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COMPENSATION COUPLES AND ISOPERIMETRIC ESTIMATES FOR VECTOR FIELDS

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Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set, and let $X_1, \ldots, X_p$ be real smooth vector fields defined in a neighborhood of $\overline{\Omega}$. We will say that $X_1, \ldots, X_p$ satisfy *Hörmander's condition of order* m, or that they are *of type* m, if $X_1, \ldots, X_p$ together with their commutators of length at most m span $\mathbb{R}^n$ at each point of $\overline{\Omega}$. It is well known that it is possible to associate with $X_1, \ldots, X_p$ a canonical metric ϱ as follows ([FP], [NSW]): we say that an absolutely continuous curve $\gamma : [0, T] \to \overline{\Omega}$ is a *subunit curve* if

$$|\langle \gamma'(t), \xi \rangle|^2 \leq \sum_j |\langle X_j(\gamma(t)), \xi \rangle|^2$$

for all $\xi \in \mathbb{R}^n$ and a.e. $t \in [0, T]$, and we define $\varrho(x, y)$ for $x, y \in \overline{\Omega}$ by

$$\varrho(x, y) = \inf\{T : \exists \text{ a subunit curve } \gamma : [0, T] \to \overline{\Omega} \text{ with } \gamma(0) = x,\ \gamma(T) = y\}.$$

The geometry of the metric space $(\overline{\Omega}, \varrho)$ is fully described in [NSW]; in particular, it is shown there that $(\overline{\Omega}, \varrho)$ is a metric space of homogeneous type with respect to Lebesgue measure, i.e. and there exist $C > 0$ and $\delta_0 > 0$ such that

$$|B(x, 2\delta)| \leq C|B(x, \delta)| \tag{1}$$

for all $x \in \overline{\Omega}$ and $\delta < \delta_0$, where $B(x, r) = \{y \in \overline{\Omega} : \varrho(x, y) < r\}$ is a metric ball, and for any measurable set E, $|E|$ denotes its Lebesgue measure. In particular, it follows from the doubling property (1) that there exist $\alpha \geq n$ and $c > 0$ such that

$$|B(x, \delta t)| \geq ct^\alpha |B(x, \delta)| \tag{2}$$

for all $x \in \overline{\Omega}$, $t \in (0, 1)$ and $\delta < \delta_0$.

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The exponent α in (2) plays an important role in many critical inequalities associated with the vector fields, by replacing the dimension n of Ω as a manifold. In particular, in the last few years, isoperimetric inequalities have been proved for $(\overline{\Omega}, \varrho)$ in which α gives an estimate of the isoperimetric dimension ([FGaW1,2], [FLW], [CDG], [G]). In its simplest form, the isoperimetric inequality can be stated as follows:

THEOREM 0. *Let E be an open, bounded, connected subset of Ω whose boundary ∂E is an oriented C^1 manifold such that E lies locally on one side of ∂E. If r_0 is sufficiently small and $B = B(x,r)$ is any ball with $x \in \overline{\Omega}$ and $0 < r < r_0$, then*

$$\min\{|B\cap E|, |B\setminus E|\}^{(\alpha-1)/\alpha} \le c \int_{\partial E\cap B} \Big(\sum_j \langle X_j, \nu\rangle^2\Big)^{1/2} dH_{n-1}, \tag{3}$$

where α is the exponent in (2), ν is the unit normal to ∂E, and the constants c, r_0 are independent of E and B.

It is possible to show that if the exponent α is sharp in (2), then the isoperimetric inequality cannot be improved. However, the result is not in general fully satisfying because the dimension can change from point to point, so that a global result can only be expressed in terms of the "worst exponent". To illustrate this phenomenon, let us consider the following two situations, which exemplify the "good" situation and the "bad" situation. First of all, let $n = 3$, and put $X_1 = \partial_1 + 2x_2\partial_3, X_2 = \partial_2 - 2x_1\partial_3$ (Heisenberg group). Here it is easy to see by Theorem 1 of [NSW] that $|B(x,\delta)| \sim \delta^4$ for $x \in \overline{\Omega}$, so that the dimension is uniformly equal to 4 and the isoperimetric inequality is satisfying. Next choose $n = 2$, $X_1 = \partial_1$, $X_2 = x_1^\beta\partial_2$ for $\beta \in \mathbb{N}$ (Grushin vector fields). In this case, $|B(x,\delta)| \sim \delta^2(|x_1|^\beta + \delta^\beta)$ if $x = (x_1, x_2)$, so that in order to obtain a global estimate we must choose $\alpha = 2+\beta$. This exponent cannot be improved if we consider sets around the origin as in [FGaW1,2], but it is not sharp for small balls away from the line $x_1 = 0$, where the natural dimension is 2.

If we try to sharpen the estimate (3) by working locally, i.e., by allowing α to depend on the size and position of the ball B, then the constant c that appears on the right side of (3) may also vary. In fact, the argument in [FLW] shows that this constant can be chosen independent of B only by using a value of α which works globally. By re-examining the argument in [FLW], we find in the general case that, for a given ball $B = B(x,r)$, the corresponding value of the constant c in (3) is actually $c_1 r|B|^{-1/\alpha}$, where c_1 depends only on $\overline{\Omega}$, r_0 and the constant c in (2) restricted to subballs of B. This "constant" clearly varies with α and B. For example, in the Grushin case mentioned above,

$$c_1 r|B|^{-1/\alpha} = c_1 r[r^2(|x_1| + r)^\beta]^{-1/\alpha}; \tag{4}$$

if $\alpha = 2+\beta$, then c_1 can be chosen independent of x_1 and r, and (4) equals

$$c_1(|x_1|/r+1)^{-\beta/(2+\beta)} \le c_1,$$

but if $\alpha = 2$ and r is small compared to $|x_1|$, then (4) is essentially $c_1|x_1|^{-\beta/2}$ with c_1 independent of x_1.

The fact that the constant c on the right in (3) can be chosen to be $c_1 r|B|^{-1/\alpha}$ ($= c_1 r|B|^{(1/q)-1}$ in the notation of [FLW]) is not explicitly stated in [FLW] but can be proved by following the reasoning there. In particular, we use the estimate preceding (4.3) of [FLW] but leave that estimate in terms of B rather than the larger balls of radius r_0 described there.

In general, we can then localize (3), and so obtain a more precise estimate, by dividing both sides of (3) by $r|B|^{-1/\alpha}$ and rewriting the estimate as

$$(3)' \qquad \frac{|B|^{1/\alpha}}{r}\min\{|B\cap E|, |B\setminus E|\}^{(\alpha-1)/\alpha} \le c_1 \int_{\partial E\cap B}\Big(\sum_j \langle X_j,\nu\rangle^2\Big)^{1/2} dH_{n-1}$$

where c_1 depends only on $\overline{\Omega}$, r_0 and the constant c in (2) restricted to subballs of B. The value of α may of course vary with B.

By applying the weighted isoperimetric inequalities proved in [FGaW1,2] and [FLW], we will show that there are cases when it is possible to stabilize (3)$'$ by replacing the left side by

$$\min\{\mu(B\cap E), \mu(B\setminus E)\}^{1/s},$$

where μ is a *fixed* measure and s is chosen *globally* with $s>1$. Moreover, the constant on the right side of the estimate will be a *global* constant. This occurs when there exists what we shall call a *compensation couple* (μ, s). The aim of the paper is to discuss this idea and to show that such a couple exists in many important examples, such as the case of the Grushin vector fields we considered above. Compensation couples also exist for vector fields of the type studied in [F], which are not smooth and so not of Hörmander type (see the remark after the proof of Proposition 5 below). Whenever $|B\cap E|$ and $|B\setminus E|$ are comparable, the version of the isoperimetric inequality involving (μ, s) will be equivalent to (3)$'$, and we will eventually show there are cases when it is better. On the other hand, there are also situations when the Lebesgue estimate (3)$'$ is sharper.

At this point, we make a short remark intended to help avoid misinterpretation of our starting inequality (3) in Riemannian settings. We would like to point out that the volume which appears in the isoperimetric inequality (3) does not coincide with the Riemannian volume when the distance ϱ comes from a Riemannian metric, i.e., when $p=n$ and $X_1,\dots,X_p$ are linearly independent. In fact, the Riemannian measure in this case is absolutely continuous with respect to Lebesgue measure with a density given by the square root of the reciprocal of the determinant of the matrix associated with the quadratic form $\sum_j \langle X_j, \xi\rangle^2$. On the other hand, in more general situations which are not Riemannian in nature, this natural weight is not suitable, since it is easy to see in many elementary situations (e.g., the Heisenberg group in $\mathbb{R}^3$ with its two standard vector fields) that such a weight would be identically ∞. This surprising phenomenon can be explained in two ways: first, the Lebesgue measure which appears in (3) reflects the fact that Lebesgue measure appears in Sobolev and Poincaré inequalities associated with the vector fields, and this measure in turn arises from the weak formulation of the equation $\sum_j X_j^2 = f \in L^2(\Omega)$. But perhaps a deeper explanation of the reason that the natural volume form is infinite lies in the fact that the "true" dimension of (Ω, ϱ) is in general much larger than its dimension n as a manifold (as we can see in all our dimensional inequalities); thus, it is not surprising that the formal n-dimensional Riemannian measure, which is a lower dimensional measure, is infinite.

We will further discuss some facts and examples related to the two isoperimetric estimates in §2. A point $x_0 \in \partial E$ is called a *characteristic point* of ∂E for $\{X_j\}$ if ∂E is a C^1 manifold in a neighborhood of x_0 and if each $X_j(x_0)$ lies in the tangent space of ∂E at x_0, i.e., if

$$\sum_j \langle X_j(x_0), \nu(x_0)\rangle^2 = 0,$$

where $\nu(x_0)$ is the unit normal to ∂E at x_0. We will give a simple proof of the fact that the characteristic points of a smooth manifold are few in the sense that the set of characteristic points has Hausdorff dimension at most $n-2$ (see Theorem 8). Moreover, in Theorem 7, we will show that if B is a small ball centered at a noncharacteristic point of ∂E, then $|B \cap E|$ and $|B \setminus E|$ are comparable, and consequently the two isoperimetric estimates for B are the same. Examples showing when these two estimates are not comparable are given at the end of the paper.

Finally, we note that the existence of a compensation couple can be used in other situations to deal with problems arising from the fact that the isoperimetric dimension fails to be constant. For instance, some of the estimates in [FGuW] that involve studying the continuity of operators of potential type whose kernels are related to strong-A_∞ weights rely on the existence of a compensation couple ($\mu = \lambda^{m/(N-1)}$ and $s = N/(N-1)$ in the notation used there).

1. Facts about compensation couples. First of all, let us recall the formula given in Theorem 1 of [NSW] which expresses the measure of a generic ball $B(x,r)$. In order to state this formula, let $\{Y_1,\ldots,Y_q\}$ be the set of all commutators of $X_1,\ldots,X_p$ of order $\le m$, and let us set length $Y_j = d_j$; if $I=(i_1,\ldots,i_n)$ is an n tuple of indices in $\{1,\ldots,q\}$, set $|I| = d_{i_1}+\ldots+d_{i_n}$. Then we have, for a suitable $L \ge n$,

$$0 < c_1 \le \frac{|B(x,\delta)|}{\sum_{j=n}^{L} \lambda_j(x)\delta^j} \le c_2, \tag{5}$$

where

$$\lambda_j(x) = \Big(\sum_{|I|=j} [\det(Y_{i_1},\ldots,Y_{i_n})(x)]^2\Big)^{1/2}. \tag{6}$$

Note that each λ_j is continuous since the Y_j are smooth, and that $\lambda_j(x) \ge 0$.

Throughout this paper, if w is a weight function (i.e., $w \in L^1_{\mathrm{loc}}(\Omega)$, $w \ge 0$) we put $w(E) = \int_E w\,dx$ for any measurable set E. Moreover, if $f \in L^1_{\mathrm{loc}}(E)$, we put $⨍_E f(x)\,dx = (1/|E|)\int_E f(x)\,dx$.

We can now define the notion of a compensation couple.

Definition. Let $\mu \in L^1_{\mathrm{loc}}(\Omega)$ be a nonnegative function, and let $s > 1$ be a real number. We say that (μ,s) is a *compensation couple* if there exist constants $c, C > 0$ such that

$$c(|B|/\delta)^s \le \int_B \mu(y)\,dy \le C(|B|/\delta)^s \tag{7}$$

for every ball $B = B(x,\delta)$, $x \in \overline{\Omega}$, $\delta < \delta_0$.

A motivation for the definition is that in case $|B\cap E|$ and $|B\setminus E|$ are comparable (and so are each comparable to $|B|$), (7) implies that the left side of $(3)'$ is $\mu(B)^{1/s}$ for any value of α.

Although we restrict our attention to metrics in Euclidean space associated with Hörmander vector fields, a similar definition could be given in any space of homogeneous type in the sense of [CW].

With the notation of (5), (6), we have:

Proposition 1. *Let $d = \min\{j \in \{n,\ldots,L\} : \lambda_j(x) \not\equiv 0$ in $\Omega\}$. If a compensation couple (μ,s) exists, then $s = d/(d-1)$ and we can choose $\mu(x) = \lambda_d(x)^{1/(d-1)}$.*

Proof. Let x be a Lebesgue point of μ such that $\lambda_d(x) \ne 0$. Then by (7) and (5) we have

$$\mu(x) = \lim_{\delta\to 0} ⨍_{B(x,\delta)} \mu(y)\,dy \sim \lim_{\delta\to 0} \sum_{j=d}^{L} \lambda_j(x)^{s-1}\delta^{j(s-1)-s}.$$

Since $\mu(x) < \infty$, we must have $d(s-1) \ge s$, i.e., $s \ge d/(d-1)$. On the other hand, since the equivalence above holds at any Lebesgue point of μ, if we had $s > d/(d-1)$ then μ would vanish a.e. on Ω, which is impossible by (7). Thus, $s = d/(d-1)$ and $\mu(x) \sim \lambda_d(x)^{s-1} = \lambda_d(x)^{1/(d-1)}$ a.e.

By A_∞ and A_p, $1 \le p < \infty$, we mean the corresponding weight function classes with respect to metric balls, and we refer to [C] for a discussion of these classes. We also say that $\mu \in RH_\infty$ with respect to metric balls if

$$(RH_\infty) \qquad \operatorname*{ess\,sup}_B \mu \sim \fint_B \mu\, dy;$$

see [F], [FGuW] and [CUN] for facts about this class.

Proposition 2. *If (μ, s) is a compensation couple, then $\mu \in RH_\infty$ with respect to metric balls. In particular, $\mu \in A_\infty$ with respect to metric balls.*

Proof. Let $y \in B = B(x,\delta)$. By Proposition 1,

$$\begin{aligned} \mu(y) &= \lambda_d(y)^{1/(d-1)} = \delta^{-d/(d-1)}[\delta^d \lambda_d(y)]^{1/(d-1)} \\ &\le c\delta^{-d/(d-1)}|B(y,\delta)|^{1/(d-1)} \\ &\le c\delta^{-s}|B|^{1/(d-1)} \qquad \text{by (5) and doubling} \\ &\le c|B|^{-s}\int_B \mu \cdot |B|^{1/(d-1)} \qquad \text{by (7)} \\ &= c\frac{1}{|B|}\int_B \mu, \end{aligned}$$

and the result follows.

Proposition 3. *With the notation of* (5) *and* (6), *if a compensation couple exists, then for any $j \in \{d, \ldots, L\}$ we have*

$$(8) \qquad \delta^j \Big(\fint_B \lambda_j(y)^{1/(d-1)}\, dy\Big)^{d-1} \le c\delta^d \Big(\fint_B \lambda_d(y)^{1/(d-1)}\, dy\Big)^{d-1}$$

for any metric ball $B = B(x,\delta)$, $x \in \overline{\Omega}$, $\delta < \delta_0$. Conversely, if (8) *holds for all j, then $(\lambda_d^{1/(d-1)}, d/(d-1))$ is a compensation couple. Moreover,* (8) *is equivalent to*

$$(8)' \qquad \sup_{y\in B(x,\delta)} \delta^j \lambda_j(y) \le c\delta^d \Big(\fint_{B(x,\delta)} \lambda_d(y)^{1/(d-1)}\, dy\Big)^{d-1}.$$

Proof. By (5), for all $y \in B = B(x,\delta)$,

$$\lambda_j(y)\delta^j \le \frac{1}{c_1}|B(y,\delta)| \le c|B|$$

by doubling. Hence, taking the power $1/(d-1)$ and integrating over B, we get

$$(9) \qquad \delta^{j/(d-1)} \fint_B \lambda_j(y)^{1/(d-1)}\, dy \le c|B|^{1/(d-1)}.$$

If a compensation couple (μ, s) exists, then by Proposition 1 and (7),

$$|B|^{1/(d-1)} = \delta^s \frac{|B|^{s-1}}{\delta^s} \le c\delta^{d/(d-1)} \fint_B \lambda_d(y)^{1/(d-1)}\, dy,$$

and the first assertion follows.

Suppose now that (8) holds. By (9) with $j = d$ and $B = B(x,\delta)$,

$$\int_B \lambda_d(y)^{1/(d-1)}\, dy \le c(|B|/\delta)^{d/(d-1)}.$$

Thus it will be enough to prove the reverse inequality. Again by (5) and doubling,

$$|B|^{1/(d-1)} \le c \sum_j \lambda_j(y)^{1/(d-1)} \delta^{j/(d-1)}, \qquad y \in B.$$

Integrating over B, we get

$$\begin{aligned} |B|^{d/(d-1)} &\le c \sum_j \int_B \lambda_j(y)^{1/(d-1)}\, dy \cdot \delta^{j/(d-1)} \\ &\le c\delta^{d/(d-1)} \int_B \lambda_d(y)^{1/(d-1)}\, dy \qquad \text{by (8)}, \end{aligned}$$

and the second assertion follows.

Finally, to see that (8) implies the stronger estimate $(8)'$, note that since $\delta^j \lambda_j(y) \le c|B_\delta(y)| \le c|B_\delta(z)|$ for all $y, z \in B(x,\delta) = B$, we have

$$\begin{aligned} \sup_B \delta^j \lambda_j(y) &\le c\Big(\fint_B |B_\delta(z)|^{1/(d-1)}\, dz\Big)^{d-1} \\ &\le c\delta^d \Big(\fint_B \lambda_d(z)^{1/(d-1)}\, dz\Big)^{d-1} \qquad \text{if (8) holds}, \end{aligned}$$

which gives $(8)'$. This completes the proof of Proposition 3.

Compensation couples do not always exist. Consider for instance the following simple situation in $\mathbb{R}^3$: $X_1 = \partial_1 + 2x_2\partial_3, X_2 = \partial_2 - 2x_1\partial_3, X_3 = x_1\partial_2$. Then $[X_1, X_2] = -4\partial_3$ and

$$|B(x,\delta)| \sim x_1^2\delta^3 + \delta^4,$$

so that if a compensation couple exists, then by Proposition 1, $s = 3/2$ and $\mu(x) = |x_1|$. Choose now a ball $B(0,\delta)$; by [NSW], Theorem 7, we can assume that

$$B(0,\delta) = (-\delta,\delta) \times (-\delta,\delta) \times (-\delta^2,\delta^2),$$

so that

$$\int_{B(0,\delta)} |x_1|\, dx \sim \delta^5, \quad \text{whereas} \quad (|B(0,\delta)|/\delta)^{3/2} \sim \delta^{9/2}.$$

We now give some examples of vector fields for which compensation couples exist.

PROPOSITION 4. *Suppose the vector fields* $Y_1, \ldots, Y_q$ *are free of order* m, *i.e.*, *the commutators of length at most* m *satisfy no linear relationships other than antisymmetry and the Jacobi identity. Then* $(1, Q/(Q-1))$ *is a compensation couple*, *where* $Q = \sum_{i=1}^m im_i$, m_i *denoting the number of linearly independent commutators of length* i.

Proof. The proof is trivial once we note that $\lambda_j(x) = 0$ if $j < Q$.

PROPOSITION 5. *Suppose that* $X_1, \ldots, X_p$ *satisfy Hörmander's condition with* $p = n$ *and* $X_j = \mu_j(x)\partial_j, j = 1, \ldots, n$. *Then* $(\prod_j |\mu_j(x)|^{1/(n-1)}, n/(n-1))$ *is a compensation couple. Moreover*, $|\mu_j| \in RH_\infty$ *for* $j = 1, \ldots, n$.

Proof. We can use the characterization of metric balls given in [F], Theorem 2.3. To this end, if $x \in \Omega, r > 0$ and $j = 1, \ldots, n$, let

$$c_j(x,r) = \{u_j(t) : 0 \le t \le r, \text{ where } u = (u_1, \ldots, u_n) \text{ is any subunit curve with } u(0) = x\}.$$

Then we set

$$M_k(x,r) = \sup\Big\{|\mu_k(s)| : s \in \prod_{j=1}^n c_j(x,r)\Big\},$$

$$Q(x,r) = \prod_{k=1}^n (x_k - rM_k(x,r), x_k + rM_k(x,r)).$$

It follows from [F], Theorem 2.3, that there exists $b > 1$ such that

$$Q(x, r/b) \subset B(x,r) \subset Q(x,r) \tag{10}$$

for any $x \in \overline{\Omega}$, $r < r_0$.

We will prove that

$$B(x,r) \subset \prod_j c_j(x,r) \subset B(x, 2br). \tag{11}$$

Obviously, $B(x,r) \subset \prod_j c_j(x,r)$. Let us now show that $\prod_j c_j(x,r) \subset \overline{Q(x,r)}$; then (11) will follow from (10). Arguing by contradiction, suppose that this assertion does not hold. Then there would be $j \in \{1, \ldots, n\}$ and a subunit curve $\gamma : [0,T] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma(T) = y$ with $T \le r$ and $|x_j - y_j| > rM_j(x,r)$. Then $y \notin \overline{Q(x,r)}$, but $y \in \overline{B(x,r)} \subset \overline{Q(x,r)}$, which is a contradiction. This proves (11).

By (11), if $k \in \{1, \ldots, n\}$,

$$\sup_{B(x,r)} |\mu_k(y)| \le \sup_{\prod_j c_j(x,r)} |\mu_k(y)| = M_k(x,r),$$

and

$$\sup_{B(x,r)} |\mu_k(y)| \geq \sup_{\prod_j c_j(x,r/(2b))} |\mu_k(y)| = M_k(x, r/(2b)).$$

On the other hand, $M_k(x,r)$ is doubling in r uniformly with respect to x since

$$\begin{aligned} M_k(x,2r) &= \frac{|Q(x,2r)|}{(4r)^n \prod_{j\neq k} M_j(x,2r)} \leq \frac{|B(x,2br)|}{(4r)^n \prod_{j\neq k} M_j(x,2r)} \\ &\leq c\frac{|B(x,r)|}{r^n \prod_{j\neq k} M_j(x,2r)} \leq c\frac{|Q(x,r)|}{r^n \prod_{j\neq k} M_j(x,r)} \leq cM_k(x,r). \end{aligned}$$

Hence,

$$cM_k(x,r) \leq \sup_{B(x,r)} |\mu_k| \leq M_k(x,r), \tag{12}$$

and therefore

$$|B(x,r)| \sim r^n \prod_{k=1}^{n} \sup_{B(x,r)} |\mu_k|. \tag{13}$$

Due to the diagonal nature of the X_k and the Hörmander condition, it follows that $M_k(x,r_0) \geq c > 0$ for some c and all $x \in \overline{\Omega}$ and $k \in \{1,\ldots,n\}$. Therefore, by the doubling property of $M_k(x,r)$ in r, there exists $\alpha_k \geq 0$ so that

$$M_k(x,r) \geq cr^{\alpha_k} \quad \text{for all } x \in \overline{\Omega} \text{ and } r < r_0. \tag{14}$$

Using the Taylor expansion of μ_k with center x and order $\nu - 1 = [\alpha_k]$, we have

$$\mu_k(y) = P_\nu(x;y) + R_\nu(x;y),$$

with

$$|R_\nu(x;y)| \leq c|x-y|^\nu \quad \text{for all } y \in \overline{\Omega},$$

so that

$$\int_{B(x,r)} |\mu_k(y)|\,dy \geq \int_{B(x,r)} |P_\nu(x;y)|\,dy - c\int_{B(x,r)} |x-y|^\nu\,dy. \tag{15}$$

On the other hand, by (10),

$$\begin{aligned} \Phi_k(x,r) := \sup_{B(x,r)} |\mu_k(y)| &\leq \sup_{Q(x,r)} |\mu_k(y)| \leq \sup_{Q(x,r)} |P_\nu(x;y)| + c\sup_{Q(x,r)} |x-y|^\nu \\ &\leq \sup_{Q(x,r)} |P_\nu(x;y)| + cr^\nu, \end{aligned}$$

since $Q(x,r) \subset B(x,br) \subset B_{\mathrm{euc}}(x,cr)$, where for the last inclusion we use the fact that $\varrho(x,y) \geq c|x-y|$ by Proposition 1.1 of [NSW]. By (12) and

(14), and since $\nu = \varepsilon + \alpha_k$ where $\varepsilon = [\alpha_k] - \alpha_k + 1 > 0$, there exists $r_0 > 0$ such that if $r < r_0$ then $Cr^\nu < \frac{1}{2}\Phi_k(x,r)$, and hence

$$\Phi_k(x,r) \le 2 \sup_{Q(x,r)} |P_\nu(x;y)|. \tag{16}$$

Analogously,

$$\int\limits_{Q(x,r)} |x-y|^\nu \, dy \le c|B(x,r)|r^\nu \le cr^\varepsilon |B(x,r)|\Phi_k(x,r). \tag{17}$$

Now, since the space of all polynomials of degree $\le \nu - 1$ on $\{|\eta_j| \le 1 : j = 1,\dots,n\}$ is finite-dimensional and hence all norms are equivalent, we have

$$\int\limits_{Q(x,r/b)} |P_\nu(x;y)| \, dy = \int\limits_{|y_j - x_j| < (r/b)M_j(x,r/b)} \left| \sum_{|\beta|<\nu} \frac{D^\beta \mu_k(x)}{\beta!}(y-x)^\beta \right| dy.$$

On putting $y_j = x_j + \eta_j (r/b) M_j(x,r/b)$, $j = 1,\dots,n$, this equals

$$\begin{aligned} cr^n \prod_i M_i(x,r/b) &\int\limits_{|\eta_j|\le 1} \Bigg| \sum_{|\beta|<\nu} \frac{D^\beta \mu_k(x)}{\beta!} \left(\frac{r}{b}\right)^{|\beta|} \\ &\quad \times M_1^{\beta_1}(x,r/b) \cdots M_n^{\beta_n}(x,r/b) \eta_1^{\beta_1} \cdots \eta_n^{\beta_n} \Bigg| \, d\eta \\ &\ge c|B(x,r)| \sum_{|\beta|<\nu} \left| \frac{D^\beta \mu_k(x)}{\beta!} \right| r^{|\beta|} M_1^{\beta_1}(x,r/b) \cdots M_n^{\beta_n}(x,r/b) \\ &\ge c|B(x,r)| \sup_{Q(x,r)} |P_\nu(x;y)|. \end{aligned}$$

Hence by (15), (10) and (17),

$$\begin{aligned} \fint\limits_{B(x,r)} |\mu_k(y)| \, dy &\ge \frac{1}{|B(x,r)|} \int\limits_{Q(x,r/b)} |P_\nu(x;y)| \, dy - cr^\varepsilon \Phi_k(x,r) \\ &\ge c_1 \sup_{Q(x,r)} |P_\nu(x;y)| - cr^\varepsilon \Phi_k(x,r) \\ &\ge \frac{c_1}{2} \Phi_k(x,r) - cr^\varepsilon \Phi_k(x,r) \qquad \text{by (16)} \\ &\ge \frac{c_1}{4} \Phi_k(x,r) = \frac{c_1}{4} \sup_{B(x,r)} |\mu_k|, \end{aligned}$$

if $r < r_0$, r_0 sufficiently small. This proves the last assertion in Proposition 5.

Now, if $B = B(x,r)$,

$$\begin{aligned} \int\limits_B |\mu_1 \cdots \mu_n|^{1/(n-1)} \, dy &\le (\sup_B |\mu_n| \cdots \sup_B |\mu_n|)^{1/(n-1)} |B| \\ &\le c(|B|/r)^{n/(n-1)} \qquad \text{by (13)}, \end{aligned}$$

and then the second inequality in the definition of compensation couple is proved.

To prove the opposite inequality, recall the following facts shown in [FGuW], Proposition 2.3:

(i) if $w \in RH_\infty$ and $u \in A_\infty$, then $wu \in A_\infty$;
(ii) if $w \in RH_\infty$, then $w^\beta \in RH_\infty$ for any $\beta > 0$.

Hence we can write

$$\int_B |\mu_1 \cdots \mu_n|^{1/(n-1)}\, dy = \int_B |\mu_1|^{1/(n-1)} |\mu_2 \cdots \mu_n|^{1/(n-1)}\, dy,$$

where $|\mu_1|^{1/(n-1)} \in RH_\infty$, $|\mu_2 \cdots \mu_n|^{1/(n-1)} \in A_\infty$, and thus $|\mu_2 \cdots \mu_n|^{1/(n-1)} \in A_p$ for some $p \geq 1$. Then

$$\begin{aligned}
\int_B |\mu_1|^{1/(n-1)} &|\mu_2 \cdots \mu_n|^{1/(n-1)}\, dy \\
&\geq c\Big(\fint_B |\mu_1|^{1/p(n-1)}\, dy\Big)^p \int_B |\mu_2 \cdots \mu_n|^{1/(n-1)}\, dy \\
&\geq c(\sup_B |\mu_1|)^{1/(n-1)} \int_B |\mu_2 \cdots \mu_n|^{1/(n-1)}\, dy \quad \text{by (ii)} \\
&\geq c(\sup_B |\mu_1| \cdots \sup_B |\mu_n|)^{1/(n-1)} |B| \quad \text{by iterating the same argument} \\
&\geq c(|B|/r)^{n/(n-1)} \quad \text{by (13)},
\end{aligned}$$

and Proposition 5 is completely proved. ■

Remark. It follows from the proof of the previous result that the existence of a compensation couple for Hörmander vector fields $\mu_1\partial_1, \ldots, \mu_n\partial_n$ relies on the fact that the functions $|\mu_1|, \ldots, |\mu_n|$ are RH_∞ weights with respect to the metric ϱ, and the main point of the proof consists in showing that the supremum of $|\mu_j|$ on a metric ball is bounded by its average on the same ball. Thus, a compensation couple still exists if we drop the smoothness assumptions on $\mu_1, \ldots, \mu_n$ provided that the metric ϱ associated with them is finite and continuous with respect to the Euclidean topology, and that $|\mu_1|, \ldots, |\mu_n|$ are weight functions in RH_∞. This happens for instance if $\mu_1, \ldots, \mu_n$ satisfy the assumptions (H2), (H3) and (H4) of [F]. The continuity of ϱ follows in fact from Remark 4, p. 133, therein. Thus, let us prove that μ_j (≥ 0) belongs to RH_∞ for $j = 1, \ldots, n$. Let $u = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$, and denote by D the set of points ξ of the form $\xi = \lambda\eta$, with $\lambda > 0, |\eta - u| < 1/(2\sqrt{n})$ and such that $1/2 \leq |\xi| \leq 1$. If $|\eta - u| < 1/(2\sqrt{n})$, then $\eta_j \geq 1/(2\sqrt{n})$; on the other hand, if $\xi = \lambda\eta \in D$, then $1/2 \leq |\xi| \leq \lambda|\eta| \leq 3\lambda/2$, so that $1 \geq \xi_j \geq 1/\sigma\sqrt{n} = \varepsilon_0$, and then, with the notations of [F], $D \subset B(0,1) \cap \varDelta_{\varepsilon_0}$.

Now, by [F], Remark 3 on p. 133 and Proposition 3.1, we have for $x \in \overline{\Omega}$ and $\delta \in (0, \delta_0)$,

$$\begin{aligned}
\fint_{B(x,\delta)} \mu_j(y)\,dy &\ge \frac{1}{|B(x,\delta)|} \int_{H(\delta/\sqrt{n},x,D)} \mu_j(y)\,dy \\
&= \frac{1}{|B(x,\delta)|} \int_D \left| \det \frac{\partial H}{\partial \xi}(\delta/\sqrt{n}, x, \xi) \right| \mu_j(H(\delta/\sqrt{n}, x, \xi))\,d\xi \\
&\ge c \int_D \mu_j(H(\delta/\sqrt{n}, x, \xi))\,d\xi \\
&= c \int_{D\cap S^{n-1}} d\omega \int_{1/2}^{1} \mu_j(H(\delta/\sqrt{n}, x, \varrho\omega))\,d\varrho \\
&= c \int_{D\cap S^{n-1}} d\omega \int_{1/2}^{1} \mu_j(H(\varrho\delta/\sqrt{n}, x, \omega))\,d\varrho,
\end{aligned}$$

since $H(\theta s, x, \xi) = H(s, x, \theta\xi)$, by the definition of H and the uniqueness of the Cauchy problem. On the other hand, putting $\varrho\delta/\sqrt{n} = s$, we see that the last integral is bounded below by

$$\begin{aligned}
c\frac{1}{\delta} \int_{D\cap S^{n-1}} d\omega \int_0^{\delta/\sqrt{n}} \mu_j(H(s, x, \omega))\,ds \\
&\ge c M_j(x, \delta/\sqrt{n}) \quad \text{by hypothesis (H.4) in [F]} \\
&\ge c M_j(x, \delta),
\end{aligned}$$

by doubling ([F], Proposition 2.5). Since all constants in the above inequality are independent of δ and (locally) of x, we have proved that $|\mu_j| \in RH_\infty$ for $j = 1, \ldots, n$.

2. Isoperimetric estimates. Let us now show how the existence of a compensation couple can be used to improve the isoperimetric inequality (3)$'$. We first recall from [FLW] that if w_1, w_2 are doubling weights such that w_1 is continuous and belongs to A_1, and w_2 is doubling (i.e., the measure $w_2(x)\,dx$ is doubling), then the isoperimetric inequality (3)$'$ can be replaced by

$$(3^*)\qquad \frac{w_1(B)}{r w_2(B)^{1/q}} \min\{w_2(B\cap E), w_2(B\setminus E)\}^{1/q} \le c_1 \int_{\partial E\cap B} \Big(\sum_j \langle X_j, \nu\rangle^2\Big)^{1/2} w_1\,dH_{n-1}$$

where q is such that

$$\frac{r(I)}{r(J)}\left(\frac{w_2(I)}{w_2(J)}\right)^{1/q} \le c\frac{w_1(I)}{w_1(J)} \tag{18}$$

for all metric balls I, J with $I \subset J \subset B$, where $r(I)$ denotes the radius of I and c_1 depends only on the constant c in (18) corresponding to the particular ball B, and the (local) A_1 constant of w_1 and doubling constant of w_2. Again, as in the case when $w_1 = w_2 \equiv 1$, this estimate is not explicitly stated in [FLW], but it follows for two weights w_1, w_2 in the same way that we indicated in the introduction when $w_1, w_2 \equiv 1$.

If (μ, s) is a compensation pair and we choose $w_1 \equiv 1$ and $w_2 = \mu$ (which is doubling since it belongs to A_∞), then (18) takes the form

$$\left(\frac{r(J)}{|J|}\cdot\frac{|I|}{r(I)}\right)^{(s/q)-1} \le c_1, \tag{19}$$

which is trivially satisfied uniformly in B if $q = s$. Thus we have the following result.

Theorem 6. *Let E, B and ν be as in Theorem 0. If a compensation couple (μ, s) exists, then*

$$\min\left\{\int\limits_{B\cap E}\mu(y)\,dy, \int\limits_{B\setminus E}\mu(y)\,dy\right\}^{1/s} \le c\int\limits_{\partial E\cap B}\left(\sum_j\langle X_j,\nu\rangle^2\right)^{1/2} dH_{n-1} \tag{20}$$

with c independent of E and B.

Starting from the relative isoperimetric inequality (20), by a covering argument, we can pass to a global one. This global result can also be obtained directly from the corresponding weighted global inequalities in [FGaW1,2].

We now discuss some facts concerning relationships between the two isoperimetric estimates $(3)'$ and (20). As mentioned earlier, the two are equivalent if $|B\cap E|$ and $|B\setminus E|$ are comparable. We first prove the following result concerning the noncharacteristic points of a smooth boundary ∂E. As always, $\{X_j\}$ denotes the fixed collection of Hörmander vector fields.

Theorem 7. *Let Σ denote the boundary of E, where E is a bounded, connected subset of Ω lying on one side of Σ. If $x_0 \in \Sigma, \Sigma$ is of class C^1 in a neighborhood U of x_0, and x_0 is not a characteristic point of Σ for $\{X_j\}$ (i.e., $\sum_j\langle X_j(x_0), \nu(x_0)\rangle^2 > 0$, where $\nu(x_0)$ is the outer unit normal to Σ), then $|B(x_0,r)\cap E|$ and $|B(x_0,r)\setminus E|$ are comparable to $|B(x_0,r)|$ for sufficiently small $r > 0$.*

Proof. Without loss of generality, we may assume that $x_0 = 0$. Let $D(0)$ denote the vector space generated by $X_1(0), \dots, X_p(0)$. By hypothesis, $\nu(0)$ is not orthogonal to $D(0)$, so that if we express

$$\nu(0) = \nu_X + u \quad \text{ with } \nu_X \in D(0), u \perp D(0),$$

then $\nu_X \neq 0$.

Let ψ solve the Cauchy problem $\dot\psi = \sum_j \lambda_j X_j(\psi), \psi(0) = 0$, where the λ_j are chosen so that $\nu_X = \sum_j \lambda_j X_j(0)$. We will prove that $d_{\mathrm{euc}}(\psi(r), \Sigma \cap U) \sim r$ as $r \to 0+$, where d_{euc} denotes the usual Euclidean distance. Using a rotation and shrinking U if necessary, we may assume that there exists $f \in C^1(U, \mathbb{R})$ such that $\Sigma \cap U = \{x \in U : f(x) = 0\}$ and $\nabla f(0) = e_n = (0, \dots, 0, 1)$. Then $\nu_X = (\xi', \theta)$, with $\xi' \in \mathbb{R}^{n-1}$ and $\theta \in \mathbb{R}, \theta \neq 0$: indeed, if $\theta = 0$, then we would have

$$0 = \langle \nu(0), \nu_X \rangle = |\nu_X|^2 + \langle \nu_X, u \rangle = |\nu_X|^2,$$

a contradiction. Thus we can consider the map $F : U \to \mathbb{R}^n$ defined by

$$F(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, f(x)).$$

Keeping in mind that the Jacobian matrix of F at $x = 0$ is the identity, and by shrinking U if necessary, we obtain

$$|F(x) - F(y)| \sim |x - y| \quad \text{ if } x, y \in U.$$

It is easy to see that $F(\Sigma \cap U) = F(U) \cap \{y_n = 0\}$ and that

$$d_{\mathrm{euc}}(F(\psi(r)), F(\Sigma \cap U)) = f(\psi(r)) = r \langle \nabla f(\psi(t_r)), \dot\psi(t_r) \rangle$$

for a suitable $t_r \in (0, r)$. But when $r \to 0+$,

$$\langle \nabla f(\psi(t_r)), \dot\psi(t_r) \rangle \to \langle \nabla f(0), \nu_X \rangle = \theta \neq 0,$$

and then $d_{\mathrm{euc}}(\psi(r), \Sigma \cap U) \sim r$ as $r \to 0+$.

On the other hand, by definition of ψ, there exists a positive constant c such that $\psi(ct)$ is a subunit curve, and then

$$\varrho(\psi(r), 0) \le c_1 r$$

(in fact, it is easy to see that $\varrho(\psi(r), 0) \sim r$). For $\sigma, \varepsilon > 0$ to be chosen, consider the ball $\widetilde B = B(\psi(\varepsilon r), \sigma r)$. If $y \in \widetilde B$, then

$$\varrho(y, 0) \le \varrho(y, \psi(\varepsilon r)) + \varrho(\psi(\varepsilon r), 0) < \sigma r + c_1 \varepsilon r < r$$

if σ, ε are small enough, and hence $\widetilde B \subset B(0, r)$. Also, since $|x - y| \le c_2 \varrho(x, y)$, $\widetilde B \subset B_{\mathrm{euc}}(\psi(\varepsilon r), c_2 \sigma r)$. Thus, for any $y \in \widetilde B$ and $z \in U \cap \Sigma$, we have

$$\begin{aligned} |y - z| &\ge |z - \psi(\varepsilon r)| - |\psi(\varepsilon r) - y| \ge |z - \psi(\varepsilon r)| - c_2 \sigma r \\ &\ge c_3 \varepsilon r - c_2 \sigma r \ge \tfrac{1}{2} c_3 \varepsilon r > 0 \end{aligned}$$

if σ is small enough. Hence, $\widetilde{B} \cap \Sigma = \emptyset$ and then $\widetilde{B} \subset B(0,r) \setminus E$. On the other hand, by doubling, $|\widetilde{B}| \sim |B(0,r)|$. We may repeat the same argument with $\nu(0)$ replaced by $-\nu(0)$, and then we have proved that both $|B(0,r) \cap E|$ and $|B(0,r) \setminus E|$ are equivalent to $|B(0,r)|$. This completes the proof of Theorem 7.

Let us now prove a result concerning the size of the set of characteristic points. We use H_γ to denote γ-dimensional Hausdorff measure in $\mathbb{R}^n$.

THEOREM 8. *Let Σ be a C^1 manifold of codimension* 1 *in* $\mathbb{R}^n$, *and let Σ_0 be the set of characteristic points of Σ* (*i.e.*, *the set of points $x \in \Sigma$ such that $X_j(x)$ is tangent to Σ at x for $j = 1, \ldots, p$*). *If $n > 2$, then for any $\varepsilon > 0, H_{n-2+\varepsilon p}(\Sigma_0) = 0$. If $n = 2$, the set Σ_0 consists of isolated points. Moreover*, *these results are sharp.*

Proof. To show that the result is sharp, we consider the following example in $\mathbb{R}^3$:

$$X_1 = \partial_1, \qquad X_2 = x_1\partial_2, \qquad X_3 = \partial_3$$

and $\Sigma = \{x_2 = 0\}$. In this case, $\Sigma_0 = \{x_1 = x_2 = 0\}$ is a linear manifold of dimension 1 in $\mathbb{R}^3$.

Let $\overline{x} \in \Sigma_0$ be a characteristic point. By a local change of variables, we can map $\overline{x}$ into the origin and Σ into $\{x_n = 0\}$. Thus, since the Hörmander condition is invariant under diffeomorphisms, we can restrict ourselves to proving the assertion in a neighborhood U of the origin where $\Sigma = \{x_n = 0\}$, so that if we use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, and let $c_{nj}(x)$ denote the coefficient of ∂_n in X_j, then Σ_0 is given by

$$\Sigma_0 = \{x = (x', 0) \in U : c_{n1}(x', 0) = \ldots = c_{np}(x', 0) = 0\}.$$

Next, we will prove that there exists at least one index $i \in \{1, \ldots, p\}$ and a multi-index $\beta \in (\mathbb{N} \cup \{0\})^{n-1}$ such that

$$D^\beta_{x'} c_{ni}(0) \neq 0. \tag{21}$$

To show this, we will argue by contradiction, proving that if this statement does not hold, then the Hörmander condition fails to be true at the origin. More precisely, we will prove that each iterated commutator of $X_1, \ldots, X_p$ has zero nth component at the origin, and so lies in Σ, a contradiction since then ∂_n does not belong to the Lie algebra generated by $X_1, \ldots, X_p$ at 0. It will be enough to prove the following assertion:

Let $X = \sum_l c_l \partial_l$ and $Y = \sum_l d_l \partial_l$ be smooth vector fields. If $D^\alpha_{x'} c_n(0) = D^\alpha_{x'} d_n(0) = 0$ for every $\alpha \in (\mathbb{N} \cap \{0\})^{n-1}$, then the same is true for $[X, Y]$, i.e.,

$$D^\alpha_{x'}([X, Y]_n)(0) = 0 \quad \textit{for every } \alpha \in (\mathbb{N} \cup \{0\})^{n-1}.$$

Now
$$D^\alpha_{x'}[X,Y]_n = D^\alpha_{x'}\Big(\sum_l c_l(\partial_l d_n)\Big) - D^\alpha_{x'}\Big(\sum_l d_l(\partial_l c_n)\Big) = I_1 - I_2.$$
Let us prove that, for instance, I_1 vanishes at $x=0$. By Leibniz' formula,
$$\begin{aligned} I_1 &= \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\sum_l (D^\gamma_{x'}c_l)(D^{\alpha-\gamma}_{x'}\partial_l d_n) \\ &= \sum_{l\le n-1}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}(D^\gamma_{x'}c_l)(D^{\alpha-\gamma}_{x'}\partial_l d_n) \\ &\quad+\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}(D^\gamma_{x'}c_n)(D^{\alpha-\gamma}_{x'}\partial_n d_n) = 0\end{aligned}$$
at $x=0$, since $(D^\gamma_{x'}c_n)(0)=0$ and $(D^{\alpha-\gamma}_{x'}\partial_l d_n)(0)=0$ where $l\le n-1$, since the derivative is taken only with respect to the first $n-1$ variables. Thus (21) is proved.

Suppose now $n>2$. Since $\Sigma_0\subset\{(x',0)\in U : c_{ni}(x',0)=0\}$, where i is as in (21), it will be enough to prove that the function f defined in a neighborhood U' of the origin in $\mathbb{R}^{n-1}$ by
$$f(x') = c_{ni}(x',0)$$
vanishes on a set of Hausdorff dimension at most $n-2$. Arguing as in [S], p. 343, there is a vector $\xi'\in\mathbb{R}^{n-1}$, $|\xi'|=1$, such that
$$|(\xi'\cdot\nabla_{x'})^k f(0)|>0 \quad \text{ for } k=|\beta|.$$
Then, by using a rotation and denoting the generic point $x'\in\mathbb{R}^{n-1}$ by (t,x''), $t\in\mathbb{R}$, $x''\in\mathbb{R}^{n-1}$, we may assume that
$$\frac{\partial^k}{\partial t^k}f(t,x'')\Big|_{(t,x'')=(0,0)} > 0.$$
Moreover, without loss of generality, we may also assume that
$$\frac{\partial^l}{\partial t^l}f(t,x'')\Big|_{(t,x'')=(0,0)} = 0 \quad \text{ for } l=0,1,\dots,k-1,$$
so that we can apply Malgrange's preparation theorem (see [H], Theorem 7.5.5) and write, by shrinking U' if necessary,
$$f(t,x'') = g(t,x'')(t^k + a_{k-1}(x'')t^{k-1}+\ldots+a_0(x''))$$
with $g(t,x'')>0$ in U' and $a_j(0)=0$ for $j=0,\dots,k-1$. Thus, the theorem will be completely proved for $n>2$ by showing that
$$\{(t,x'')\in U' : t^k + a_k(x'')t^{k-1}+\ldots+a_0(x'') \coloneqq p_k(t,x'')=0\}\subset\bigcup_{j=1}^k V_j, \tag{22}$$
where each V_j is an $(n-2)$-dimensional manifold.

To prove (22), we will argue by induction on k. If $k = 1$, the set $\{t + a_0(x'') = 0\}$ is an $(n-2)$-dimensional manifold in $\mathbb{R}^{n-1}$. Suppose now that (22) holds for k and let us prove that it also holds for $k+1$. We can write

$$\{p_{k+1}(t,x'') = 0\} = \left\{p_{k+1}(t,x'') = 0,\ \frac{\partial}{\partial t}p_{k+1}(t,x'') \neq 0\right\} \\ \cup \left\{p_{k+1}(t,x'') = 0,\ \frac{\partial}{\partial t}p_{k+1}(t,x'') = 0\right\}.$$

The first set above is an $(n-2)$-dimensional manifold. The other one is contained in $\left\{\frac{\partial}{\partial t}p_{k+1}(t,x'') = 0\right\}$, which is contained in the union of k submanifolds of dimension $n-2$ by the induction hypothesis since $\frac{\partial}{\partial t}p_{k+1}$ is a polynomial in t with the same structure as p_k. This completes the proof for $n > 2$.

In case $n = 2$, by applying Rolle's theorem repeatedly, it follows that the set of zeros of $f(x')$ cannot have accummulation points, or else the derivatives of all orders of f would vanish at 0. Hence the point $x' = 0$ is isolated, and the proof of Theorem 8 is complete.

Let us show that inequality (20) improves some known isoperimetric inequalities for Grushin vector fields, and let us consider the case $n = 2$ for simplicity. If $X_1 = \partial_1, X_2 = x_1^\beta \partial_2, \beta \in \mathbb{N}$, then by Proposition 5, the pair $(\mu, s) = (|x_1|^\beta, 2)$ is a compensation couple. We will check (20) for sets that are not far away from the degeneration line $x_1 = 0$ and that do not intersect this line in a large set. (However, note that in any case $1/s = 1/2 \leq (\alpha-1)/\alpha = (\beta+1)/(\beta+2)$ with $\alpha = \beta+2$ as in (2), for B with center 0.) Thus, consider (by a simple limit argument) sets E of the form $E_{\gamma,\delta} = \{(x_1,x_2) : 0 \leq x_1 \leq \delta,\ |x_2| \leq x_1^\gamma\}$ for $0 < \delta < 1$ and $\gamma \geq \beta+1$. We can choose, by the characterization of the balls for Grushin type vector fields ([FL], [F]), $B = [-\delta,\delta] \times [-\delta^{\beta+1}, \delta^{\beta+1}]$, so that the left-hand side of (20) takes the form

$$\left(\int_0^\delta |x_1|^{\beta+\gamma}\, dx_1\right)^{1/2} \sim \delta^{(\beta+\gamma+1)/2},$$

while the left-hand side of (3)$'$ or (3) is

$$\left(\int_0^\delta |x_1|^{\gamma}\, dx_1\right)^{(\beta+1)/(\beta+2)} \sim \delta^{(\gamma+1)(\beta+1)/(\beta+2)}.$$

Since $(\beta+\gamma+1)/2 < (\gamma+1)(\beta+1)/(\beta+2)$ when $\gamma > \beta+1$ and since $0 < \delta < 1$, it follows that (20) is a better estimate than (3)$'$ in the sense that (20) implies (3)$'$.

On the other hand, if we instead pick E to be the set $x_2 > |x_1|^\gamma$ and choose $\gamma < \beta + 1$, we obtain an example where $(3)'$ is better than (20): in fact,

$$\frac{|B|^{1/\alpha}}{\delta}\min\{|B\cap E|, |B\setminus E|\}^{(\alpha-1)/\alpha} \sim |B\cap E|^{(\beta+1)/(\beta+2)} \sim \Big(\int_0^{\delta^{\beta+1}} x_2^{1/\gamma}\,dx_2\Big)^{(\beta+1)/(\beta+2)} \sim \delta^{(\beta+1)^2(\gamma+1)/\gamma(\beta+2)},$$

but

$$\min\{\mu(B\cap E), \mu(B\setminus E)\}^{1/s} = \mu(B\cap E)^{1/2} = \Big(\iint_{B\cap E} |x_1|^\beta\,dx_1\,dx_2\Big)^{1/2} \sim \Big(\int_0^{\delta^{\beta+1}}\Big(\int_0^{x_2^{1/\gamma}} x_1^\beta\,dx_1\Big)\,dx_2\Big)^{1/2} \sim \delta^{(\beta+1)(\beta+\gamma+1)/(2\gamma)}.$$

Since $\beta + 1 > \gamma$, this last exponent is larger than the other one, and consequently $(3)'$ implies (20) for small δ. Note that by choosing $\gamma > 1$, ∂E is a C^1 curve in this case and the origin is a characteristic point for X_1, X_2.

REFERENCES

[C] A. P. Calderón, *Inequalities for the maximal function relative to a metric*, Studia Math. 57 (1976), 297–306.

[CDG] L. Capogna, D. Danielli and N. Garofalo, *The geometric Sobolev embedding for vector fields and the isoperimetric inequality*, Comm. Anal. Geom. 2 (1994), 203–215.

[CW] R. R. Coifman et G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. 242, Springer, 1971, 1–158.

[CUN] D. Cruz-Uribe, SFO, and C. J. Neugebauer, *The structure of the reverse Hölder classes*, Trans. Amer. Math. Soc. 347 (1995), 2941–2960.

[FP] C. Fefferman and D. H. Phong, *Subelliptic eigenvalue estimates*, in: Conference on Harmonic Analysis in Honor of A. Zygmund, Chicago, 1980, W. Beckner *et al.* (eds.), Wadsworth, 1981, 590–606.

[F] B. Franchi, *Weighted Sobolev–Poincaré inequalities and pointwise estimates for a class of degenerate elliptic equations*, Trans. Amer. Math. Soc. 327 (1991), 125–158.

[FGaW1] B. Franchi, S. Gallot et R. L. Wheeden, *Inégalités isopérimétriques pour des métriques dégénérées*, C. R. Acad. Sci. Paris Sér. I Math. 317 (1993), 651–654.

[FGaW2] —, —, —, *Sobolev and isoperimetric inequalities for degenerate metrics*, Math. Ann. 300 (1994), 557–571.

[FGuW] B. Franchi, C. E. Gutiérrez and R. L. Wheeden, *Weighted Sobolev–Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations 19 (1994), 523–604.

[FLW] B. Franchi, G. Lu and R. L. Wheeden, *Representation formulas and weighted Poincaré inequalities for Hörmander vector fields*, Ann. Inst. Fourier (Grenoble) 45 (1995), 577–604.

[G] M. Gromov, *Carnot–Carathéodory spaces seen from within*, Prépublications de l'IHES (1994).

[H] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Grundlehren Math. Wiss. 256, Springer, Berlin, 1983.

[NSW] A. Nagel, E. M. Stein and S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math. 155 (1985), 103–147.

[S] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, 1993.

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