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REPRESENTATION RING OF THE SEQUENCE OF ALTERNATING GROUPS

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1. Introduction. It is known (see [L]) that the sequence $S_1, S_2, \ldots$ of symmetric groups and canonical inclusion maps $S_p \times S_q \to S_{p+q}$ gives rise to a graded bicommutative Hopf algebra $(R(S_\infty), \phi, \psi)$, where $R(S_\infty) = \bigoplus_{n=0}^{\infty} R(S_n)$ is the direct sum of the additive groups of the complex representation rings of S_n (in degree $2n$), $R(S_0) = \mathbb{Z}$. The multiplication $\phi : R(S_\infty) \otimes R(S_\infty) \to R(S_\infty)$ comes from the induction maps

$$\mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} : R(S_p) \otimes R(S_q) \approx R(S_p \times S_q) \to R(S_{p+q})$$

and the comultiplication $\psi : R(S_\infty) \to R(S_\infty) \otimes R(S_\infty)$ comes from the restriction maps

$$\mathrm{Res}_{S_p \times S_q}^{S_{p+q}} : R(S_{p+q}) \to R(S_p \times S_q) \approx R(S_p) \otimes R(S_q).$$

The Hopf algebra $(R(S_\infty), \phi, \psi)$ is isomorphic to the Hopf algebra $\mathbb{Z}[y_1, y_2, \ldots]$ with the usual multiplication ($y_1, y_2, \ldots$ being algebraically independent over $\mathbb{Z}$); moreover, $\psi(y_n) = \sum_{p+q=n} y_p \otimes y_q$, each y_n has degree $2n$ and corresponds to a trivial representation of S_n.

In the present paper we apply a similar construction to the sequence $A_1, A_2, \ldots$ of alternating groups. The canonical inclusion maps $A_p \times A_q \to A_{p+q}$, $A_q \times A_p \to A_{p+q}$ are not conjugate (for odd p, q), as they are in the case of symmetric groups. This results in the non-commutativity of the ring $R(A_\infty) = \bigoplus_{n=0}^{\infty} R(A_n)$. The structure of this ring is described in Theorems 1 and 2. It is also shown that the canonical comultiplication does not induce a Hopf algebra structure on $R(A_\infty)$.

2. Notations. We denote by A_n, $n = 1, 2, \ldots$, the group of even permutations on n letters. The canonical inclusion maps $A_p \times A_q \to A_{p+q}$ give

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rise to the induction maps

$$\operatorname{Ind}_{A_p\times A_q}^{A_{p+q}} : R(A_p)\otimes R(A_q)\approx R(A_p\times A_q)\to R(A_{p+q})$$

which are associative in an obvious sense; thus we have the graded ring $R(A_\infty)=\bigoplus_{n=0}^{\infty} R(A_n)$, where $R(A_n)$ is the additive group of the complex representation ring of the group A_n, having degree $2n$, $R(A_0)=\mathbb{Z}$ and multiplication $R(A_\infty)\otimes R(A_\infty)\to R(A_\infty)$ comes from the above induction maps.

For any partition α of n (written $\alpha\vdash n$) let α' denote the conjugate partition and let $[\alpha]$ be the corresponding equivalence class of an irreducible representation of S_n in $R(S_n)$ which is a free $\mathbb{Z}$-module with basis $\{[\alpha]\}_{\alpha\vdash n}$. The maps $\omega_n : R(S_n)\to R(S_n)$, $[\alpha]\mapsto[\alpha']$, determine an automorphism $\omega=\bigoplus\omega_n$ of the Hopf algebra $R(S_\infty)$ (see [L]).

We denote by $\iota_n : R(A_n)\to R(S_n)$ and $r_n : R(S_n)\to R(A_n)$ the induction and restriction maps. It is clear that $\iota=\bigoplus\iota_n : R(A_\infty)\to R(S_\infty)$ is a ring homomorphism. It is well known (see [JK, 2.5.7]) that for $\alpha\vdash n$,

(1) if $\alpha\neq\alpha'$ then $r_n([\alpha])=r_n([\alpha'])$ is the class of an irreducible representation of A_n; we denote it by $a_\alpha=a_{\alpha'}$ ($[\alpha]\downarrow A_n$ in the notation of [JK]);

(2) if $\alpha=\alpha'$ and $n\geq 2$ then $r_n([\alpha])$ is a sum of two distinct classes of irreducible representations of A_n which we denote by a_α^+ and a_α^- ($[\alpha]^+$, $[\alpha]^-$ in the notation of [JK]).

In this way, for $\alpha\vdash n$ we get all classes of irreducible representations of A_n. Moreover,

(1′) if $\alpha\neq\alpha'$ then $\iota_n(a_\alpha)=[\alpha]+[\alpha']$;

(2′) if $\alpha=\alpha'$ and $n>2$ then $\iota_n(a_\alpha^+)=\iota_n(a_\alpha^-)=[\alpha]$.

To describe the characters ζ^{α^+}, ζ^{α^-} of the representations a_α^+, a_α^- let $h(\alpha)=(h_1^\alpha,\ldots,h_s^\alpha)$ be the decreasing sequence of the lengths of the main hooks (i.e. (i,i)-hooks) of the Young diagram associated with α. The conjugacy class $C^{h(\alpha)}$ of S_n, consisting of all permutations with cycle partition $h(\alpha)$, is the only class which splits over A_n into two classes $C^{h(\alpha)^+}$, $C^{h(\alpha)^-}$ on which each of the characters ζ^{α^+}, ζ^{α^-} takes distinct values. The class $C^{h(\alpha)^+}$ contains an element $(1,2,\ldots,h_1)(h_1+1,\ldots,h_1+h_2)\ldots$ and $C^{h(\alpha)^-}$ is conjugate to $C^{h(\alpha)^+}$ by any transposition. We have (see [JK, 2.5.13])

$$\zeta^{\alpha^\pm}_{h(\alpha)^+}=u_\alpha\pm v_\alpha,\qquad \zeta^{\alpha^\pm}_{h(\alpha)^-}=u_\alpha\mp v_\alpha, \tag{3}$$

where $u_\alpha=\frac12\zeta^\alpha_{h(\alpha)}$, $v_\alpha=\frac12(\zeta^\alpha_{h(\alpha)}\prod_{i=1}^s h_i^\alpha)^{1/2}$ and ζ^α_γ denotes the value of the character ζ^α of $[\alpha]$ on the conjugacy class C^γ of S_n and similarly for

$\zeta^{\alpha^\pm}$. Moreover,

$$\zeta^{\alpha^\pm}_\gamma = \zeta^\alpha_\gamma/2 \quad \text{for } \gamma \neq h(\alpha). \tag{4}$$

For a finite group G we denote by $R(G)$ the complex representation ring of G and by $(\ ,\ )_G$, or briefly $(\ ,\ )$, the Schur inner product of representations as well as that of their classes or characters. The classes of irreducible representations form an orthonormal basis of $R(G)$ with respect to this product.

3. Lemmas. A crucial role is played by the following

LEMMA 1. *Let ξ, η be irreducible characters of the groups A_p, A_q and let ζ^{ε^+}, ζ^{ε^-} be the irreducible characters of A_{p+q} corresponding to a self-conjugate partition $\varepsilon = \varepsilon'$ of $p+q$. Let $h(\varepsilon) = (h^\varepsilon_1, \ldots, h^\varepsilon_k)$ be the decreasing sequence of the lengths of the main hooks of the Young diagram associated with ε. Denote by $\xi\eta$ the character induced on A_{p+q} by the character $\xi \otimes \eta$ of $A_p \times A_q$.*

If $p \geq 2$ and $q \geq 2$ then

$$(\xi\eta, \zeta^{\varepsilon^+} - \zeta^{\varepsilon^-})_{A_{p+q}} = (\pm 1)_\alpha (\pm 1)_\beta \operatorname{sgn}(I, J) \tag{5}$$

for $\xi = \zeta^{\alpha^\pm}$, $\eta = \zeta^{\beta^\pm}$, where α, β are self-conjugate partitions $\alpha = \alpha' \vdash p$, $\beta = \beta' \vdash q$ with $h(\alpha) = (h^\varepsilon_{i_1}, \ldots, h^\varepsilon_{i_s})$, $h(\beta) = (h^\varepsilon_{j_1}, \ldots, h^\varepsilon_{j_r})$ such that $I = \{i_1, \ldots, i_s\}$, $J = \{j_1, \ldots, j_r\}$ form a decomposition of $\{1, \ldots, k\}$. For all other pairs of characters ξ, η the inner product (5) *is zero.*

If $h^\varepsilon_k = 1$ and either $p \geq 2$, $q = 1$ or $p = 1$, $q \geq 2$ then for a self-conjugate partition γ such that $h(\gamma) = (h^\varepsilon_1, \ldots, h^\varepsilon_{k-1})$ we have

$$(\zeta^{\gamma^\pm}\zeta^1, \zeta^{\varepsilon^+} - \zeta^{\varepsilon^-}) = (\pm 1)_\gamma = (-1)^{k-1}(\zeta^1\zeta^{\gamma^\pm}, \zeta^{\varepsilon^+} - \zeta^{\varepsilon^-}),$$

where ζ^1 denotes a trivial character of A_1.

If either $h^\varepsilon_k > 1$ or ξ is an irreducible character different from $\zeta^{\gamma^\pm}$ then

$$(\xi\zeta^1, \zeta^{\varepsilon^+} - \zeta^{\varepsilon^-}) = (\zeta^1\xi, \zeta^{\varepsilon^+} - \zeta^{\varepsilon^-}) = 0.$$

Proof. To simplify notation we write h_i instead of h^ε_i. Since the characters ζ^{ε^+}, ζ^{ε^-} differ only on elements of $C^{h(\varepsilon)}$, by Frobenius reciprocity we get

$$\begin{aligned}(\xi\eta, \zeta^{\varepsilon^+} - \zeta^{\varepsilon^-})_{A_{p+q}} &= (\operatorname{Ind}^{A_{p+q}}_{A_p\times A_q}(\xi\otimes\eta), \zeta^{\varepsilon^+} - \zeta^{\varepsilon^-})_{A_{p+q}} \\ &= (\xi\otimes\eta, \operatorname{Res}^{A_{p+q}}_{A_p\times A_q}(\zeta^{\varepsilon^+} - \zeta^{\varepsilon^-}))_{A_p\times A_q} \\ &= m_{pq}\sum \xi(t')\eta(t'')[\zeta^{\varepsilon^+}(t) - \zeta^{\varepsilon^-}(t)],\end{aligned}$$

where $m_{pq} = |A_p \times A_q|^{-1}$ and $t = (t', t'')$ runs over $(A_p \times A_q) \cap C^{h(\varepsilon)}$.

Define $C_+ = (A_p \times A_q) \cap C^{h(\varepsilon)^+}$ and $C_- = (A_p \times A_q) \cap C^{h(\varepsilon)^-}$. Then (3) implies

$$(\xi\eta, \zeta^{\varepsilon^+} - \zeta^{\varepsilon^-}) = m_{pq}\Big[\sum_{C_+} \xi(t')\eta(t'')2v_\varepsilon + \sum_{C_-} \xi(t')\eta(t'')(-2v_\varepsilon)\Big]$$
$$= 2v_\varepsilon m_{pq}\Big[\sum_{C_+} \xi(t')\eta(t'') - \sum_{C_-} \xi(t')\eta(t'')\Big].$$

To compute this sum we shall represent C_+ and C_- as unions of products of conjugacy classes of A_p and of A_q.

It is clear that if $h_{i_1}+\ldots+h_{i_s} \neq p$ for each sequence $1 \le i_1 < \ldots < i_s \le k$ then the set $(A_p \times A_q) \cap C^{h(\varepsilon)}$ is empty and thus the inner product is zero. Let us denote by $\mathcal{I}$ the set of all sequences $I = \{1 \le i_1 < \ldots < i_s \le k\}$ such that $h_{i_1} + \ldots + h_{i_s} = p$. Any such sequence has a complementary sequence $J = \{1 \le j_1 < \ldots < j_r \le k\}$ such that $r + s = k$, $h_{j_1} + \ldots + h_{j_r} = q$.

Consider a fixed sequence I. The conjugacy class $C^{h(\varepsilon)^+}$ of A_{p+q} is determined by its element

$$t_{p+q}(\varepsilon) = (1, 2, \ldots, h_1)(h_1 + 1, \ldots, h_1 + h_2) \ldots (\ldots, p + q) \tag{6}$$

and consists of all elements of the form $(t_{p+q}(\varepsilon))^\sigma$ for $\sigma \in A_{p+q}$; the class $C^{h(\varepsilon)^-}$ consists of all elements of the same form for $\sigma \in S_{p+q}\backslash A_{p+q}$ and $C^{h(\varepsilon)^-} = (C^{h(\varepsilon)^+})^{\tau'}$, where $\tau' = (1, 2)$ is a transposition.

For $I \in \mathcal{I}$, denote by $h(I)$ the partition $(h_{i_1}, \ldots, h_{i_s})$ of p and by $h(J)$ the partition $(h_{j_1}, \ldots, h_{j_r})$ of q. They determine, as in (6), elements

$$t'_p = t'_p(I) = (1, 2, \ldots, h_{i_1}) \ldots (\ldots, p) \in C^{h(I)^+} \subseteq A_p \times \{1\},$$
$$t''_q = t''_q(J) = (p + 1, \ldots, p + h_{j_1}) \ldots (\ldots, p + q) \in C^{h(J)^+} \subseteq \{1\} \times A_q.$$

The element $t'_p(I)t''_q(J)$ has the same cycle partition as $t_{p+q}(\varepsilon)$. Hence it is equal to $(t_{p+q}(\varepsilon))^\tau$ for some $\tau \in S_{p+q}$. It is easy to check that since all h_i are odd, we have $\mathrm{sgn}(\tau) = \mathrm{sgn}(I, J) = \mathrm{sgn}(i_1, \ldots, i_s, j_1, \ldots, j_r)$.

Assume now that p and q are both ≥ 2. In this case we have

$$(A_p \times A_q) \cap C^{h(\varepsilon)} = \bigcup C^{h(I)} \times C^{h(J)}$$
$$= \bigcup (C^{h(I)^+} \times C^{h(J)^+} \cup C^{h(I)^-} \times C^{h(J)^-})$$
$$\cup \bigcup (C^{h(I)^+} \times C^{h(J)^-} \cup C^{h(I)^-} \times C^{h(J)^+}),$$

where I runs over $\mathcal{I}$ and it is easy to verify that if $\mathrm{sgn}(I, J) = 1$ (resp. -1) then the first union is contained in C_+ (resp. C_-) and the second one in C_- (resp. C_+). Moreover, we know that $C^{h((I)^-} = (C^{h(I)^+})^{\tau'}$, $C^{h(J)^-} = (C^{h(J)^+})^{\tau''}$, where $\tau' = (1, 2)$ and $\tau'' = (p + q - 1, p + q)$ are transpositions.

We can continue the computation of the inner product:

$$
\begin{aligned}
(\xi\eta, \zeta^{\varepsilon^+} - \zeta^{\varepsilon^-})_{A_{p+q}}
&= 2v_\varepsilon m_{pq} \sum_{I\in\mathcal{I}} \operatorname{sgn}(I,J) \sum [\xi(t')\eta(t'') + \xi((t')^{\tau'})\eta((t'')^{\tau''}) \\
&\quad - \xi(t')\eta((t'')^{\tau''}) - \xi((t')^{\tau'})\eta(t'')] \\
&= 2v_\varepsilon m_{pq} \sum_{I\in\mathcal{I}} \operatorname{sgn}(I,J) \sum [\xi(t') - \xi((t')^{\tau'})][\eta(t'') - \eta((t'')^{\tau''})],
\end{aligned}
$$

where (t', t'') runs over $(C^{h(I)^+}) \times (C^{h(J)^+})$.

An irreducible character ξ of the group A_p satisfies $\xi(t') = \xi((t')^{\tau'})$ for $t' \in A_p$ with the only exception when $\xi = \zeta^{\gamma^\pm}$ and t' belongs to the conjugacy class $C^{h(\gamma)}$ of elements with cycle partition $h(\gamma)$; similarly for η. Hence a summand in the last sum corresponding to I is non-zero only if $\xi = \zeta^{\alpha^\pm}$ and $\eta = \zeta^{\beta^\pm}$, where α denotes a self-conjugate partition of p with main hook lengths $h(\alpha) = h(I)$ and $\beta \vdash q$ is similarly related to J, $h(\beta) = h(J)$. In this case all the remaining terms are zero. For $\xi = \zeta^{\alpha^\pm}$, $\eta = \zeta^{\beta^\pm}$ we get

$$
\begin{aligned}
&(\xi\eta, \zeta^{\varepsilon^+} - \zeta^{\varepsilon^+}) \\
&\quad = 2v_\varepsilon m_{pq} \operatorname{sgn}(I,J) \frac{1}{2}|C^{h(\alpha)}| \frac{1}{2}|C^{h(\beta)}| [\xi(t'_p) - \xi((t'_p)^{\tau'})][\eta(t''_q) - \eta((t''_q)^{\tau''})].
\end{aligned}
$$

Using (3) and the formulas

$$
\zeta^\alpha_{h(\alpha)} = (-1)^{(p-s)/2}, \qquad |C^{h(\alpha)}| = p!/\prod_{m=1}^{s} h_{i_m}
$$

and a similar one for β (see [JK, 2.5.12 and 1.2.15]) we get

$$
(\zeta^{\alpha^\pm}\zeta^{\beta^\pm}, \zeta^{\varepsilon^+} - \zeta^{\varepsilon^-}) = (\pm 1)_\alpha (\pm 1)_\beta \operatorname{sgn}(I,J)
$$

and the first part of the lemma is proved.

A similar computation applies also to the second part of the lemma. If $q = 1$ and $h_k = 1$ then $C_+ = C^{h(\gamma)^+} \times A_1$ and $C_- = C^{h(\gamma)^-} \times A_1$. If $p = 1$ and $h_k = 1$ then we have $C_+ = A_1 \times C^{h(\gamma)^+}$ and $C_- = A_1 \times C^{h(\gamma)^-}$ when k is odd, and C_+, C_- are to be interchanged for even k. ∎

R e m a r k 1. In the notation of Lemma 1 we have $s \equiv p$ and $r \equiv q \pmod 2$ because all h_i are odd. Thus the relations $\operatorname{sgn}(I,J) = (-1)^{sr} \operatorname{sgn}(I,J) = (-1)^{pq} \operatorname{sgn}(I,J)$ and (5) imply that $\zeta^{\alpha^\pm}$ and $\zeta^{\beta^\pm}$ do not commute if p, q are odd. Consequently, the ring $R(A_\infty)$ is not commutative.

Using the Littlewood–Richardson Rule one can prove by an easy induction on $r + s$ the following

LEMMA 2. *If* ε, α, β *are as in Lemma* 1 *then* $([\alpha][\beta],[\varepsilon]) = 1$.

4. The ring $R(A_\infty)$. We compute the structure constants of the ring $R(A_\infty)$ in terms of those of the ring $R(S_\infty)$ using the above lemmas.

THEOREM 1. *The structure constants of the ring* $R(A_\infty)$ *in the basis consisting of elements* $1, a_1 \in R(A_1), a_\varepsilon^+, a_\varepsilon^-$ *for* $\varepsilon = \varepsilon' \vdash n$, $n = 3, 4, \ldots$ *and elements* $a_\delta = a_{\delta'}$ *for* $\delta \neq \delta' \vdash n$, $n = 2, 3, \ldots$ *are as follows.*

Let p, q *be positive integers*, $\alpha \vdash p$, $\beta \vdash q$, $\varepsilon, \delta \vdash (p+q)$, $\varepsilon = \varepsilon'$, $\delta \neq \delta'$.

(i) *If* $\alpha = \alpha'$, $\beta = \beta'$, $p \geq 2$, $q \geq 2$ *then*

$$(a_\alpha^\pm a_\beta^\pm, a_\delta) = ([\alpha][\beta],[\delta]).$$

If the Young diagrams associated with α *and* β *have all main hooks of distinct lengths and* α, β, ε *are related as in the first part of Lemma* 1 *then* $(a_\alpha^\pm a_\beta^\pm, a_\varepsilon^\varrho) = 1$ *and* $(a_\alpha^\pm a_\beta^\pm, a_\varepsilon^{-\varrho}) = 0$ *for* $\varrho = (\pm 1)_\alpha (\pm 1)_\beta \operatorname{sgn}(I,J)$.

In the opposite case we have

$$(a_\alpha^\pm a_\beta^\pm, a_\varepsilon^\pm) = \tfrac{1}{2}([\alpha][\beta],[\varepsilon]).$$

(ii) *If* $\alpha = \alpha'$, $\beta \neq \beta'$, $p \geq 2$ *then*

$$(a_\alpha^\pm a_\beta, a_\delta) = (a_\beta a_\alpha^\pm, a_\delta) = ([\alpha]([\beta]+[\beta']),[\delta]),$$
$$(a_\alpha^\pm a_\beta, a_\varepsilon^\pm) = (a_\beta a_\alpha^\pm, a_\varepsilon^\pm) = ([\alpha][\beta],[\varepsilon]),$$

and consequently $a_\alpha^\pm a_\beta = a_\beta a_\alpha^\pm$.

(iii) *If* $\alpha \neq \alpha'$, $\beta \neq \beta'$ *then*

$$(a_\alpha a_\beta, a_\delta) = (a_\beta a_\alpha, a_\delta) = (([\alpha]+[\alpha'])([\beta]+[\beta']),[\delta]),$$
$$(a_\alpha a_\beta, a_\varepsilon^\pm) = (a_\beta a_\alpha, a_\varepsilon^\pm) = ([\alpha]([\beta]+[\beta']),[\varepsilon]) = (([\alpha]+[\alpha'])[\beta],[\varepsilon]),$$

and consequently $a_\alpha a_\beta = a_\beta a_\alpha$.

(iv) *If* $\gamma = \gamma' \vdash p$, $p \geq 2$ *then*

$$(a_\gamma^\pm a_1, a_\delta) = (a_1 a_\gamma^\pm, a_\delta) = ([\gamma][1],[\delta]).$$

If the Young diagram associated with γ *does not have a main hook of length one and* γ, ε *are related as in the second part of Lemma* 1 *then*

$$(a_\gamma^\pm a_1, a_\varepsilon^\lambda) = 1 \quad \textit{and} \quad (a_\gamma^\pm a_1, a_\varepsilon^{-\lambda}) = 0 \quad \textit{for } \lambda = (\pm 1)_\gamma;$$
$$(a_1 a_\gamma^\pm, a_\varepsilon^\mu) = 1 \quad \textit{and} \quad (a_1 a_\gamma^\pm, a_\varepsilon^{-\mu}) = 0 \quad \textit{for } \mu = (\pm 1)_\gamma (-1)^{k-1}.$$

In the opposite case we have

$$(a_\gamma^\pm a_1, a_\varepsilon^+) = (a_\gamma^\pm a_1, a_\varepsilon^-) = (a_1 a_\gamma^\pm, a_\varepsilon^+) = (a_1 a_\gamma^\pm, a_\varepsilon^-) = 0.$$

(v) $a_1 a_1 = a_2$, *where* a_2 *denotes the class of trivial representations of* A_2.

Proof. Let $x \in R(A_p)$, $y \in R(A_q)$ and suppose $xy \in R(A_{p+q})$ has in our basis a representation

$$xy = \sum (m_\varepsilon^+ a_\varepsilon^+ + m_\varepsilon^- a_\varepsilon^-) + \sum m_\delta a_\delta \quad \text{for } m_\varepsilon^+, m_\varepsilon^-, m_\delta \in \mathbb{Z}.$$

Since $\iota_{p+q}(xy) = \iota_p(x)\iota_q(y)$, by (1′), (2′) we get

$$\iota_p(x)\iota_q(y) = \sum (m_\varepsilon^+ + m_\varepsilon^-)[\varepsilon] + \sum m_\delta([\delta] + [\delta'])$$

and consequently

$$\begin{aligned} m_\varepsilon^+ + m_\varepsilon^- &= (xy, a_\varepsilon^+ + a_\varepsilon^-) = (\iota_p(x)\iota_q(y), [\varepsilon]), \\ m_\delta &= (xy, a_\delta) = (\iota_p(x)\iota_q(y), [\delta]) = (\iota_p(x)\iota_q(y), [\delta']). \end{aligned} \tag{7}$$

Hence the first formula in (i) follows immediately. To prove the second one we use Lemma 1 to get $(a_\alpha^\pm a_\beta^\pm, a_\varepsilon^+ - a_\varepsilon^-) = \varrho$. Since by (7) and by Lemma 2 we have

$$(a_\alpha^\pm a_\beta^\pm, a_\varepsilon^+ + a_\varepsilon^-) = ([\alpha][\beta], [\varepsilon]) = 1,$$

our result follows. The last formula in (i) follows from Lemma 1, because in this case we have $(a_\alpha^\pm a_\beta^\pm, a_\varepsilon^+ - a_\varepsilon^-) = 0$ and $(a_\alpha^\pm a_\beta^\pm, a_\varepsilon^+ + a_\varepsilon^-) = ([\alpha][\beta], [\varepsilon])$.

The remaining parts of the theorem can be proved in the same way using Lemma 1, the equality $([\alpha][\beta], [\gamma]) = ([\alpha'][\beta'], [\gamma'])$ and the Littlewood–Richardson Rule. ∎

The next theorem presents a better insight into the structure of the ring $R(A_\infty)$.

Let $\bigwedge = \bigwedge(z_1, z_3, z_5, \ldots)$ be the graded exterior $\mathbb{Z}$-algebra on a free $\mathbb{Z}$-module with a basis $\{z_1, z_3, z_5, \ldots\}$ and grading $\deg z_{2j+1} = 2(2j+1)$, $j = 0, 1, \ldots$ Let Γ be an ideal of $\bigwedge$ with a $\mathbb{Z}$-basis consisting of all monomials $2z_{l_1} \wedge z_{l_2} \wedge \ldots \wedge z_{l_k}$ in $\bigwedge$ such that $l_1 > \ldots > l_k$ and either $k > 1$, or $k = 1$ and $l_1 > 1$. We define a homomorphism of $\mathbb{Z}$-modules $g : \Gamma \to R(A_\infty)$ by the formula

$$g(2z_{l_1} \wedge z_{l_2} \wedge \ldots \wedge z_{l_k}) = a_\alpha^+ - a_\alpha^-,$$

where α denotes the self-conjugate partition of $l_1 + \ldots + l_k$ with main hook lengths $l_1, \ldots, l_k$.

Theorem 2. *The image of the homomorphism*

$$\iota = \bigoplus \iota_n : R(A_\infty) \to R(S_\infty)$$

is a subring $T = \bigoplus_{n=0}^\infty T_n$ of the ring $R(S_\infty)$, where

$$T_n = \{x_n \in R(S_n) : \omega_n(x_n) = x_n\}$$

is a free $\mathbb{Z}$-module with the basis consisting of the elements of the form $[\varepsilon]$ for $\varepsilon = \varepsilon' \vdash n$ and $[\delta] + [\delta']$ for $\delta \neq \delta' \vdash n$.

The kernel $L = \bigoplus_{n=3}^\infty L_n$ of the homomorphism ι is a free $\mathbb{Z}$-module with basis consisting of the elements of the form $a_\varepsilon^+ - a_\varepsilon^-$ for all self-conjugate partitions ε. The homomorphism of $\mathbb{Z}$-modules $g : \Gamma \to R(A_\infty)$ maps Γ onto L and induces a ring isomorphism of Γ and L.

Proof. The formulas $\omega_n([\beta]) = [\beta']$ and (1′), (2′) imply the first part of the theorem and the description of L. Hence g induces an isomorphism of $\mathbb{Z}$-modules Γ and L. To prove that it is an isomorphism of rings (without unity) it is sufficient to prove that

$$(a_\alpha^+ - a_\alpha^-)(a_\beta^+ - a_\beta^-) = 0 \tag{8}$$

if α, β are self-conjugate partitions and $h_i^\alpha = h_j^\beta$ for some i, j and

$$(a_\alpha^+ - a_\alpha^-)(a_\beta^+ - a_\beta^-) = 2\,\mathrm{sgn}(I,J)(a_\varepsilon^+ - a_\varepsilon^-) \tag{9}$$

if the self-conjugate partitions α, β, ε satisfy the conditions of the first part of Lemma 1. Theorem 1(i) implies that in case (8) we have

$$(a_\alpha^+ - a_\alpha^-)a_\beta^+ = (a_\alpha^+ - a_\alpha^-)a_\beta^- = 0$$

and the formula follows.

To prove (9) let us remark that since L is an ideal, $(a_\alpha^+ - a_\alpha^-)(a_\beta^+ - a_\beta^-)$ is a linear combination of $a_\gamma^+ - a_\gamma^-$ for self-conjugate partitions γ. Theorem 1(i) implies that the only term that can occur with a non-zero coefficient corresponds to $\gamma = \varepsilon$. Define $\varrho = \mathrm{sgn}(I,J)$ (in the notation of Lemma 1). Then we have

$$((a_\alpha^+ - a_\alpha^-)(a_\beta^+ - a_\beta^-), a_\varepsilon^\varrho) = (a_\alpha^+ a_\beta^+ + a_\alpha^- a_\beta^- - a_\alpha^- a_\beta^+ - a_\alpha^+ a_\beta^-, a_\varepsilon^\varrho) = 2$$

because $(a_\alpha^+ a_\beta^+, a_\varepsilon^\varrho) = (a_\alpha^- a_\beta^-, a_\varepsilon^\varrho) = 1$ and $(a_\alpha^- a_\beta^+, a_\varepsilon^\varrho) = (a_\alpha^+ a_\beta^-, a_\varepsilon^\varrho) = 0$. Hence the formula follows. ∎

Remark 2. The natural comultiplication $\psi : R(A_\infty) \to R(A_\infty) \otimes R(A_\infty)$ induced by the restriction maps $R(A_{p+q}) \to R(A_p) \otimes R(A_q)$ is not a ring homomorphism, thus $R(A_\infty)$ is not a Hopf algebra. In fact, we have $a_2 = a_1 a_1$ and $\psi(a_2) = a_2 \otimes 1 + a_1 \otimes a_1 + 1 \otimes a_2$ but $\psi(a_1)\psi(a_1) = a_1 a_1 \otimes 1 + 2a_1 \otimes a_1 + 1 \otimes a_1 a_1$ hence $\psi(a_1 a_1) \neq \psi(a_1)\psi(a_1)$.

Remark 3. The ring $T \subset R(S_\infty)$ is not closed with respect to the comultiplication ψ of $R(S_\infty)$. In fact, we have

$$\begin{aligned}\psi([2,2]) = [2,2] \otimes 1 + 1 \otimes [2,2] + [2,1] \otimes [1] \\ + [1] \otimes [2,1] + [2] \otimes [2] + [1,1] \otimes [1,1]\end{aligned}$$

by general formulas or by an easy computation. The component in $R(S_2) \otimes R(S_2)$ of this element is

$$[2] \otimes [2] + [1,1] \otimes [1,1] = y_1^2 \otimes y_1^2 - y_1^2 \otimes y_2 - y_2 \otimes y_1^2 + 2y_2 \otimes y_2 \tag{10}$$

because $[2] + [1,1] = y_1^2$ and $[2] = y_2$. The component T_2 is generated by y_1^2 hence (10) does not belong to $T_2 \otimes T_2$.

Remark 4. The ring T is not generated by $[\varepsilon]$ for self-conjugate ε, neither are these elements algebraically independent. In fact, there are six monomials in $[\varepsilon]$'s which belong to T_6, namely

$$[1]^6, \quad [2,1][1]^3, \quad [2,1]^2, \quad [2,2][1]^2, \quad [3,1,1][1], \quad [3,2,1],$$

and we have a relation of linear dependence among them:

$$[3,2,1] + [2,1]^2 = [2,2][1]^2 + [3,1,1][1].$$

Nevertheless T_6 is a free $\mathbb{Z}$-module on six free generators.

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