A^- Z$ is not a predecessor of Y. In [18] (or see also [11], [9]) it is shown that $\Sigma(\to Y)$ is a section. It is a (+)-section by construction.

If $\mathcal{S}$ is a (+)-section in $\mathcal{C}$, it is well-known that $\mathcal{S}$ is not of Dynkin type; see [9]. ∎

1.2. Let $H = k\Delta$ be a hereditary algebra. We recall that a *tilting module* T of H is a module satisfying:

(i) $\mathrm{Ext}^1_H(T,T) = 0$;

(ii) the number of indecomposable pairwise non-isomorphic direct summands of T is the number of vertices of Δ.

The algebra $\mathrm{End}_H(T)$ is then said to be a *tilted algebra*.

Part (i) of the following result is taken from [18] (see also [9]). For part (ii) see [9].

PROPOSITION. *Let $\mathcal{C}$ be a directing component of Γ_A and $\mathcal{S}$ be a (+)-section. Consider the translation subquiver $\mathcal{C}^+(\mathcal{S})$ of $\mathcal{C}$ formed by all predecessors of $\mathcal{S}$. Then*:

(i) *There is a hereditary algebra H of infinite representation type and a tilting module T of H without preinjective direct summands such that the tilted algebra $B = \mathrm{End}_H(T)$ is a full convex subcategory of A and $\mathcal{C}^+(\mathcal{S})$ is a translation subquiver of Γ_B closed with respect to predecessors.*

(ii) *There are functionals $f^+_{\mathcal{S}}$, $g^+_{\mathcal{S}} : K_0(A) \to \mathbb{R}$ such that for any indecomposable B-module X we have $X \in \mathcal{C}^+(\mathcal{S})$ if and only if $f^+_{\mathcal{S}}(\mathbf{dim}\, X) > 0$ and $g^+_{\mathcal{S}}(\mathbf{dim}\, X) > 0$.*

Proof. We indicate the main steps of the proof.

(i) Let B be the full subcategory of A formed by the support of $\mathcal{S}$, that is, by those vertices $i \in Q_0$ such that $X(i) \neq 0$ for some $X \in \mathcal{S}$. By a well-known argument, B is convex in A. Then $\mathcal{S}$ is a section in Γ_B and, in fact, a *slice* in the sense of [14]. Hence, B is a tilted algebra $\mathrm{End}_H(T)$ such that the vertices of $\mathcal{S}$ are of the form $\mathrm{Hom}_H(T, I^0_j)$, where I^0_j is the indecomposable injective $H = k\Delta$-module corresponding to $j \in \Delta_0$ (see [14]). If a direct summand T_j of T is of the form $\tau^n_H I^0_j$, then $\mathrm{Hom}_H(T, T_j)$ is a projective in $\mathcal{C}$ which is a predecessor of $\mathrm{Hom}_H(T, I^0_j) \in \mathcal{S}$, a contradiction. It is not difficult to see that $\mathcal{C}^+(\mathcal{S})$ is formed by B-modules (see [9]).

(ii) Let ϱ be the spectral radius of the Coxeter matrix ϕ_Δ of $H = k\Delta$ (see [12], [15] for definitions). If Δ is of extended Dynkin type, then $\varrho = 1$; otherwise $\varrho > 1$ (recall from 1.1 that Δ is not Dynkin). By [12] and [15], there is an eigenvector $y^+ \in \mathbb{N}^{\Delta_0}$ of ϕ_Δ with eigenvalue ϱ, all of whose coordinates are positive. By [12], an indecomposable H-module N is preinjective if and only if $\langle y^+, \mathbf{dim}\, N\rangle_H > 0$.

Since $B = \mathrm{End}_H(T)$ is a tilted algebra, there is an isometry $\sigma : K_0(H) \to K_0(B)$, $\mathbf{dim}\, N \mapsto \mathbf{dim}\,\mathrm{Hom}_H(T,N) - \mathbf{dim}\,\mathrm{Ext}^1_H(T,N)$, preserving the Euler bilinear form. Let $i : K_0(B) \to K_0(A)$ be the natural inclusion and define $f^+(z) = \langle i\sigma(y^+), z\rangle_A$. Further, we set $g^+(z) = \sum_{Y\in\mathcal{S}} \langle z, \mathbf{dim}\, Y\rangle_A$.

Let $X \in \mathcal{C}^+(\mathcal{S})$. Since X is a predecessor of $\mathcal{S}$, we have $X = \mathrm{Hom}_H(T, N)$ for some preinjective H-module N. Then $\mathbf{dim}\, X = i\sigma(\mathbf{dim}\, N)$, hence $f^+(\mathbf{dim}\, X) = \langle\sigma(y^+), \sigma(\mathbf{dim}\, N)\rangle_B = \langle y^+, \mathbf{dim}\, N\rangle_H > 0$; clearly, $g^+(\mathbf{dim}\, X) = \sum_{j\in\Delta_0} \langle \mathbf{dim}\, N, \mathbf{dim}\, I_j^0\rangle_H > 0$. Conversely, assume that for an indecomposable B-module X we have $f^+(\mathbf{dim}\, X) > 0$ and $g^+(\mathbf{dim}\, X) > 0$. Assume $X = \mathrm{Ext}^1_H(T, M)$ for some indecomposable H-module M. Then there is a path from M to some direct summand T_j of T. Since T_j is not preinjective by (i), neither is M. Hence $g^+(\mathbf{dim}\, X) = \sum_{j\in\Delta_0} \langle -\sigma(\mathbf{dim}\, M), \sigma(\mathbf{dim}\, I_j^0)\rangle_B < 0$, a contradiction. Then $X = \mathrm{Hom}_M(T, N)$ for some indecomposable H-module N. Clearly, if N is not preinjective, then $f^+(\mathbf{dim}\, X) < 0$. Therefore, N is preinjective and X is a predecessor of $\mathcal{S}$ in $\mathcal{C}$. That is, $X \in \mathcal{C}^+(\mathcal{S})$ as desired. ■

Let $\mathcal{S}_1, \ldots, \mathcal{S}_m$ be a set of $(+)$-sections intersecting exactly once all τ-stable orbits of a directing component $\mathcal{C}$. We denote by $\mathcal{C}^+$ the full translation subquiver of $\mathcal{C}$ formed by the modules in $\bigcup_{i=1}^m \mathcal{C}^+(\mathcal{S}_i)$, and we call it the $(+)$-*stable part* of $\mathcal{C}$. Dually we define $\mathcal{C}^-$, the $(-)$-*stable part* of $\mathcal{C}$.

1.3. We recall that a connected component $\mathcal{C}$ of Γ_A is called *convex* if any path $X = X_0 \to X_1 \to \ldots \to X_s = Y$ between indecomposable modules in mod_A with X and Y in $\mathcal{C}$ has all its modules X_i in $\mathcal{C}$. Moreover, $\mathcal{C}$ is said to be *standard* if the full subcategory $[\mathcal{C}]$ of mod_A defined by $\mathcal{C}$ is equivalent to the mesh category $k(\mathcal{C})$ of $\mathcal{C}$ (see [14]).

As shown in [18], there are examples of directing components in Γ_A which are neither convex nor standard. Nevertheless, the following holds.

PROPOSITION. *Let $\mathcal{C}$ be a directing component of Γ_A.*

(a) *If $\mathcal{C}$ is convex, then $\mathcal{C}$ is standard.*

(b) *Assume* $\mathrm{Hom}_A(\mathcal{C}^-, P_j) = 0$ *for every projective P_j, then $\mathcal{C}$ is convex.*

Proof. (a) We define a functor $F : k(\mathcal{C}) \to [\mathcal{C}]$. We will consider each vertex x of $\mathcal{C}$ as the isomorphism class of an indecomposable and we shall fix an indecomposable $F(x) \in x$. For each $Y \in x$ we set an isomorphism $\phi_Y : Y \to F(x)$. We shall define for each arrow $x \xrightarrow{\alpha} y$ in $\mathcal{C}$ an irreducible map $F(\alpha) : F(x) \to F(y)$ such that any mesh

$$\begin{tikzcd} & y_1 \arrow[dr, "\alpha_1"] & \\ \tau_A x \arrow[ur, "\beta_1"] \arrow[dr, "\beta_s"'] & \vdots & x \\ & y_s \arrow[ur, "\alpha_s"'] & \end{tikzcd}$$

is sent in $\sum_{i=1}^s F(\alpha_i)F(\beta_i) = 0$. This is an easy task that we just sketch.

Let $\mathcal{C}_0$ be a finite convex and connected subquiver of $\mathcal{C}$ containing modules of all the τ-orbits in $\mathcal{C}$ (this is possible because of 1.1). Since there are no cycles in $\mathcal{C}_0$ we may explicitly define F in $\mathcal{C}_0$ satisfying the above condition (see [14, 2.3] for indications how to proceed). Now assume we have defined finite connected convex subquivers $\mathcal{C}_0, \mathcal{C}_1^+, \dots, \mathcal{C}_n^+$ of $\mathcal{C}$ satisfying $\mathcal{C}_{i-1}^+ \cup \tau_A \mathcal{C}_{i-1}^+ \subset \mathcal{C}_i^+$, for $i = 1, \dots, n$; F is defined in $\mathcal{C}_n^+$ satisfying the mesh relations. Consider those $y_1, \dots, y_s$ in $\mathcal{C}$ such that $y_i \notin \mathcal{C}_n^+$ but $\tau_A^- y_i \in \mathcal{C}_n^+$; moreover, assume that $y_i \to y_j$ in $\mathcal{C}$ implies $i < j$. For the mesh

$$\begin{tikzcd} & z_1 \arrow[dr, "\beta_1"] & \\ y_1 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_s"'] & \vdots & \tau_A^- y_1 \\ & z_s \arrow[ur, "\beta_s"'] & \end{tikzcd}$$

all $z_1, \dots, z_s$ are in $\mathcal{C}_n^+$ and $F(\beta_i)$ are defined. Take any Auslander–Reiten sequence $0 \to \tau_A F(\tau_A^- y_1) \xrightarrow{(f_i)_i} \bigoplus_{i=1}^s F(z_i) \to F(\tau_A^- y_1) \to 0$. Then we set $F(\alpha_i) = f_i \phi^{-1}_{\tau_A F(\tau_A^- y_1)}$. Inductively we define F in the quiver $\mathcal{C}_{n+1}^+$ formed by $\mathcal{C}_n^+$ and $y_1, \dots, y_s$. In this way we define F in $\bigcup_{n\in\mathbb{N}} \mathcal{C}_n^+$, and dually in $\bigcup_{n\in\mathbb{N}} \mathcal{C}_n^-$, to get the desired functor $F : k(\mathcal{C}) \to [\mathcal{C}]$.

The proof that F is faithful is exactly as in [14]. We shall prove that F is full. We divide the proof in several steps.

(i) For $X, Y \in \mathcal{C}$, we have $\mathrm{rad}_A^\infty(X, Y) = 0$.

Suppose that $h \in \mathrm{rad}_A^\infty(X, Y)$. Considering the source map $X = X_0 \xrightarrow{s_1} E_1$, there is a factorization $h = f_1' s_1$. Hence there is an irreducible map $X_0 \xrightarrow{\alpha_1} X_1$ and a non-zero map $f_1 \in \mathrm{rad}_A^\infty(X_1, Y)$ with $f_1\alpha_1 \neq 0$. Proceeding inductively we get a chain $X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} X_2 \to \dots$ in $\mathcal{C}$ and maps $0 \neq f_i \in \mathrm{rad}_A^\infty(X_i, Y)$ with $f_i \alpha_i \dots \alpha_1 \neq 0$. Then $\mathcal{C}$ admits a $(-)$-section $\mathcal{S}$ such that $X_i \in \mathcal{C}^-(\mathcal{S})$ for $i \geq N$ for some $N \in \mathbb{N}$.

Since $\dim_k Y < \infty$ we may assume that $\operatorname{Im} f_i = L$ for all $i \geq N$.

By 1.2, there is a hereditary algebra H and a tilting H-module T without postprojective direct summands such that $\mathcal{C}^-(\mathcal{S})$ is formed by postprojective B-modules, where $B = \mathrm{End}_H(T)$ is the corresponding tilted algebra. Since $\mathrm{Hom}_B(X_i, L) \neq 0$ for $i \geq N$, it follows that L is not postprojective as B-module. There is a regular B-module R such that f_i factorizes as $X_i \xrightarrow{f_i'} R \xrightarrow{f_i''} L$. Therefore $0 \neq f_i'' f_i' \alpha_i \dots \alpha_1 : X \to Y$ factorizes through R, which is not in $\mathcal{C}$. This contradicts the convexity of $\mathcal{C}$. Hence $\mathrm{rad}_A^\infty(X, Y) = 0$.

(ii) Let $X, Y \in \mathcal{C}$ be such that $\mathrm{Hom}_A(X, Y) \neq 0$. We claim that there is a path $X = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_s} X_s = Y$ in $\mathcal{C}$. Indeed, if $0 \neq h \in \mathrm{rad}_A^s(X, Y) - \mathrm{rad}_A^{s+1}(X, Y)$, we may find a path of length s from X to Y in $\mathcal{C}$.

(iii) Let $0 \neq h \in \mathrm{Hom}_A(F(x), F(y))$ and let s be the maximal length of paths from x to y. By induction on s, we prove that $h = F(w)$ for some element $w \in k(\mathcal{C})(x, y)$.

If $s = 0$, since $\mathcal{C}$ is directing, $h = \lambda \mathbf{1}_x$ for some $\lambda \in k$ and $h = F(\lambda \mathbf{1}_x)$. If $s = 1$, there are arrows $\alpha_1, \ldots, \alpha_s$ from x to y such that $F(\alpha_1), \ldots, F(\alpha_s)$ are representatives of a basis of $\mathrm{rad}_A(F(x), F(y))/\mathrm{rad}_A^2(F(x), F(y))$. Moreover, h is irreducible, since otherwise there is a path of length ≥ 2 from x to y. Hence $h = \sum_{i=1}^s \lambda_i F(\alpha_i) = F(\sum_{i=1}^s \lambda_i \alpha_i)$ for some $\lambda_i \in k$.

Assume $s > 1$. Then h is not an isomorphism since X is directing. Consider the source map $F(x) \xrightarrow{(F(\alpha_i))_i} \bigoplus_{i=1}^s F(y_i)$. There are maps $h_i \in \mathrm{Hom}_A(F(y_i), F(y))$ such that $h = \sum_{i=1}^s h_i F(\alpha_i)$. By induction hypothesis, $h_i = F(v_i)$ for some $v_i \in k(\mathcal{C})(y_i, y)$ and

$$h = \sum_{i=1}^s F(v_i) \cdot F(\alpha_i) = F\Big(\sum_{i=1}^s v_i \alpha_i\Big).$$

(b) Assume that $\mathrm{Hom}_A(\mathcal{C}^-, P_j) = 0$ for all projectives P_j. Let $X = Y_0 \xrightarrow{f} Y_1 \to \ldots \to Y_s = Y$ be a path in mod_A between indecomposable modules with X and Y in $\mathcal{C}$ and $Y_1 \notin \mathcal{C}$. Hence $0 \neq f \in \mathrm{rad}_A^\infty(X, Y_1)$ and, as in part (i) of (a), we get a $(-)$-section $\mathcal{S}$ in $\mathcal{C}$, a path $X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} X_2 \to \ldots$ of irreducible maps in $\mathcal{C}$ with $X_i \in \mathcal{C}^-(\mathcal{S})$ for $i \geq N$ and maps $0 \neq f_i \in \mathrm{rad}_A^\infty(X_i, Y_1)$ such that $0 \neq f_i \alpha_i \ldots \alpha_1$. Let $B = \mathrm{End}_H(T)$ be the tilted algebra such that $\mathcal{C}^-(\mathcal{S})$ is contained in the postprojective component of B, where T is a tilting H-module without postprojective direct summands. We shall show that for any Z which is a successor of $\mathcal{S}$ in mod_B and a map $0 \neq g : Z \to N$ between indecomposable A-modules we have $N \in \mathrm{mod}_B$. In particular, this shows that X_N and Y_1 belong to the postprojective component of B, which is a contradiction.

By 1.2, B is convex in A. Let D be the maximal convex subcategory of A containing B such that any successor of $\mathcal{S}$ in mod_D is a B-module. We shall show that $D = A$. Otherwise, there is a D-module M such that there is a convex subcategory E of A either of the form $D[M]$ or $[M]D$. In case $E = [M]D$ with $M = I_y/\mathrm{soc}\, I_y|_D$, the maximality of D implies that there is a direct summand N of M which is a successor of $\mathcal{S}$; but then $\mathrm{Hom}_D(\mathcal{S}, N) \neq 0$ and $\mathcal{C}^-(\mathcal{S})$ should contain modules Z with $Z(y) \neq 0$, a contradiction. In case $E = D[M]$ with $M = \mathrm{rad}\, P_x|_D$, the maximality of D implies that there is a summand N of M which is a successor of $\mathcal{S}$ and then $\mathrm{Hom}_A(\mathcal{S}, P_x) \neq 0$, contradicting the hypothesis. Hence $D = A$ and we are done. ∎

1.4. Of course, we say that a component is *sincere* when there are modules $X_1, \ldots, X_s$ in $\mathcal{C}$ such that $\bigoplus_{i=1}^s X_i$ is sincere.

COROLLARY 1. *Let $\mathcal{C}$ be a sincere directing component of Γ_A. Then $\mathcal{C}$ is convex if and only if* $\mathrm{Hom}_A(\mathcal{C}^-, P_j) = 0$ *for every projective* P_j.

Proof. Assume that $\mathcal{C}$ is convex and $\mathrm{Hom}_A(X, P_j) \neq 0$ for some $X \in \mathcal{C}^-$. Since $P_j \notin \mathcal{C}^-$, we have $\mathrm{rad}_A^\infty(X, P_j) \neq 0$. If $P_j \in \mathcal{C}$, then $\mathcal{C}$ is not standard, contradicting part (i) of Proposition 1.3. Then $P_j \notin \mathcal{C}$. But $\mathcal{C}$ being sincere implies the existence of $Y \in \mathcal{C}$ with $\mathrm{Hom}_A(P_j, Y) \neq 0$ and hence $\mathcal{C}$ is not convex. ∎

COROLLARY 2. *Let $\mathcal{C}$ be a directing and convex component of Γ_A. Then*:

(a) *The number of τ-orbits of $\mathcal{C}$ is at most n (= number of vertices of Q_0).*

(b) *The number of τ-orbits of $\mathcal{C}$ is exactly n if and only if $\mathcal{C}$ is sincere.*

Proof. (a) Let $\mathcal{S}_1, \ldots, \mathcal{S}_s$ be a set of (+)-sections such that $\mathcal{C}^+ = \bigcup_{i=1}^s \mathcal{C}^+(\mathcal{S}_i)$. Let $t(\mathcal{S}_i)$ be the set of τ-orbits of $\mathcal{S}_i$, $i = 1, \ldots, s$. We may assume that $t(\mathcal{S}_i) \cap t(\mathcal{S}_j) = \emptyset$ for $i \neq j$. For $i \in \{1, \ldots, s\}$, 1.2 yields a tilted algebra B_i such that $\mathcal{S}_i$ is a slice in Γ_{B_i}. Since $\mathcal{C}$ is convex, the proof of Proposition 1.3(b) shows that any indecomposable A-module which is a predecessor of $\mathcal{S}_i$ is indeed a B_i-module. Moreover, B_i and B_j have no common vertices for $i \neq j$. To prove the claim, it is enough to show that there are no projective B_i-modules P in $\mathcal{C}$. Otherwise, $\mathrm{Hom}_A(P, X) \neq 0$ for some indecomposable B_i-module $X \in \mathcal{C}^+(\mathcal{S}_i)$. Since by 1.3, $\mathcal{C}$ is standard, there is a path in $\mathcal{C}$ from P to X and $P \in \mathcal{C}^+(\mathcal{S}_i)$, which is a contradiction.

(b) By the construction above, if there are n τ-orbits of $\mathcal{C}$, then for each indecomposable projective P_j, either $P_j \in \mathcal{C}$ or $\mathrm{Hom}_A(P_j, \mathcal{C}^+(\mathcal{S})) \neq 0$ for some (+)-section $\mathcal{S}$ of $\mathcal{C}$. Hence $\mathcal{C}$ is sincere. Conversely, assume $\mathrm{Hom}_A(P, \mathcal{C}) \neq 0$ for an indecomposable projective A-module $P \notin \mathcal{C}$. Then, clearly, $\mathrm{Hom}_A(P, \mathcal{C}^+(\mathcal{S})) \neq 0$ for some (+)-section $\mathcal{S}$ of $\mathcal{C}$. Hence the result follows. ∎

COROLLARY 3. *The quiver Γ_A has at most two sincere directing convex components. If it has two, then A is a concealed algebra (and the two sincere directing convex components are the postprojective and the preinjective components).*

Proof. Let $\mathcal{C}$ be a sincere directing convex component. By 1.3, $\mathcal{C}$ is generalized standard (in the sense of [17]), that is, $\mathrm{rad}_A^\infty(X, Y) = 0$ for $X, Y \in \mathcal{C}$. Moreover, $\mathcal{C}$ is faithful (since there are modules $X_1, \ldots, X_n$ in $\mathcal{C}$ admitting a monomorphism $0 \to {}_AA \to \bigoplus_{i=1}^n X_i$). Hence Theorem 4 of [17] applies. ∎

1.5. We shall deal with the directing convex components of Γ_A. We give the following characterization of these components.

The *componental quiver* $Q(\Gamma_A)$ has as vertices the components of Γ_A and there is an arrow $\mathcal{C} \to \mathcal{D}$ if there are modules $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ such that $\operatorname{rad}_A^\infty(X, Y) \neq 0$ (see [6] and [16]).

PROPOSITION. *Let $\mathcal{C}$ be a directing component of Γ_A. Then $\mathcal{C}$ is convex if and only if $\mathcal{C}$ is not in a cycle in the componental quiver $Q(\Gamma_A)$.*

Proof. Assume that $\mathcal{C}$ is directing and convex. If $\mathcal{C} = \mathcal{C}_0 \to \mathcal{C}_1 \to \mathcal{C}_2 \to \ldots \to \mathcal{C}_s = \mathcal{C}$ is a cycle in $Q(\Gamma_A)$, there are modules $X_0 \in \mathcal{C}$, $X_i \in \mathcal{C}_i$ and $\operatorname{rad}_A^\infty(X_{i-1}, X_i) \neq 0$, $i = 1, \ldots, s$. If $s = 1$, this contradicts the fact that $\mathcal{C}$ is standard (cf. 1.3). If $s > 1$, this contradicts the convexity of $\mathcal{C}$.

Assume that $\mathcal{C}$ is not in a cycle in $Q(\Gamma_A)$. Suppose that $\mathcal{C}$ is directing and not convex. Let $X = X_0 \xrightarrow{f} X_1 \to \ldots \to X_s = Y$ be a path of non-zero maps in mod_A between indecomposable modules with $X, Y \in \mathcal{C}$ and $X_1 \notin \mathcal{C}$. As in part (i) of 1.3(a) we may construct a module L not in $\mathcal{C}$ with $\operatorname{rad}_A^\infty(X, L) \neq 0$ and $\operatorname{Hom}_A(L, X_1) \neq 0$. This yields the desired cycle in $Q(\Gamma_A)$. ∎

2. Some remarks about tilted algebras

2.1. We recall from [14] that a *slice* $\mathcal{S}$ in a component $\mathcal{C}$ of Γ_A is a section in $\mathcal{C}$ (that is, satisfies conditions (i) to (iii) in 1.1) which is sincere and path closed in mod_A. The latter condition is obviously difficult to check. An algebra is tilted if and only if it admits a slice [14].

We rephrase here the following characterization of tilted algebras essentially proved in [11] and [17] (independently).

THEOREM. *Let A be an algebra. The following are equivalent*:

(a) *A is a tilted algebra.*

(b) *Γ_A admits a sincere directing convex component $\mathcal{C}$ containing a section which intersects each τ-orbit of $\mathcal{C}$.*

(c) *Γ_A admits a directing convex component $\mathcal{C}$ containing a sincere section.*

(d) *Γ_A admits a component $\mathcal{C}$ having a faithful section $\mathcal{S}$ such that* $\operatorname{Hom}_A(X, \tau_A Y) = 0$ *for all modules X and Y in $\mathcal{S}$.* ∎

2.2. Let B be a tilted algebra of the form $\operatorname{End}_H(T)$, where T is a tilting module over the hereditary algebra H. Moreover, B is *concealed* if T is a direct sum of postprojective H-modules. A representation-infinite algebra B is concealed if and only if it has two components $\mathcal{P}$ and $\mathcal{I}$ of Γ_B admitting slices; in that case $\mathcal{P}$ is postprojective and $\mathcal{I}$ preinjective. Since postprojective (and, dually, preinjective) components of Γ_B are easy to identify and construct (see [4] and [9]), it follows that concealed algebras are easy to identify.

THEOREM [20] (see also [10]). *Let B be an algebra. The following are equivalent*:

(a) *B is tilted of the form* $\mathrm{End}_H(T)$, *where T is a tilting H-module without preinjective direct summands.*

(b) *B admits a convex subcategory C such that C is a concealed algebra and the postprojective component of Γ_C is also postprojective in Γ_B. Moreover*, *Γ_B has a connecting component without projective modules.* ■

3. Construction of directing components

3.1. The main feature which will allow the inductive construction of algebras with directing components is the following.

Recall that A is a *one-point extension* of B by the B-module M if we may write $A = B[M] := \left[\begin{smallmatrix} B & M \\ O & k \end{smallmatrix}\right]$ with the usual addition and multiplication of matrices. In this case $M = \mathrm{rad}\, P_a$ for some source a of Q.

PROPOSITION. *Let $A = B[M]$ with $M = \mathrm{rad}\, P_a$ and assume that P_a belongs to a sincere directing and convex component $\mathcal{C}$ of Γ_A. Consider the decomposition of $B = B_1 \amalg \ldots \amalg B_s$ into irreducible factor algebras and the corresponding decomposition $M = M_1 \oplus \ldots \oplus M_s$ with* $\mathrm{supp}\, M_i \subset B_i$. *Then*:

(a) *there is a directing convex component $\mathcal{C}_i$ of Γ_{B_i} where all direct summands of M_i lie*, $i = 1, \ldots, s$;

(b) *each $\mathcal{C}_i$ is sincere in* mod_{B_i}, $i = 1, \ldots, s$.

Proof. (a) Let $\mathcal{D}_1, \ldots, \mathcal{D}_t$ be all the components of Γ_{B_1} intersecting $\mathcal{C}$. We shall show that all $\mathcal{D}_i$ are directing convex components of Γ_{B_1} and $t = 1$.

(1) First observe that each component $\mathcal{D}_i$ contains a direct summand of M.

Indeed, let $X \in \mathcal{D}_i \cap \mathcal{C}$ and a chain of irreducible maps $X = X_0 \xrightarrow{\alpha_i} X_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_s} X_s = M'$ in $\mathcal{C}$, where α_i has some orientation and M' is an indecomposable direct summand of M. Assume $\mathrm{Hom}_A(P_a, X_i) = 0$ for $i = 0, \ldots, j$ and $\mathrm{Hom}_A(P_a, X_{j+1}) \neq 0$. Then we get a chain $X = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_j} X_j$ of irreducible maps in Γ_{B_1}; moreover, there is a direct summand Y of the restriction of X_{j+1} to B_1 such that $\mathrm{Hom}_A(X_j, Y) \neq 0$ or $\mathrm{Hom}_A(Y, X_j) \neq 0$ and $\mathrm{Hom}_A(M'', Y) \neq 0$ for some direct summand M'' of M. Hence, M'' is a B_1-module. Moreover, M'' lies in $\mathcal{D}_i$, since $\mathrm{rad}^{\infty}_{B_1}(M'', Y) \subset \mathrm{rad}^{\infty}_A(M'', X_{j+1}) = 0$, by 1.3.

(2) Let $X \in \mathcal{D}_1$. We will show that for some $m_0 \geq 0$, and every $m \geq m_0$, either $\tau^m_{B_1} X$ is projective or $\tau^m_{B_1} X \in \mathcal{C}^+$. For this purpose, let $X = Y_0 \xrightarrow{\beta_1} Y_1 \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_s} Y_s = M'$ be a chain of irreducible maps in $\mathcal{D}_1$, where M' is an indecomposable direct summand of M (see (1)). We proceed by induction on s.

If $s = 0$, then $X = M'$ and all $\tau_A^i X$ $(i \geq 0)$, if defined, belong to mod_{B_1} (otherwise there would be a cycle through P_a in mod_A). Hence $\tau_{B_1}^i X = \tau_A^i X$ for all $i \geq 0$ where the last module is well defined. Thus the claim follows in this case. Assume that $s > 0$ and that the claim holds for all Y_i, $1 \leq i \leq s$.

If $\tau_{B_1}^m Y_1 \in \mathcal{C}^+$, then either one of $\tau_{B_1}^j X$ is projective for some $0 \leq j \leq m$, or $\tau_{B_1}^{m+1} X \in \mathcal{C}^+$. Otherwise, $\tau_{B_1}^m Y_1 = P_b$ is projective for some m and we may assume that $Z = \tau_{B_1}^m X \to \tau_{B_1}^m Y_1 = P_b$ is well defined. Since $\mathcal{C}$ is sincere and $P_b \in \mathcal{D}_1$ we have $P_b \in \mathcal{C}$ (otherwise $0 \neq \mathrm{rad}_A^\infty(P_b, \mathcal{C}^+) = \mathrm{rad}_{B_1}^\infty(P_b, \mathcal{C}^+)$). Thus $Z \in \mathcal{C}$. We show that either $\tau_{B_1}^i Z$ is projective or $\tau_{B_1}^i Z \in \mathcal{C}$ for $i \geq 0$, hence proving our claim. Indeed, assume $\tau_{B_1}^i Z$ is non-projective in $\mathcal{C}$ but $\tau_{B_1}^{i+1} Z \notin \mathcal{C}$. This means that there is some direct summand M'' of M with $\mathrm{Hom}_{B_1}(M'', \tau_{B_1}^{i+1} Z) \neq 0$. By convexity of $\mathcal{C}$, we should have $\tau_{B_1}^{i+1} Z \in \mathcal{C}$, a contradiction.

(3) We claim that $\mathcal{D}_1$ (and also $\mathcal{D}_2, \ldots, \mathcal{D}_t$) is a directing convex component of Γ_{B_1}. Indeed, by (2), $\mathcal{D}_1$ has only finitely many τ_{B_1}-orbits. Moreover, we may assume that $\mathcal{D}_1^+ \subset \mathcal{C}^+$.

Assume there is a cycle $X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_s} X_s = X$ of morphisms between indecomposable B_1-modules and $X \in \mathcal{D}_1$. If no $f_i \in \mathrm{rad}_{B_1}^\infty(X_{i-1}, X_i)$, $i = 1, \ldots, s$, then we may assume that the cycle is formed by irreducible maps.

Applying τ_{B_1} repeatedly, by (2), either we find a cycle through a projective $P_b \in \mathcal{C}$ or through some module in $\mathcal{C}^+$; in any case, a contradiction. Otherwise, we have $f_j \in \mathrm{rad}_{B_1}^\infty(X_{j-1}, X_j)$, which yields morphisms $0 \neq g \in \mathrm{rad}_{B_1}^\infty(X_{j-1}, Y)$ and $0 \neq h \in \mathrm{Hom}_{B_1}(Y, X_j)$ with $Y \in \mathcal{D}_1^+ \subset \mathcal{C}^+$, also a contradiction.

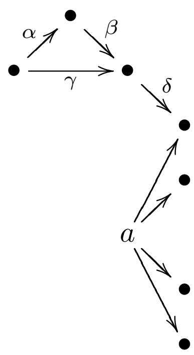
The convexity of $\mathcal{D}_1$ is shown in a similar way.

(4) Finally, we show that $t = 1$. Indeed, assume $t > 1$ and let $M' \in \mathcal{D}_1$ and $M'' \in \mathcal{D}_2$ be two indecomposable direct summands of M. Since B_1 is connected, there are vertices $b_1, \ldots, b_m$ different from a in B_1 such that $\mathrm{Hom}_{B_1}(P_{b_1}, M') \neq 0$, $\mathrm{Hom}_{B_1}(P_{b_m}, M'') \neq 0$ and there is a chain of non-zero maps $P_{b_1} \overset{f_1}{\text{---}} P_{b_2} \overset{f_2}{\text{---}} P_{b_3} \overset{f_3}{\text{---}} \ldots \overset{f_{m-1}}{\text{---}} P_{b_m}$ (maps in some direction). If $P_{b_1} \notin \mathcal{C}$, then $\mathrm{Hom}_{B_1}(P_{b_1}, \mathcal{D}_1^+) \neq 0$ and $b_1 \in \operatorname{supp} \mathcal{D}_1$. Similarly, if $P_{b_m} \notin \mathcal{C}$, then $b_m \in \operatorname{supp} \mathcal{D}_2$ and $\operatorname{supp} \mathcal{D}_1 \cap \operatorname{supp} \mathcal{D}_2 = \emptyset$ by 1.3. Therefore, there is some $1 \leq j \leq m$ such that $b_{j-1} \in \operatorname{supp} \mathcal{D}_1$, $P_{b_j} \in \mathcal{C}$ and $b_{j+1} \in \operatorname{supp} \mathcal{D}_2$. Then we can find $Y_1 \in \mathcal{D}_1$ and $Y_2 \in \mathcal{D}_2$ with non-zero maps $Y_1 \overset{g_1}{\text{---}} P_{b_j} \overset{g_2}{\text{---}} Y_2$ (for example, if $\mathrm{Hom}_{B_1}(P_{b_{j-1}}, \mathcal{D}_1^+) \neq 0$ and $P_{b_{j_1}} \xrightarrow{f_j} P_{b_j}$, then f_j factorizes through $\mathcal{D}_1^+$). Using standardness of $\mathcal{C}$ (cf. 1.3), it follows that $\mathcal{D}_1 = \mathcal{D}_2$.

(b) follows immediately by counting orbits and by Corollary 3 in 1.4. ∎

R e m a r k s. (1) We can view 3.1 as a generalization of the corresponding representation-finite situation shown in [14, (6.5)].

(2) Sincerity in 3.1 is needed, as the following example shows. Let $A = kQ/I$, where Q is as follows:



and $I = (\beta\alpha, \delta\beta, \delta\gamma)$. In this case $A = B[M]$ for $M = \operatorname{rad} P_a$ and B decomposes as four representation-finite algebras, one of which is not directing.

3.2. We need to recall some concepts. Following [8], we say that an A-module M with indecomposable decomposition $\bigoplus_{i=1}^{m} M_i$ is *directing* if there is no pair (i,j) and a non-projective module Y such that M_i is a predecessor of $\tau_A Y$ and Y a predecessor of M_j in mod_A.

We recall that if a is a source in Q, $M = \operatorname{rad} P_a$ and $A = B[M]$, then P_a is directing in mod_A if and only if M is directing in mod_B (cf. [8]).

Let $A = B[M]$ be as above, and let $h : X \to Y$ be an irreducible map in mod_B. Following [4], we say that h is *M-finite* if $h \notin \operatorname{rad}_A^{\infty}(X,Y)$. An indecomposable B-module Y is *M-finite* if there is a walk $M' = Y_0 \overset{\alpha_1}{\text{---}} Y_1 \overset{\alpha_2}{\text{---}} \ldots \overset{\alpha_s}{\text{---}} Y_s = Y$ in Γ_B, where M' is an indecomposable direct summand of M and where each α_i is M-finite, $1 \le i \le s$.

Assume that the direct summands of M lie in directing convex components $\mathcal{C}_1, \ldots, \mathcal{C}_s$ of Γ_B. Then we may consider the *vector space category* $\operatorname{Hom}_B(M, \mathcal{C}')$, where $\mathcal{C}' = \bigcup_{i=1}^{s} \mathcal{C}_i$; see [13, 14]. Define $|Y| = \operatorname{Hom}_B(M,Y)$ for $Y \in \mathcal{C}'$. Then the full subcategory of $\operatorname{Hom}_B(M,\mathcal{C}')$ whose objects are those $Y \in \mathcal{C}'$ with $|Y| \ne 0$, forms a poset $\mathcal{C}'_M$. Indeed, $|X| \le |Y|$ in $\mathcal{C}'_M$ implies that $X \le Y$ in the path order of $\mathcal{C}'$.

Lemma. *Let $h : X \to Y$ be an irreducible map in $\mathcal{C}'$. Then h is M-infinite if and only if the following two conditions hold*:

(i) *there are infinitely many pairwise non-isomorphic indecomposable A-modules of the form $(V, L, \gamma : V \to \operatorname{Hom}_B(M, L))$, where $V \in \operatorname{mod}_k$ and L lies in a B-module with X as a direct summand and γ is linear*;

(ii) *there is a morphism $0 \ne g \in \operatorname{Hom}_B(M,X)$ with $hg = 0$.*

The proof is the same as the one given in [4, (2.2)] for the case where $\mathcal{C}' = \mathcal{P}$ is a postprojective component.

3.3. We now prove the basic result on the "formation" of directing components by means of one-point extensions. Compare with [4, (2.3)] and [19, (2.8)].

THEOREM. *Let $A = B[M]$ be a one-point extension with $M = \operatorname{rad} P_a$. Assume $B = B_1 \amalg \ldots \amalg B_s$ is a decomposition with irreducible factors and let $M = M_1 \oplus \ldots \oplus M_s$ be the corresponding decomposition of M as in* 3.1. *Assume that for each $i = 1, \ldots, s$, there is a directing convex and sincere component $\mathcal{C}_i$ of Γ_{B_i} such that every indecomposable direct summand of M_i lies in $\mathcal{C}_i$. Then P_a belongs to a sincere directing convex component of Γ_A if and only if the following conditions hold*:

(i) *M is directing as a B-module*;

(ii) *each indecomposable projective B_i-module $P_y \in \mathcal{C}_i$ is M-finite. Moreover, the set of predecessors of P_y in Γ_A which are not B-modules is finite and formed by directing modules.*

Proof. Assume first that $\mathcal{C}$ is a directing convex and sincere component of Γ_A containing P_a. Therefore M is directing in mod_B (cf. 3.2). Let $P_y \in \mathcal{C}_i$. Since $\mathcal{C}$ is sincere, either $\operatorname{Hom}_A(P_y, \mathcal{C}^+) \neq 0$ or $P_y \in \mathcal{C}$. By the proof of 3.1(a), the first possibility implies $\operatorname{Hom}_A(P_y, \mathcal{C}_i^+) \neq 0$ and $P_y \notin \mathcal{C}_i$. Hence $P_y \in \mathcal{C}$. Therefore there is a walk $M' = X_0 \overset{\alpha_1}{\text{—}} X_1 \overset{\alpha_2}{\text{—}} X_2 \overset{\alpha_3}{\text{—}} \ldots \overset{\alpha_s}{\text{—}} X_s = P_y$ in Γ_A. We may assume that $\operatorname{Hom}_A(P_a, X_j) \neq 0$ and $\operatorname{Hom}_A(P_a, X_i) = 0$ for $j+1 \leq i \leq s$. Then there is an indecomposable direct summand Y of the restriction of X_j to B_i and a map $Y \overset{g}{\text{—}} X_{j+1}$ in mod_{B_i}, and $\operatorname{Hom}_{B_i}(M'', Y) \neq 0$ for some indecomposable direct summand M'' of M. We get a chain of non-zero maps $M'' \overset{f}{\text{—}} Y \overset{g}{\text{—}} X_{j+1} \overset{\alpha_{j+2}}{\text{—}} \ldots \overset{\alpha_s}{\text{—}} X_s = P_y$ in mod_{B_i}. Since $\operatorname{rad}_{B_i}^{\infty}(M'', Y) \subset \operatorname{rad}_A^{\infty}(M, X_j) = 0$ by 1.3, all these modules belong to $\mathcal{C}_i$. Again by 1.3, this yields a walk of M-finite maps between M'' and P_y. Hence P_y is M-finite.

Finally, let X be a predecessor of P_y in Γ_A which is not a B-module. Therefore, $\operatorname{Hom}_A(P_a, X) \neq 0$ and by convexity of $\mathcal{C}$, we get $X \in \mathcal{C}$. Obviously, there are only finitely many such modules, all of them directing.

For the converse, assume that (i) and (ii) are satisfied. We proceed to prove the result in several steps.

(1) We construct the component $\mathcal{C}$ of Γ_A where P_a lies and show that $\mathcal{C}$ has no oriented cycles. Indeed, we will define inductively full convex subquivers $\mathcal{C}_n$ of Γ_A satisfying:

(a) $\mathcal{C}_n$ is connected, contains no oriented cycle and is closed under predecessors, and

(b) $\tau_A^{-1}\mathcal{C}_n \cup \mathcal{C}_n \subset \mathcal{C}_{n+1}$.

Moreover, $P_a \in \mathcal{C}_0$ and hence $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ has no oriented cycles (indeed, taking a finite walk P_a — $\ldots$ — Y — X in $\mathcal{C}$, we may assume $Y \in \mathcal{C}_n$ and $Y \to X$. If X is not projective, $X \in \tau_A^{-1}\mathcal{C}_n \subset \mathcal{C}_{n+1}$. If X is projective, then Y is not injective, $\tau_A^{-1}Y \in \mathcal{C}_{n+1}$ and $X \in \mathcal{C}_{n+1}$).

Define $\mathcal{C}'_i$ to be the maximal full convex connected subquiver of $\mathcal{C}_i$ formed by all the predecessors of direct summands of M_i and modules which are non-comparable (in the path order of $\mathcal{C}_i$) with any direct summand of M_i. We form $\mathcal{C}_0$ by attaching to the $\mathcal{C}_i$ the irreducible maps from direct summands of M to P_a.

Before continuing the construction, let us check that indeed $\mathcal{C}_0$ is a full subquiver of Γ_A. It is enough to show that $\operatorname{Hom}_{B_i}(M_i, \tau_{B_i}Y) = 0$ for every $Y \in \mathcal{C}'_i$. But otherwise, we would get two paths $M' \to \ldots \to \tau_{B_i}Y$ and $Y \to \ldots \to M''$ in $\mathcal{C}'_i$, with M' and M'' direct summands of M, therefore contradicting assumption (i). Clearly, $\mathcal{C}_0$ satisfies (a).

Assume $\mathcal{C}_n$ is defined and let $X_1, \ldots, X_t$ be those modules in $\mathcal{C}_n$ with $\tau_A^{-1}X_i \notin \mathcal{C}_n$, numbered in such a way that $i < j$ whenever X_i precedes X_j in $\mathcal{C}_n$. Define $D_0 = \mathcal{C}_n$, and D_{i+1} as the full subquiver of Γ_A consisting of D_i and the predecessors of $\tau_A^{-1}X_{i+1}$ in Γ_A, and $\mathcal{C}_{n+1} := D_t$. It is enough to show inductively that D_i satisfies condition (a) above. Assume that D_i satisfies (a) and consider the almost split sequence $0 \to X_{i+1} \to X \to \tau_A^{-1}X_{i+1} \to 0$. We shall prove that each indecomposable direct summand Y of X has only finitely many predecessors in Γ_A which are not B-modules and there are no cycles in Γ_A passing through them.

Indeed, if Y is not projective, then Y belongs to D_i and we are done. Assume that Y is projective. By the arguments above we may assume that $Y = P_b$ with $b \neq a$. Since all the components $\mathcal{C}_i$ are sincere, there is some j such that $P_b \in \mathcal{C}_j$. By (ii), P_b is M-finite and the set of predecessors in Γ_A which are not B-modules is finite and formed by directing modules. This concludes the construction of $\mathcal{C}$.

(2) We show that $\mathcal{C}$ is directing convex and sincere. By (1), it is enough to show that $\mathcal{C}$ is convex and sincere.

Let $X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_s} X_s = Y$ be a sequence of non-zero maps in mod_A between indecomposable modules with $X, Y \in \mathcal{C}$. We may assume that $X_1, \ldots, X_{s-1}$ are not in $\mathcal{C}$. By construction of $\mathcal{C}$, there are only finitely many τ_A-orbits, hence there are non-zero maps $X_{s-1} \to Y' \to Y$ with $Y' \in \mathcal{C}^+$ and we may assume $Y' \in \mathcal{C}_i^+$. Without loss of generality we may also assume that $X \in \operatorname{mod}_B$ (otherwise $\operatorname{Hom}_A(M, X_0) \neq 0$, since $X_0 \neq S_a$, and we may start the chain of maps with $M' \xrightarrow{f_0} X_0$, where $M' \in \mathcal{C}_j$). If all X_i are B-modules, then we get $X \in \mathcal{C}_i$ and by convexity $X_j \in \mathcal{C}_i$ for $0 \leq j \leq s$, and then we may assume that all f_j are irreducible. By induction we show that all $X_j \in \mathcal{C}$ $(0 \leq j \leq s)$. Assume $X_j \in \mathcal{C}_n$

for $t \leq j \leq s$. Since $\mathcal{C}_n$ is closed under predecessors in Γ_A, if $X_{j-1} \notin \mathcal{C}$, then $0 \neq \mathrm{rad}_A^\infty(X_{j-1}, \mathcal{C}^+) = \mathrm{rad}_{B_i}^\infty(X_{j-1}, \mathcal{C}_i^+)$, a contradiction. Hence also $X_{j-1} \in \mathcal{C}$. In the other case, there is some X_i not a B-module. We may choose the last j such that $\mathrm{Hom}_A(P_a, X_j) \neq 0$. Since also $X_j \neq S_a$, we have $\mathrm{Hom}_B(M'', Y) \neq 0$ and $\mathrm{Hom}_B(Y, X_{j+1}) \neq 0$ for some indecomposable direct summand Y of X_j and M'' of M. Then our path may be substituted by $M'' \to Y \to X_{j+1} \to \ldots \to X_s = Y$ and we are as in the first situation. This shows that $\mathcal{C}$ is convex.

To see that $\mathcal{C}$ is sincere we shall prove that $P_a \in \mathcal{C}$ (by definition) and for all $P_b \in \mathcal{C}_i$, also $P_b \in \mathcal{C}$. But this follows from (ii). The proof is complete. ■

3.4. Clearly, 3.1 and 3.3 may be put together to provide the following method of construction of algebras with sincere directing convex components (and the components themselves).

Theorem. *Let $A = kQ/I$ be a k-algebra such that Q is connected. Then the following are equivalent*:

(a) *Γ_A admits a sincere directing convex component.*

(b) *There is a sequence $A_0, A_1, \ldots, A_s = A$ of convex subcategories of A satisfying the following conditions*:

  (1) *A_0 is either the trivial algebra k or a tilted algebra with a connecting component without projective or injective modules*;

  ($2i$) *each algebra A_i decomposes in a sum of irreducible factors $B_1^i \amalg \ldots \amalg B_{s(i)}^i$ and each $\Gamma_{B_j^i}$ admits a sincere directing convex component $\mathcal{C}_j^i$, $j = 1, \ldots, s(i)$.*

  For each $i = 0, \ldots, s-1$, the condition ($3i$) or its dual ($3i$)* holds:

  ($3i$) *$A_{i+1} = A_i[M_i]$ is a one-point extension with $M_i = \mathrm{rad}\, P_i$, where M_i decomposes as $M_1^i \oplus \ldots \oplus M_{s(i)}^i$ with $M_j^i \in \mathrm{mod}_{B_j}^i$, $j = 1, \ldots, s(i)$, and*

  - *M_i is a directing A_i-module*;
  - *each indecomposable projective B_j^i-module $P_j \in \mathcal{C}_j^i$ is M_i-finite. Moreover, the set of predecessors of P_j in $\mathrm{mod}_{A_{i+1}}$ which are not B_j^i-modules is finite and formed by directing modules.* ■

3.5. Finally, we prove that the fact that an indecomposable A-module X belongs to a directing convex component of $\mathcal{C}$ can be read from the values of a finite set of (easily constructible) linear functionals (compare with [9] and [12]).

Let $\mathcal{C}$ be a sincere directing convex component of Γ_A.

Let $\mathcal{S}_1^+, \ldots, \mathcal{S}_t^+$ be a set of (+)-sections intersecting exactly once all the τ-stable orbits of $\mathcal{C}$. Let $\mathcal{S}_1^-, \ldots, \mathcal{S}_m^-$ be the corresponding set of (−)-sections.

Let P be the direct sum of all projectives in $\mathcal{C}$ and I be the direct sum of all injectives in $\mathcal{C}$.

Consider $i = 1, \ldots, t$. Let B_i^+ be the support algebra of $\mathcal{C}^+(\mathcal{S}_i^+)$. By 1.2, there are functionals $f_i^+, g_i^+ : K_0(A) \to \mathbb{R}$ such that an indecomposable B_i^+-module X lies in $\mathcal{C}^+(\mathcal{S}_i^+)$ if and only if $f_i^+(\mathbf{dim}\, X) > 0$ and $g_i^+(\mathbf{dim}\, X) > 0$. Moreover, $g_i^+(z) = \sum_{Y \in \mathcal{S}_i^+} \langle z, \mathbf{dim}\, Y \rangle$. Dually, for $j = 1, \ldots, m$ consider the support algebra B_j^- of $\mathcal{C}^-(\mathcal{S}_j^-)$ and functionals $f_j^-, g_j^- : K_0(A) \to \mathbb{R}$ such that an indecomposable B_j^--module Y lies in $\mathcal{C}^-(\mathcal{S}_j^-)$ if and only if $f_j^-(\mathbf{dim}\, Y) > 0$ and $g_j^-(\mathbf{dim}\, Y) > 0$. Moreover, $g_j^-(z) = \sum_{X \in \mathcal{S}_j^-} \langle \mathbf{dim}\, X, z \rangle$.

We need also the functionals $p = \langle \mathbf{dim}\, P, - \rangle$ and $q = \langle -, \mathbf{dim}\, I \rangle$.

THEOREM. *An indecomposable A-module X lies in $\mathcal{C}$ if and only if one of the following conditions holds*:

(a) $p(\mathbf{dim}\, X) = 0$ *and for some* $i = 1, \ldots, t$, $f_i^+(\mathbf{dim}\, X) > 0$ *and* $g_i^+(\mathbf{dim}\, X) > 0$;

(b) $q(\mathbf{dim}\, X) = 0$ *and for some* $j = 1, \ldots, m$, $f_j^-(\mathbf{dim}\, X) > 0$ *and* $g_j^-(\mathbf{dim}\, X) > 0$;

(c) $p(\mathbf{dim}\, X) > 0$ *and for some* $j = 1, \ldots, m$, $g_j^-(\mathbf{dim}\, X) < 0$;

(d) $q(\mathbf{dim}\, X) > 0$ *and for some* $i = 1, \ldots, t$, $g_i^+(\mathbf{dim}\, X) < 0$;

(e) $p(\mathbf{dim}\, X) > 0$ *and* $q(\mathbf{dim}\, X) > 0$;

(f) *for some* $i = 1, \ldots, t$ *and* $j = 1, \ldots, m$, *we have* $g_i^+(\mathbf{dim}\, X) < 0$ *and* $g_j^-(\mathbf{dim}\, X) < 0$.

Proof. Let $X \in \Gamma_A$. We first show the following claim.

CLAIM. $X \in \mathcal{C}^+(\mathcal{S}_i^+)$ *if and only if* $p(\mathbf{dim}\, X) = 0$, $f_i^+(\mathbf{dim}\, X) > 0$ *and* $g_i^+(\mathbf{dim}\, X) > 0$.

Indeed, if $X \in \mathcal{C}^+(\mathcal{S}_i^+)$, then $p(\mathbf{dim}\, X) = \dim_k \mathrm{Hom}_A(P, X) = 0$ by 1.3. Moreover, by definition, $f_i^+(\mathbf{dim}\, X) > 0$ and $g_i^+(\mathbf{dim}\, X) > 0$. For the converse, observe that $p(\mathbf{dim}\, X) = 0$ implies that X is a module over $B_1^+ \amalg \ldots \amalg B_t^+$. If $f_i^+(\mathbf{dim}\, X) > 0$, then $\mathrm{supp}\, X$ intersects B_i^+ and hence $X \in \mathrm{mod}\, B_i^+$, and the result follows from the definitions.

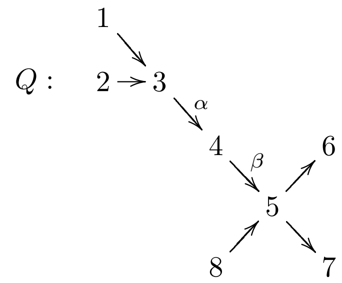
Similarly, we have $X \in \mathcal{C}^-(\mathcal{S}_j^-)$ for some $1 \le j \le m$ if and only if (b) holds.

Assume that X is in $\mathcal{C}$ but not in any $\mathcal{C}^+(\mathcal{S}_i^+)$ or $\mathcal{C}^-(\mathcal{S}_j^-)$.

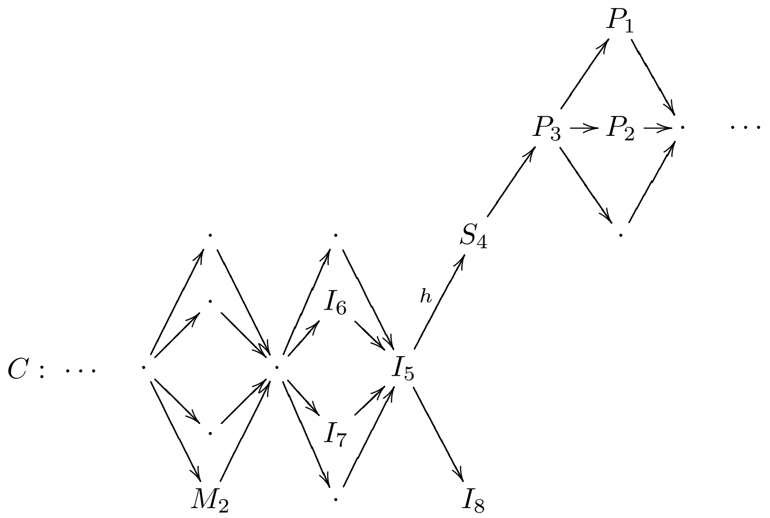
Assume that $p(\mathbf{dim}\, X) > 0$ but (e) does not hold. Then $\mathrm{Hom}_A(X, I) = 0$ and X is a $B_1^- \amalg \ldots \amalg B_m^-$-module. Assume $X \in \mathrm{mod}\, B_j^-$. Since $X \notin \mathcal{C}^-(\mathcal{S}_j^-)$, then either $f_j^-(\mathbf{dim}\, X) < 0$ or $g_j^-(\mathbf{dim}\, X) < 0$. In the first case X is not a preinjective B_j-module (see 1.2). In particular, X is a predecessor of $\mathcal{S}_j^-$ in mod_{B_j}, which is impossible. Hence $g_j^-(\mathbf{dim}\, X) < 0$ and we are in situation (c). If $p(\mathbf{dim}\, X) = 0$, by the dual argument we are either in case (d) or in case (f).

Conversely, assume X satisfies one of (a) to (f). In the situations (a) and (b) we apply 1.2. In the other cases we get modules Y_1 and Y_2 in $\mathcal{C}$ such that $\mathrm{Hom}_A(Y_1, X) \neq 0$ and $\mathrm{Hom}_A(X, Y_2) \neq 0$. Then convexity implies that $X \in \mathcal{C}$. ■

3.6. Example. Consider the algebra $B = kQ/I$ given by the quiver



and the ideal I generated by $\beta\alpha$. The Auslander–Reiten quiver Γ_B admits a sincere directing convex component $\mathcal{C}$ depicted below:



Consider the indecomposable B-modules $M_1 = P_3$ and $M_2 = \tau_B^2 I_8$, both in $\mathcal{C}$. We consider the one-point extensions $A_i = B[M_i]$, $i = 1, 2$.

First we observe that (i) and (ii) in Theorem 3.3 are satisfied by B and M_1. Indeed, M_1 is directing, P_1 and P_2 are M_1-finite (since the maps $P_3 \to P_i$ are irreducible in mod_{A_1}, $i = 1, 2$) and all their predecessors in Γ_{A_1} are B-modules. Hence Γ_{A_1} has a sincere directing convex component, which in fact is neither postprojective nor preinjective.

For $i = 2$, we observe that h is not M_2-finite. Indeed, $\dim_k \mathrm{Hom}_B(M_2, I_5) = 2$ and $\dim_k \mathrm{Hom}_B(M_2, S_4) = 1$ and therefore the conditions in 3.2 are satisfied. In particular, P_3 is not M_2-finite and by 3.3, Γ_{A_2} does not admit a sincere directing convex component. It is easy to see that the component of Γ_{A_2} containing M_2 is directing and convex.

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